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NON-VANISHING OF SYMMETRIC SQUARE L-FUNCTIONS

YUK-KAM LAU

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ABSTRACT. Given a complex number s with $0 < \Re e\, s < 1$, we study the existence of a cusp form of large even weight for the full modular group such that its associated symmetric square L-function $L(\mathrm{sym}^2 f, s)$ does not vanish. This problem is also considered in other articles.

1. INTRODUCTION

Let k be an even positive integer and f a holomorphic cusp form of weight k with respect to the full modular group. We represent the Fourier expansion of f (at the cusp ∞) by

$$f(z) = \sum_{n=1}^{\infty} \psi_f(n) n^{(k-1)/2} e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$. Assume that $f(z)$ is an eigenfunction for all Hecke operators T_n, with $T_n f = \lambda_f(n) n^{(k-1)/2} f$. Note that $\lambda_f(n)$ is real and has the Deligne's bound

$$|\lambda_f(n)| \le \tau(n) \tag{1.1}$$

where $\tau(n) = \sum_{d|n} 1$ is the divisor function. We normalize f so that $\psi_f(1) = 1$; then we have $\psi_f(n) = \lambda_f(n)$. Such an f is called a primitive form. Associated to each primitive f, the Rankin-Selberg convolution L-function $L(f \otimes f, s)$ and the symmetric square L-function $L(\mathrm{sym}^2 f, s)$ are respectively defined as, for $\Re e\, s > 1$,

$$L(f \otimes f, s) = \sum_{n=1}^{\infty} \lambda_f(n)^2 n^{-s}$$

and

$$L(\mathrm{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s} \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta-function. These two L-functions are closely linked by the relation (see [5, (0.2) and (0.4)])

$$\zeta(s) L(\mathrm{sym}^2 f, s) = \zeta(2s) L(f \otimes f, s).$$

In this paper, we are concerned with the non-vanishing results of $L(\mathrm{sym}^2 f, s)$ in the critical strip. Li [4] showed that for a given complex number $\rho \neq 1/2$

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satisfying $0 < \Re e\, \rho < 1$ and $\zeta(\rho) \neq 0$, there are infinitely many primitive forms f of different weight such that $\zeta(2s)L(f \otimes f, s)$ do not vanish at $s = \rho$, or equivalently, $L(\mathrm{sym}^2 f, \rho) \neq 0$. In addition, Kohnen and Sengutpa [3] have recently showed that for any fixed $s = \sigma + it$ with $0 < \sigma < 1$ and $\sigma \neq 1/2$, and for all sufficiently large k, there exists a primitive form f of weight k such that $L(\mathrm{sym}^2 f, s) \neq 0$. The approaches used in [4] and [3] are different: the former utilizes an approximate functional equation for an averaged sum of $L(\mathrm{sym}^2 f, \rho)$ while the latter relies on a formula of Zagier. Here, we shall use another method to prove the theorem below, which includes the results in [3] and [4].

Theorem. *For any fixed $s \in \mathbf{C}$ with $0 < \Re e\, s < 1$, there exist infinitely many even k such that $L(sym^2 f, s) \neq 0$ for some primitive form f of weight k. Furthermore, when $\Re e\, s \neq 1/2$ or $s = 1/2$, there exists a constant $k_0(s)$ depending on s such that for all even $k \geq k_0(s)$, $L(sym^2 f, s)$ does not vanish for some primitive form f of weight k.*

Remark. The case $s = 1/2$ is not treated in either [3] or [4]. Moreover, our alternative proof is somewhat simpler than [4], and seems more ‘elementary’ than [3] (without using Zagier's formula).

