PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3243–3247
S 0002-9939(02)06708-4
Article electronically published on June 11, 2002

ON MULHOLLAND'S INEQUALITY

WITOLD JARCZYK AND JANUSZ MATKOWSKI

(Communicated by Jonathan M. Borwein)

ABSTRACT. H.P. Mulholland has presented a sufficient condition for a generalization of the Minkowski inequality and another such condition was given by R.M. Tardiff. We show that Mulholland's condition implies Tardiff's, but that the converse is false.

In 1950 H.P. Mulholland [4] (see also [1, Theorem VIII.8.1]) proved the following:

If $\varphi : (0,\infty) \to (0,\infty)$ *is a continuous function with* $\lim_{x\to 0} \varphi(x) = 0$*, and if the functions* φ *and* $\log\circ\varphi\circ\exp$ *are convex, then*

$$\varphi^{-1}(\varphi(x_1+y_1)+\varphi(x_2+y_2)) \le \varphi^{-1}(\varphi(x_1)+\varphi(x_2)) + \varphi^{-1}(\varphi(y_1)+\varphi(y_2)) \tag{1}$$
$$\textit{for all} \quad x_1, x_2, y_1, y_2 \in (0,\infty).$$

Another result of this type was proved in 1984 by R.M. Tardiff [6] and reads as follows:

If $\varphi : (0,\infty) \to (0,\infty)$ *is a differentiable function with* $\varphi' > 0$ *and* $\lim_{x\to 0} \varphi(x) = 0$*, and if the functions* φ *and* $\log\circ\varphi'\circ\exp$ *are convex, then* (1) *holds.*

It is quite obvious that for $\varphi(x) = x^p$ with $p \ge 1$ each of the above results yields the classical Minkowski inequality. The function φ given by $\varphi(x) = \exp x - 1$ for $x \in (0,\infty)$ serves as an example to show that both Mulholland's and Tardiff's results yield inequalities distinct from Minkowski's. For an integral version of Mulholland's inequality the reader is referred to [3].

It is easy to verify that a continuous function $f : (0,\infty) \to (0,\infty)$ is such that $\log\circ f\circ\exp$ is convex if and only if

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)} \qquad \text{for all} \quad x \in (0,\infty),$$

i.e., if and only if f is convex with respect to the geometric mean. Accordingly, following the terminology introduced in [2], we will say that any such function f is *geometrically convex*. Note that geometrically convex functions have the same limiting and differentiability properties as ordinary convex functions (cf. [1, Chapter VII]). Since the sum of two convex functions is convex, it is also clear that the product of two geometrically convex functions is geometrically convex. Moreover, if $f : (0,\infty) \to (0,\infty)$ is increasing and $\log\circ f$ is convex, i.e. f is *logarithmically convex*, then f is geometrically convex. In fact, making use of the inequality

Received by the editors June 4, 2001.
2000 *Mathematics Subject Classification.* Primary 39B72, 26D15; Secondary 26A51.
Key words and phrases. Generalizations of the Minkowski inequality, Mulholland's inequality, convex functions, geometrically convex functions.

$0 < \sqrt{xy} \le \frac{x+y}{2}$ for all $x, y \in (0, \infty)$, we have

$$f(\sqrt{xy}) = \exp(\log f(\sqrt{xy})) \le \exp\left(\log f\left(\frac{x+y}{2}\right)\right) \le \exp\frac{\log f(x) + \log f(y)}{2}$$
$$= \sqrt{f(x)f(y)}$$

for all $x, y \in (0, \infty)$.

Recently, B. Schweizer (private communication; see also [5]) posed the problem of comparing Mulholland's and Tardiff's results. His question can be formulated more precisely in the following way:

Let $\varphi : (0, \infty) \to (0, \infty)$ be a differentiable convex function with $\varphi' > 0$ and $\lim_{x\to 0} \varphi(x) = 0$. Is there any relation between the geometric convexity of φ and φ'?

In this note we will prove (see the Theorem below) that for (not necessarily convex) functions having the limit zero at zero geometric convexity of the derivative implies geometric convexity of the function. Thus Tardiff's theorem follows from Mulholland's. On the other hand, as we show by an example, there are (even convex) geometrically convex functions whose derivatives are not geometrically convex.

We start with the formulation of the main result.

