PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3295–3299
S 0002-9939(02)06724-2
Article electronically published on June 11, 2002

# PRODUCTS OF UNIFORMLY NONCREASY SPACES

ANDRZEJ WIŚNICKI

(Communicated by Jonathan M. Borwein)

ABSTRACT. We show that finite products of uniformly noncreasy spaces with a strictly monotone norm have the fixed point property for nonexpansive mappings. It gives new and natural examples of superreflexive Banach spaces without normal structure but with the fixed point property.

## 1. INTRODUCTION

Let $X$ be a Banach space and $C$ be a nonempty, bounded, closed and convex subset of $X$. We say that $X$ has the fixed point property (FPP for short) if every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) has a fixed point. We say that $E$ has the weak fixed point property (wc-FPP for short) if we additionally assume that $C$ is weakly compact. The celebrated fixed point theorem of W. A. Kirk [9] asserts that every Banach space with the so-called normal structure has wc-FPP. For a detailed exposition of metric fixed point theory we refer the reader to [1], [2] and [7].

The notion of uniformly noncreasy spaces was introduced by S. Prus in [15] to describe a large class of superreflexive Banach spaces with the fixed point property. Let us recall that a real Banach space $X$ is uniformly noncreasy (UNC for short) if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $f, g \in S_{X^*}$ and $\|f - g\| \geq \varepsilon$, then $\operatorname{diam} S(f, g, \delta) \leq \varepsilon$, where

$$S(f, g, \delta) = \{x \in B_X : f(x) \geq 1 - \delta \ \wedge \ g(x) \geq 1 - \delta\}.$$

A Banach space $X$ is noncreasy if $\operatorname{diam} S(f, g, 0) = 0$ whenever $f, g \in S_{X^*}, f \neq g$. To be precise, we put $\operatorname{diam} \emptyset = -\infty$. Obviously, if $X$ is uniformly noncreasy, then it is noncreasy. It was proved in [15] that all uniformly noncreasy spaces are superreflexive and have FPP. The Bynum space $l_{2,\infty}$ (which is $l_2$ space endowed with the norm $\|x\|_{2,\infty} = \max\{\|x^+\|_2, \|x^-\|_2\}$; see [4]) and the space $X_{\sqrt{2}}$ (which is $l_2$ space endowed with the norm $\|x\|_{\sqrt{2}} = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\}$; see [3]) are examples of uniformly noncreasy spaces without normal structure.

One of the important problems in metric fixed point theory are the permanence properties of normal structure and other conditions which guarantee the fixed point property under the direct product operation (see [17] and the references given there).

---

Received by the editors June 12, 2001.
2000 *Mathematics Subject Classification.* Primary 47H09, 47H10, 46B20.
*Key words and phrases.* Nonexpansive mappings, fixed points.
This research was supported in part by KBN grant 2 PO3A 029 15.

Let $Z$ be a finite dimensional normed space $(\mathbb{R}^k, \|\|_Z)$ which has a strictly monotone norm. That is,

$$\|(x(1), ..., x(k))\|_Z < \|(y(1), ..., y(k))\|_Z$$

if $0 \leq x(i) \leq y(i)$ for $i = 1, ..., k$ and $0 \leq x(i_0) < y(i_0)$ for some $i_0$. We shall write $(X_1 \oplus ... \oplus X_k)_Z$ for the $Z$ direct product of Banach spaces $X_1, ..., X_k$ with the norm

$$\|(x(1), ..., x(k))\| = \left\|\left(\|x(1)\|_{X_1}, ..., \|x(k)\|_{X_k}\right)\right\|_Z ,$$

where $(x(1), ..., x(k)) \in X_1 \times ... \times X_k$.

We prove that $Z$ direct products of uniformly noncreasy spaces with a strictly monotone norm have the fixed point property for nonexpansive mappings. A certain extension of this theorem for spaces with the so-called property $(P)$ (see [18, 19]) is also shown. We note that our results go beyond the conditions which guarantee normal structure and give new examples of Banach spaces without (weak) normal structure but with the (weak) fixed point property.

## 2. Results

Let us recall [14] that a set $A$ of a Banach space $X$ is said to be metrically convex if for every $x, y \in A$ with $x \neq y$ there exists $z \in A$, $x \neq z \neq y$ such that $\|x - z\| + \|y - z\| = \|x - y\|$. It is not difficult to see that if $A$ is closed, this condition is equivalent to the following one: for every $x, y \in A$ there exists $z \in A$ such that $\|x - z\| = \|y - z\| = \frac{1}{2}\|x - y\|$.

We shall always assume that $(X_1 \oplus ... \oplus X_k)_Z$ is a $Z$ direct product of $X_1, ..., X_k$ with a strictly monotone norm. Denote by $P_i, i = 1, ..., k$, the coordinate projections of $(X_1 \oplus ... \oplus X_k)_Z$ onto $X_i$.