2. Preliminaries

Let $S_k(1)$ be the linear space of cusp forms of weight k for the full modular group $\Gamma = SL_2(\mathbf{Z})$. Then $S_k(1)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle = \int_{\Gamma\backslash \mathbf{H}} y^k f(z)\overline{g(z)}\, \frac{dxdy}{y^2}$$

and the set of all primitive forms $\mathcal{B}_k$ forms an orthogonal basis for $S_k(1)$. Moreover, we have the Petersson trace formula: define

$$w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle}$$

and $S(m,n,c) = \sum_{ad\equiv 1\,(c)} e((am+dn)/c)$ (the classical Kloosterman sum); then

(2.1)

$$\sum_{f\in\mathcal{B}_k} w_f \lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c\geq 1} c^{-1} S(m,n,c) J_{k-1}\big(\frac{4\pi\sqrt{mn}}{c}\big)$$

where $\delta_{m,n} = 1$ or 0 according to whether $m = n$ or not, and $J_{k-1}(x)$ is the Bessel function. From [6, (5) in Section 2·13], we have the integral representation

$$J_{k-1}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(k-1)\theta + ix\sin\theta}\, d\theta. \tag{2.2}$$

Bounding trivially, using integration by parts or the Poisson integral representation $J_{k-1}(x) = (\sqrt{\pi}\Gamma(k-1/2))^{-1}(x/2)^{k-1}\int_{-1}^{1}(1-t^2)^{k-3/2}e^{ixt}\, dt$ ([6, (3) in 2·3]), we have the following estimates: for $x \geq 0$,

(2.3)

$$\text{(i) } J_{k-1}(x) \ll 1, \quad \text{(ii) } J_{k-1}(x) \ll \frac{x}{k}, \quad \text{(iii) } J_{k-1}(x) \ll \frac{1}{\Gamma(k-1/2)}\big(\frac{x}{2}\big)^{k-1}.$$

Using the Weil bound

$$|S(m,n,c)| \leq (m,n,c)^{1/2}c^{1/2}\tau(c), \tag{2.4}$$

and $\lambda_f(1)=1$, we have with (2.3)(ii)

$$\sum_{f\in\mathcal{B}_k} w_f \ll 1 + k^{-1}\sum_{c\geq 1} c^{-3/2}\tau(c) \ll 1. \tag{2.5}$$

Define

$$\begin{aligned}\Delta(s) &= \pi^{-3s/2}\Gamma(\frac{s+1}{2})\Gamma(\frac{s+k-1}{2})\Gamma(\frac{s+k}{2})\\ &= \pi^{(1-3s)/2}2^{1-s-k}\Gamma(s+k-1)\Gamma(\frac{s+1}{2})\end{aligned} \tag{2.6}$$

(as $\Gamma(s)\Gamma(s+1/2)=\sqrt{\pi}2^{1-2s}\Gamma(2s)$) and $\Lambda(\mathrm{sym}^2 f,s)=\Delta(s)L(\mathrm{sym}^2 f,s)$. Then $\Lambda(\mathrm{sym}^2 f,s)$ is entire and satisfies the functional equation (shown by Shimura [5])

$$\Lambda(\mathrm{sym}^2 f,s)=\Lambda(\mathrm{sym}^2 f,1-s). \tag{2.7}$$

Moreover one can show that $\Lambda(\mathrm{sym}^2 f,s)\to 0$ as $|\mathrm{Im}\, s|\to\infty$ in any vertical strip $|\Re e\, s|\ll 1$.

Finally, let us explain the approach here (which is quite widely used in non-vanishing problems). Using residue theorem and the functional equation of $L(\mathrm{sym}^2 f,\cdot)$, we can express $L(\mathrm{sym}^2 f,s)$ as a convergent series. The averaging process (over all primitive forms) with Petersson trace formula yields that the (averaged) sum consists of two parts: the diagonal terms (contributed by $\delta_{m,n}$ in (2.1)) and the off-diagonal terms. (See (3.6) below.) We then obtain the asymptotic formula (3.13) after giving an estimation to the off-diagonal terms. Our result is deduced from this formula.

3. Proof of the Theorem

Assume $0<\Re e\, s\leq 1/2$. Consider the integral $(2\pi i)^{-1}\int_{\mathcal{R}}\Lambda(\mathrm{sym}^2 f,s+w)\,dw/w$ where $\mathcal{R}$ is the positively oriented rectangular contour with vertices at $\pm 2\pm iT$, we have, by residue theorem and taking $T\to\infty$, that