Theorem. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a differentiable function with $\varphi' > 0$ and $\lim_{x\to 0} \varphi(x) = 0$. If φ' is geometrically convex, then so is φ.*

In the proof we will use the following:

Lemma. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a differentiable function with $\varphi' > 0$ and $\lim_{x\to 0} \varphi(x) = 0$. If φ' is geometrically convex, then*

$$\lim_{x\to 0} x\varphi'(x) = 0.$$

Proof. Since the product of two geometrically convex functions as well as the identity function are geometrically convex, the function $(0, \infty) \ni x \mapsto x\varphi'(x)$ has the same property and, consequently, $\lim_{x\to 0} x\varphi'(x)$ exists. By the Lagrange Mean Value Theorem for every $x \in (0, \infty)$ there is a $\xi(x) \in (0, x)$ such that

$$\varphi(x) = x\varphi'(\xi(x)).$$

Thus

$$0 < \xi(x)\varphi'(\xi(x)) < x\varphi'(\xi(x)) = \varphi(x) \qquad \text{for} \quad x \in (0, \infty)$$

whence

$$\lim_{x\to 0} \xi(x)\varphi'(\xi(x)) = 0$$

and the assertion follows. $\square$

Proof of the Theorem. Since $\log \circ \varphi' \circ \exp$ is convex it is absolutely continuous and, consequently, φ' is absolutely continuous and the set D of all points of differentiability of φ' has the property that

$$\operatorname{card}((0, \infty) \setminus D) \le \aleph_0.$$

Fix an $x \in \log D$. Then, since $\log \circ \varphi' \circ \exp$ has an increasing derivative on $\log D$, for any $t \in D \cap (0, \exp x]$ we have

$$\begin{aligned}(t\varphi'(t))' &= \varphi'(t)\left(1 + t\frac{\varphi''(t)}{\varphi'(t)}\right) \\ &= \varphi'(t)(1 + (\log \circ \varphi' \circ \exp)'(\log t)) \le \varphi'(t)(1 + (\log \circ \varphi' \circ \exp)'(x)),\end{aligned}$$

whence, by the Lemma,

$$\begin{aligned}\exp x \cdot \varphi'(\exp x) = \int_0^{\exp x} (t\varphi'(t))' dt &\le (1 + (\log \circ \varphi' \circ \exp)'(x)) \int_0^{\exp x} \varphi'(t) dt \\ &= (1 + (\log \circ \varphi' \circ \exp)'(x))\varphi(\exp x).\end{aligned}$$

Thus, since

$$(\log \circ \varphi \circ \exp)'(x) = \frac{\exp x \cdot \varphi'(\exp x)}{\varphi(\exp x)},$$

for $x \in \log D$ we have

$$(\log \circ \varphi \circ \exp)'(x) \le 1 + (\log \circ \varphi' \circ \exp)'(x),$$

whence

$$0 \le 1 + \left[\log \frac{\varphi'(\exp x)}{\varphi(\exp x)}\right]' = \left[\log\left(\exp x \cdot \frac{\varphi'(\exp x)}{\varphi(\exp x)}\right)\right]' \qquad \text{for} \quad x \in \log D.$$

Consequently,

$$\left[\log\left(t\frac{\varphi'(t)}{\varphi(t)}\right)\right]' \ge 0 \qquad \text{for} \quad t \in D.$$

Since the function $(0, \infty) \ni t \mapsto \log\left(t\frac{\varphi'(t)}{\varphi(t)}\right)$ is absolutely continuous, the above condition yields that it is increasing, so that the function

$$(0, \infty) \ni t \mapsto t\frac{\varphi'(t)}{\varphi(t)}$$

has the same property. Consequently, the function $(\log \circ \varphi \circ \exp)'$ is increasing, that is to say φ is geometrically convex. $\square$

Example. We will describe a rather general procedure for the construction of a function which is convex and geometrically convex, has the limit zero at zero, but whose derivative is not geometrically convex.

Let $f : (0, \infty) \to (0, \infty)$ be an arbitrary increasing differentiable function such that the function $(0, \infty) \ni x \mapsto f(x)/x$ is integrable on each interval of the form $(0, a)$. Clearly the function $\varphi : (0, \infty) \to (0, \infty)$ defined by

$$\varphi(x) = x \exp \int_0^x \frac{f(t)}{t} dt \tag{2}$$

is twice differentiable and $\lim_{x \to 0} \varphi(x) = 0$. Since

$$\varphi'(x) = (1 + f(x)) \exp \int_0^x \frac{f(t)}{t} dt \qquad \text{for} \quad x \in (0, \infty), \tag{3}$$

the function φ' is increasing and, consequently, φ is convex. Moreover,

$$x\frac{\varphi'(x)}{\varphi(x)} = 1 + f(x) \qquad \text{for} \quad x \in (0,\infty)$$

which means that $(\log\circ\varphi\circ\exp)'$ is increasing. Thus φ is geometrically convex.