**Lemma 2.1.** *Let $X_1, ..., X_k$ be Banach spaces and let $A$ be a closed, metrically convex subset of $(X_1 \oplus ... \oplus X_k)_Z$. Then $P_i(A), i = 1, ..., k$, are metrically convex, too. Moreover, for a fixed $j \in \{1, ..., k\}$ and $x = (x(1), ..., x(k))$, $y = (y(1), ..., y(k)) \in A$ there exists $z = (z(1), ..., z(k)) \in A$ such that*

$$\|x(j) - z(j)\| = \|z(j) - y(j)\| = \frac{1}{2}\|x(j) - y(j)\|$$

*and*

$$\|x(i) - y(i)\| = \|x(i) - z(i)\| + \|z(i) - y(i)\|$$

*for each $i = 1, ..., k$.*

*Proof.* Fix $j \in \{1, ..., k\}$ and $x = (x(1), ..., x(k))$, $y = (y(1), ..., y(k)) \in A$. We may assume that $\|x - y\| = 1$. Since $A$ is closed and metrically convex, then by Menger's theorem [14] (see also [7]) there exists an isometry $\varphi : [0, 1] \to A$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Let $g(t) = \|x(j) - (P_j\varphi)(t)\|, t \in [0, 1]$. Clearly $g$ is a continuous function, $g(0) = 0$ and $g(1) = \|x(j) - y(j)\|$. By the Darboux theorem, there exists $s \in [0, 1]$ such that $g(s) = \frac{1}{2}\|x(j) - y(j)\|$. Put $z = \varphi(s)$ and observe that $\|x(i) - y(i)\| = \|x(i) - z(i)\| + \|z(i) - y(i)\|$ for $i = 1, ..., k$. Indeed, if $\|x(i_0) - y(i_0)\| < \|x(i_0) - z(i_0)\| + \|z(i_0) - y(i_0)\|$ for some $i_0$, then

$$\begin{aligned}\|x - y\| &= \|(\|x(1) - y(1)\|, ..., \|x(k) - y(k)\|)\|_Z \\ &< \|(\|x(1) - z(1)\| + \|z(1) - y(1)\|, ..., \|x(k) - z(k)\| + \|z(k) - y(k)\|)\|_Z \\ &\leq \|x - z\| + \|z - y\|\end{aligned}$$

which is a contradiction with the fact that $\varphi$ is an isometry. Moreover $\|x(j) - z(j)\| = g(s) = \frac{1}{2}\|x(j) - y(j)\| = \|z(j) - y(j)\|$ and the proof is complete. □

We will say that $w$ lies between $x$ and $y$ if $\|x - y\| = \|x - w\| + \|w - y\|$.

**Lemma 2.2.** *Let $X$ be a noncreasy Banach space, $x, y \in S_X$, $\|x - y\| = 1$ and assume that there exist $f, g \in S_{X^*}$ such that $f(x) = 1, f(y) = 0, g(x) = 0, g(y) = 1$. If $w$ lies between $x$ and $y$, then $\|w\| \le 1$.*

*Proof.* We may assume that $x \neq w \neq y$. Let $\|x - w\| = \alpha$, $\|w - y\| = \beta, \alpha, \beta > 0, \alpha + \beta = 1$. Then $|f(x) - f(w)| \le \alpha$ and therefore $f(w) = \beta$. Similarly, $g(w) = \alpha$. Hence

$$f\left(\frac{1}{\alpha}(x - w)\right) = \frac{1}{\alpha}(1 - \beta) = 1,\ f\left(\frac{1}{\beta}(w - y)\right) = 1$$

and the same holds for $-g$. This means that $\frac{1}{\alpha}(x - w), \frac{1}{\beta}(w - y) \in S(f, -g, 0)$. Moreover $\|f + g\| \ge (f + g)(x) = 1$. But $X$ is noncreasy so $\frac{1}{\alpha}(x - w) = \frac{1}{\beta}(w - y)$ (note that a similar argument is used in [15, Theorem 9]). Thus $w = \beta x + \alpha y$ and consequently $\|w\| \le 1$. □

We can now prove our main result. We recall that the ultrapower $\widetilde{X}$ (or $(X)_U$) of a Banach space $X$ is the quotient space of

$$l_\infty(X) = \left\{(x_n) : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \| (x_n) \| = \sup_n \| x_n \| < \infty\right\}$$

by $\ker \mathcal{N} = \left\{(x_n) \in l_\infty(X) : \lim_U \| x_n \| = 0\right\}$. Here $\lim_U$ denotes the ultralimit over a free ultrafilter $U \subset 2^{\mathbb{N}}$ (see [1, 7, 16] for more details).

**Theorem 2.3.** *Let $X_1, ..., X_k$ be uniformly noncreasy Banach spaces. Then $X = (X_1 \oplus ... \oplus X_k)_Z$ has the fixed point property.*

*Proof.* We join together the arguments from [15, Theorem 9] and [5, Theorem 1]. The new ingredients are Lemmas 2.1 and 2.2.