$$\begin{aligned}\Lambda(\mathrm{sym}^2 f,s) &= \frac{1}{2\pi i}(\int_{(2)}-\int_{(-2)})\Lambda(\mathrm{sym}^2 f,s+w)\,\frac{dw}{w}\\ &= \frac{1}{2\pi i}\int_{(2)}\Lambda(\mathrm{sym}^2 f,s+w)\,\frac{dw}{w}+\frac{1}{2\pi i}\int_{(2)}\Lambda(\mathrm{sym}^2 f,1-s+w)\,\frac{dw}{w}\end{aligned}$$

after using the functional equation (2.7) and changing w to $-w$. Hence, if we write

$$V_z(y)=\frac{1}{2\pi i}\int_{(2)}\zeta(2(z+w))\Delta(z+w)y^{-w}\,\frac{dw}{w}, \tag{3.1}$$

we get from (1.2) that

$$L(\mathrm{sym}^2 f,s)\Delta(s)=\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^{1-s}}V_{1-s}(n)+\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^s}V_s(n). \tag{3.2}$$

Let $z=1-s$ or s. From (2.6),

$$\Delta(z+w)\ll_{|s|} 2^{-k}\Gamma(\Re e\,(z+w)+k-1)|\Gamma(\frac{z+w+1}{2})| \tag{3.3}$$

for $\Re e\,(z+w) \geq -3/4$. Moving the line of integration to $\Re e\,(z+w) = A$, we have for $A > \max(\Re e\, z, 1/2)$,

$$V_z(y) \ll_{|s|,A} y^{\Re e\, z - A} 2^{-k}\Gamma(k+A-1). \tag{3.4}$$

Shifting to $\Re e\,(z+w) = -1/2$ (across the poles at $w=0$, $1/2-z$), we obtain with (3.3)

(3.5)
$$V_z(1) = \begin{cases} \zeta(2z)\Delta(z) + \Delta(1/2)(1/2-z)^{-1} \\ \gamma\Delta(1/2) + 2^{-1}\Delta'(1/2) \end{cases} + O(2^{-k}\Gamma(k-3/2))$$

where γ is the Euler constant. The second case corresponds to $z = 1/2$. As will be seen, the main term is given by

$$V_{1-s}(1) + V_s(1) \quad = \quad \begin{cases} \zeta(2-2s)\Delta(1-s) + \zeta(2s)\Delta(s) \\ 2\gamma\Delta(1/2) + \Delta'(1/2) \end{cases} + \cdots$$

according to $s \neq 1/2$ or $s = 1/2$. Its order of magnitude is about $2^{-k}\Gamma(k - \Re e\, s)$.

Let $0 < \nu \leq 10^{-3}$ be a fixed number. Both sums in (3.2) over $n > k^{1+5\nu}$ can be evaluated as follows: choosing $A = 1 + \nu^{-1}$ in (3.4), we have ($z = 1-s$ or s)

$$\begin{aligned} \sum_{n > k^{1+5\nu}} \frac{\lambda_f(n^2)}{n^z} V_z(n) \quad &\ll \quad 2^{-k}\Gamma(k+\nu^{-1}) \sum_{n > k^{1+5\nu}} \frac{\tau(n^2)}{n^{1+1/\nu}} \\ &\ll \quad 2^{-k}k^{-4-1/\nu}\Gamma(k+\nu^{-1}) \ll 2^{-k}k^{-1/4}\Gamma(k-1/2) \end{aligned}$$

by (1.1) and Stirling's formula ([1, Chapter 10]). Summing over all primitive forms and using (2.1), with $\lambda_f(1) = 1$,

$$\begin{aligned} &\Delta(s) \sum_{f \in \mathcal{B}_k} w_f L(\mathrm{sym}^2 f, s) \\ &\quad = V_{1-s}(1) + V_s(1) + \sum_{z=1-s,s} 2\pi i^{-k} \sum_{n \leq k^{1+5\nu}} n^{-z} V_z(n) \mathcal{J}(n) \\ &\qquad + O((\sum_f w_f) 2^{-k} k^{-1/4} \Gamma(k-1/2)) \end{aligned} \tag{3.6}$$

where $\mathcal{J}(n) = \sum_{c \geq 1} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c)$. We give an estimate for $\mathcal{J}(n)$. From (2.3)(ii) and (2.4),

$$\sum_{c > k^{1+20\nu}} c^{-1} S(1, n^2, c) J_{k-1}(\frac{4\pi n}{c}) \ll n k^{-1} \sum_{c > k^{1+20\nu}} c^{-3/2}\tau(c) \ll n/k^{3/2+9\nu}.$$