It follows from (3) and (2) that for every $x \in (0,\infty)$,

$$\varphi'(x) = (1+f(x))\frac{\varphi(x)}{x}$$

and

$$\begin{aligned} x\varphi''(x) &= x\left(f'(x) + (1+f(x))\frac{f(x)}{x}\right)\exp\int_0^x \frac{f(t)}{t}dt \\ &= (xf'(x) + f(x) + f(x)^2)\frac{\varphi(x)}{x}. \end{aligned}$$

Therefore

$$x\frac{\varphi''(x)}{\varphi'(x)} = \frac{xf'(x)}{1+f(x)} + f(x) \qquad \text{for} \quad x \in (0,\infty)$$

and to infer that φ' is not geometrically convex it suffices to choose f in such a way that the function $F : (0,\infty) \to \mathbb{R}$ given by

$$F(x) = \frac{xf'(x)}{1+f(x)} + f(x)$$

is not increasing.

To this end, let

$$f(x) = \frac{x+\sin x}{x+\sin x+1} \qquad \text{for} \quad x \in (0,\infty).$$

Clearly f is increasing and differentiable and

$$f'(x) = \frac{1+\cos x}{(x+\sin x+1)^2} \qquad \text{for} \quad x \in (0,\infty).$$

Moreover, the function $(0,\infty) \ni x \mapsto f(x)/x$ has a finite limit at zero, so it is integrable on each interval of the form $(0,a)$. For every $k \in \mathbb{N}$ we have

$$f(k\pi) = \frac{k\pi}{k\pi+1} \qquad \text{and} \qquad f'(k\pi) = \frac{1+(-1)^k}{(k\pi+1)^2},$$

whence

$$F(k\pi) = \frac{k\pi\frac{1+(-1)^k}{(k\pi+1)^2}}{1+\frac{k\pi}{k\pi+1}} + \frac{k\pi}{k\pi+1} = \frac{k\pi}{k\pi+1}\left(\frac{1+(-1)^k}{2k\pi+1}+1\right).$$

Consequently,

$$\begin{aligned} F(2k\pi) &= \frac{2k\pi}{2k\pi+1}\cdot\frac{4k\pi+3}{4k\pi+1} > \frac{2k\pi}{2k\pi+1}\cdot\frac{4k\pi+2}{4k\pi+1} \\ &= 2\cdot\frac{2k\pi}{4k\pi+1} = \frac{4k\pi}{4k\pi+1} > \frac{(2k+1)\pi}{(2k+1)\pi+1} = F((2k+1)\pi) \end{aligned}$$

for each $k \in \mathbb{N}$ which yields the desired result. $\square$

Observe that the Example also gives an answer to a question posed by Tardiff [6] who asked for a convex function which satisfies Mulholland's inequality (1) and whose derivative is not geometrically convex.

References

[1] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Uniwersytet Śląski, Polish Scientific Publishers, Warszawa–Kraków–Katowice 1985. MR **86i:**39008

[2] J. Matkowski, *L^p–like paranorms,* in: *Selected Topics in Functional Equations and Iteration Theory*, Proceedings of the Austrian–Polish Seminar, Graz, 1991, Grazer Math. Ber. 316 (1992), 103–138. MR **94g:**39010

[3] J. Matkowski, *On a generalization of Mulholland's inequality*, Abh. Mat. Sem. Univ. Hamburg 63 (1993), 97–103. MR **94g:**26032

[4] H.P. Mulholland, *On generalizations of Minkowski's inequality in the form of a triangle inequality*, Proc. London Math. Soc. (2) 51 (1950), 294–307. MR **11:**503f

[5] A. Sklar, *Remark and Problem*, Report of meeting, 37th International Symposium on Functional Equations (Huntington, 1999), Aequationes Math. 60 (2000), 187–188.

[6] R. M. Tardiff, *On a generalized Minkowski inequality and its relation to dominates for t–norms*, Aequationes Math. 27 (1984), 308–316. MR **86j:**26013

Institute of Mathematics, Technical University, Podgórna 50, PL–65–246 Zielona Góra, Poland – and – Institute of Mathematics, Silesian University, Bankowa 14, PL–40–007 Katowice, Poland

E-mail address: w.jarczyk@im.uz.zgora.pl

Institute of Mathematics, Technical University, Podgórna 50, PL–65–246 Zielona Góra, Poland – and – Institute of Mathematics, Silesian University, Bankowa 14, PL–40–007 Katowice, Poland

E-mail address: j.matkowski@im.uz.zgora.pl