By (2.3)(iii), when $n \leq k^{1-\nu}$,

$$\begin{aligned} \sum_{c \leq k^{1+20\nu}} c^{-1} S(1, n^2, c) J_{k-1}(\frac{4\pi n}{c}) \quad &\ll \quad \Gamma(k-1/2)^{-1} \sum_{c \leq k^{1+20\nu}} c^{-1/2}\tau(c)(2\pi k^{1-\nu})^{k-1} \\ &\ll \quad k^{(1-\nu)k}\Gamma(k-1/2)^{-1} \ll n/k^{3/2+9\nu}, \end{aligned}$$

by Stirling's formula. Similarly, for $k^{1-\nu} < n \leq k^{1+5\nu}$ we have

$$\sum_{k^{6\nu} < c \leq k^{1+20\nu}} c^{-1} S(1, n^2, c) J_{k-1}(4\pi n/c) \ll n/k^{3/2+9\nu}.$$

Hence,

$$\mathcal{J}(n) = \delta(n,k) \sum_{c \le k^{6\nu}} c^{-1} S(1, n^2, c) J_{k-1}(\frac{4\pi n}{c}) + O(nk^{-3/2-9\nu}) \tag{3.7}$$

where $\delta(n,k) = 0$ if $n \le k^{1-\nu}$, and 1 if $k^{1-\nu} < n \le k^{1+5\nu}$. Inserting (3.7) into (3.6), together with (2.5) and the estimate

$$k^{-3/2-9\nu} \sum_{n \le k^{1+5\nu}} |n^{1-z} V_z(n)| \ll 2^{-k} k^{-3/2-9\nu} (\log k) \Gamma(k+1) \ll 2^{-k} k^{-2\nu} \Gamma(k-1/2)$$

(following from (3.4) with $A = 2$), we see that (3.6) becomes

$$\begin{aligned} &\Delta(s) \sum_f w_f L(\mathrm{sym}^2 f, s) \\ (3.8) \quad = \quad & V_{1-s}(1) + V_s(1) + i^{-k}(\mathcal{E}_k(1-s) + \mathcal{E}_k(s)) + O(2^{-k} k^{-8\nu} \Gamma(k-1/2)) \end{aligned}$$

where

$$\mathcal{E}_k(z) = 2\pi \sum_{k^{1-\nu} < n \le k^{1+5\nu}} n^{-z} V_z(n) \sum_{c \le k^{6\nu}} c^{-1} S(1, n^2, c) J_{k-1}(\frac{4\pi n}{c}).$$

From (2.2), we have

(3.9)

$$\mathcal{E}_k(z) = \sum_{k^{1-\nu} < n \le k^{1+5\nu}} n^{-z} V_z(n) \sum_{c \le k^{6\nu}} c^{-1} S(1, n^2, c) \int_0^{\pi/2} 2\Re e\, f_k(\theta, \frac{4\pi n}{c})\, d\theta$$

where $f_k(\theta, x) = e^{ix \sin\theta}(e^{-i(k-1)\theta} - e^{i(k-1)\theta})$. When $|x| \le k^{6/5}$, we have

$$\left| \frac{d}{d\theta}(x \sin\theta \pm (k-1)\theta) \right| \asymp k \qquad \text{for } \pi/2 - k^{-1/4} \le \theta \le \pi/2,$$

whence $\int_{\pi/2-k^{-1/4}}^{\pi/2} f_k(\theta, \frac{4\pi n}{c})\, d\theta \ll k^{-1}$ for $4\pi n/c \le k^{6/5}$ by integration by parts. From (3.4) with $A = 1$ and (2.4),

$$\begin{aligned} &\sum_{k^{1-\nu} < n \le k^{1+5\nu}} n^{-z} V_z(n) \sum_{c \le k^{6\nu}} c^{-1} S(1, n^2, c) \int_{\pi/2-k^{-1/4}}^{\pi/2} \Re e\, f_k(\theta, \frac{4\pi n}{c})\, d\theta \\ \ll \quad & 2^{-k} k^{-1} \Gamma(k) \sum_{k^{1-\nu} < n \le k^{1+5\nu}} n^{-1} \sum_{c \le k^{6\nu}} c^{-1/2} \tau(c) \ll 2^{-k} k^{-8\nu} \Gamma(k-1/2). \end{aligned}$$

We put this estimate into (3.9). Then we interchange the sums in the remaining part and use the periodicity of $S(1, \cdot, c)$ to give

(3.10)

$$\begin{aligned} \mathcal{E}_k(z) &= \sum_{c \le k^{6\nu}} c^{-1} \sum_{k^{1-\nu} < n \le k^{1+5\nu}} S(1, n^2, c) n^{-z} V_z(n) \int_0^{\pi/2-k^{-1/4}} 2\Re e\, f_k(\theta, \frac{4\pi n}{c})\, d\theta \\ &\quad + O(2^{-k} k^{-8\nu} \Gamma(k-1/2)) \\ &= \sum_{c \le k^{6\nu}} \sum_{0 \le r < c} c^{-1} S(1, r^2, c) T_z(r, c) + O(2^{-k} k^{-8\nu} \Gamma(k-1/2)) \end{aligned}$$

with

$$T_z(r,c) = 2 \sum_{\substack{k^{1-\nu}<n\le k^{1+5\nu}\\ n\equiv r\,(c)}} n^{-z}V_z(n) \int_0^{\pi/2-k^{-1/4}} \Re e\, f_k(\theta, \frac{4\pi n}{c})\, d\theta.$$

From the definition of $f_k(\theta,\cdot)$ (the line below (3.9)), we see that

$$T_z(r,c) \ll \int_0^{\pi/2-k^{-1/4}} |\sum_{\substack{k^{1-\nu}<n\le k^{1+5\nu}\\ n\equiv r\,(c)}} n^{-z}V_z(n)e(\frac{2n}{c}\sin\theta)|\, d\theta$$

$$= \int_0^{\pi/2-k^{-1/4}} \left| \int_{(\kappa)} \zeta(2(z+w))\Delta(z+w) \sum_{\substack{k^{1-\nu}<n\le k^{1+5\nu}\\ n\equiv r\,(c)}} n^{-z-w}e(\frac{2n}{c}\sin\theta)\, \frac{dw}{w} \right| d\theta$$

by (3.1) with the path moved from $\Re e\, w = 2$ to $\kappa = 2 - \Re e\, z$. By (3.3), $\Delta(z+w) \ll 2^{-k}\Gamma(k+1)(|w|+1)^{-2}$ for $\Re e\, w = \kappa$. Hence,

$$\begin{aligned} T_z(r,c) &\ll 2^{-k}\Gamma(k+1) \\ &\quad \times \int_{(\kappa)} \int_0^{\pi/2-k^{-1/4}} |\sum_{K_1<m\le K_2} \frac{e(2m\sin\theta)}{(cm+r)^{z+w}}|\, d\theta\, \frac{|dw|}{(|w|+1)^3} \end{aligned} \tag{3.11}$$

where $K_1 = (k^{1-\nu} - r)/c$ and $K_2 = (k^{1+5\nu} - r)/c$. Using $\sum_{m\le M} e(2m\alpha) \ll |\sin(2\pi\alpha)|^{-1}$ with partial summation, or bounding trivially, the sum in (3.11) is

$$\ll (|w|+1)k^{2\nu-2}\min(|\sin(2\pi\sin\theta)|^{-1}, k) \tag{3.12}$$

as $\Re e\,(z+w) = 2$. After substituting (3.12) into (3.11), the θ-integral equals

$$\begin{aligned} &\int_0^{\pi/2-k^{-1/4}} \min(|\sin(2\pi\sin\theta)|^{-1}, k)\, d\theta \\ = &\int_0^{\cos(k^{-1/4})} \min(|\sin(2\pi u)|^{-1}, k)\, \frac{du}{\sqrt{1-u^2}} \\ \ll &\left(\int_0^{k^{-1}} + \int_{1/2-k^{-1}}^{1/2+k^{-1}}\right) k\, du + \left(\int_{k^{-1}}^{1/2-k^{-1}} + \int_{1/2+k^{-1}}^{3/4}\right) |\sin(2\pi u)|^{-1}\, du \\ &+ \int_{3/4}^{1-(16k)^{-1/2}} |\sin(2\pi u)|^{-1}\, \frac{du}{\sqrt{1-u}} \ll 1 + \log k + k^{1/4} \end{aligned}$$

by using $\sin\alpha \ge 2\alpha/\pi$ if $0 \le \alpha \le \pi/2$. In view of (3.11) and (3.12), we conclude that $T_z(r,c) \ll 2^{-k}k^{2\nu-7/4}\Gamma(k+1) \ll 2^{-k}k^{2\nu-1/4}\Gamma(k-1/2)$, and by (3.10) that

$$\mathcal{E}_k(z) \ll 2^{-k}k^{2\nu-1/4}\Gamma(k-1/2) \sum_{c\le k^{6\nu}} \sum_{0\le r<c} c^{-1}|S(1,r^2,c)| + 2^{-k}k^{-8\nu}\Gamma(k-1/2)$$

which is absorbed by the O-term in (3.8). Therefore, (3.8) and (3.5) yield

$$\begin{aligned} &\Delta(s) \sum_{f\in\mathcal{B}_k} w_f L(\mathrm{sym}^2 f, s) \\ = &\left\{ \begin{array}{l} \zeta(2-2s)\Delta(1-s) + \zeta(2s)\Delta(s) \\ \Delta'(1/2) + 2\gamma\Delta(1/2) \end{array} \right. + O(2^{-k}k^{-8\nu}\Gamma(k-1/2)). \end{aligned} \tag{3.13}$$

From Stirling's formula, we have $\Gamma(k+z-1) = \Gamma(k+a-1)e^{ib\log k + O(1/k)}$ $(z = a+ib)$ for $|z| \le k^{1/3}$ and $\Gamma'(k-1/2)/\Gamma(k-1/2) = \log k + O(1)$. Hence for the case $s = 1/2$, the dominating term in (3.13) is $\Delta'(1/2)$, of order $2^{-k}(\log k)\Gamma(k-1/2)$, for all large k, and we can thus conclude $\sum_{f\in\mathcal{B}_k} w_f L(\mathrm{sym}^2 f, 1/2) \neq 0$. For the case $\Re e\, s < 1/2$, the term $\zeta(2-2s)\Delta(1-s)$ $(\asymp 2^{-k}\Gamma(k-\Re e\, s))$ dominates others for all large k. (Note that $\zeta(2-2s)$ is non-zero.) When $s = 1/2+it$ and $t \neq 0$, denoting $a(t) = 2^{1/2-it}\pi^{-1/4-3it/2}\zeta(1+2it)\Gamma(3/4+it/2)$, the main term in (3.13) is

$$\begin{aligned} & \zeta(1+2it)\Delta(1/2+it) + \zeta(1-2it)\Delta(1/2-it) \\ = \;& 2^{-k}\Big(a(t)\Gamma(k-1/2+it) + a(-t)\Gamma(k-1/2-it)\Big) \\ = \;& 2^{-k}\Gamma(k-1/2)\Big(2|a(t)|\cos(t\log k + \vartheta(t)) + O(k^{-1})\Big) \end{aligned}$$

where $\vartheta(t)$ is the argument of $a(t)$. Suppose $(2\pi)^{-1}t\log 2$ is irrational. Then by Kronecker's theorem ([2, Theorem 438]), there exist infinitely many r_i (depending on t) satisfying $|r_i t\log 2 + \vartheta(t) - 2\pi m_i| \le \pi/4$ for some integer m_i. Thus, we take $k = 2^{r_i}$ for those sufficiently large r_i so that the right side of (3.13) is $\gg 2^{-k}|a(t)|\Gamma(k-1/2) > 0$. If $(2\pi)^{-1}t\log 2$ is rational, we consider instead $(2\pi)^{-1}t\log 3$ which must then be irrational. Our result follows with the previous argument. The case $1/2 < \Re e\, s < 1$ is done because of the functional equation (2.7).

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Institut Élie Cartan, Université Henri Poincaré (Nancy 1), 54506 Vandoeuvre lés Nancy Cedex, France

Current address: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

E-mail address: yklau@maths.hku.hk