It is not difficult to see that $X$ is superreflexive. Let us assume that $X$ does not have FPP. Then we can find a nonexpansive mapping $T$ and a minimal invariant set $K \subset X$ for $T$. We may assume that $0 \in K$ and diam $K = 1$. Let $\langle x_n \rangle$ be an approximate fixed point sequence in $K$ which converges weakly to 0. Then $x_n(i)$ also converges weakly to 0, where $x_n = (x_n(1), ..., x_n(k))$. It follows from the Goebel-Karlovitz lemma that we can assume that $\lim_{n,m,n\neq m} \|x_n - x_m\| = 1$. Just as in [5] (see also [17, Proposition 4.1]) we can also assume that $\lim_n \|x_n(i)\| = a(i)$ and $\lim_{n,m,n\neq m} \|x_n(i) - x_m(i)\| = l(i)$ for $i = 1, ..., k$. Since $\|(a(1), ..., a(k))\|_Z = \lim_n \|x_n\| = 1$ and $\|(l(1), ..., l(k))\|_Z = \lim_{n,m,n\neq m} \|x_n - x_m\| = 1$, it is not very difficult to see that $a(i) = l(i)$ for each $i$. Indeed, $a(i) \le l(i)$ for each $i = 1, ..., k$ because $x_n(i)$ converges weakly to 0. If $a(i_0) < l(i_0)$ for some $i_0$, then $\|(a(1), ..., a(k))\|_Z < \|(l(1), ..., l(k))\|_Z$ and we would have a contradiction.

Let $f_n(i) \in S_{X_i^*}$ be functionals such that $(f_n(i))(x_n(i)) = \|x_n(i)\|$. Applying $k$ - times the procedure from [15] we can find a subsequence $\langle x_{n_l} \rangle$ with $(f_{n_l}(i))(x_{n_m}(i)) \le \min\{\frac{1}{l}, \frac{1}{m}\}$ whenever $l \neq m$ and $i = 1, ..., k$. Passing to the ultrapower $\widetilde{X} = (\widetilde{X_1} \oplus ... \oplus \widetilde{X_k})_Z$, there exist $x = (x(1), ..., x(k))$, $y = (y(1), ..., y(k)) \in \mathrm{Fix}\,\tilde{T}$ and $f(i), g(i) \in S_{\widetilde{X_i^*}}$ such that $(f(i))(x(i)) = a(i)$, $(f(i))(y(i)) = 0$, $(g(i))(x(i)) = 0$, $(g(i))(y(i)) = a(i)$ for $i = 1, ..., k$. Moreover $\|x(i)\| = \|y(i)\| = \|x(i) - y(i)\| = a(i)$

for each $i$. Choose $j$ such that $a(j) \neq 0$. Since $\operatorname{Fix}\tilde{T}$ is closed and metrically convex, by Lemma 2.1, $P_j(\operatorname{Fix}\tilde{T})$ is metrically convex, too. Thus, there exists $z = (z(1), ..., z(k)) \in \operatorname{Fix}\tilde{T}$ such that

$$\|x(j) - z(j)\| = \|z(j) - y(j)\| = \frac{1}{2}\|x(j) - y(j)\| = \frac{1}{2}a(j).$$

Moreover

$$\|x(i) - y(i)\| = \|x(i) - z(i)\| + \|z(i) - y(i)\|$$

for each $i = 1, ..., k$. Hence we can use Lemma 2.2 to conclude that $\|z(i)\| \leq a(i), i = 1, ..., k$ (we use the fact that the ultrapowers $\widetilde{X_i}$ are uniformly noncreasy [15]). But $\|z\| = 1 = \|(\|z(1)\|, ..., \|z(k)\|)\|_Z = \|(a(1), ..., a(k))\|_Z$ and consequently $\|z(i)\| = a(i)$, $i = 1, ..., k$, since the $Z$ norm is strictly monotone. In particular $\|z(j)\| = a(j)$ and we can now follow the arguments from [15, Theorem 9] to obtain a contradiction. □

Let us notice that we have actually proved a more general statement. Recall [18] (see also [19]) that a Banach space $X$ has property $(P)$ if $\liminf \|x_n - x\| < \operatorname{diam}(x_n)$ whenever $x_n \rightharpoonup x$ and $\langle x_n \rangle$ is nonconstant. It is not difficult to see that this condition is weaker than Bynum's condition $\operatorname{WCS}(X) > 1$.

**Theorem 2.4.** *Let $X_1, ..., X_k$ be uniformly noncreasy or have property (P). Then $X = (X_1 \oplus ... \oplus X_k)_Z$ has the weak fixed point property.*

We leave the proof to the reader.

**Example 2.5.** Let $l_p^k$ denote a $Z$ space $\mathbb{R}^k$ with the norm $\|(x(1), ..., x(k))\|_p = (|x(1)|^p + ... + |x(k)|^p)^{\frac{1}{p}}$, $p \geq 1$. Obviously, this is a strictly increasing norm and our results are valid. In particular, the space $\left(l^{2,1} \oplus l^{2,\infty}\right)_2$ has the fixed point property, which was first proved in [8] by completely different methods.

**Example 2.6.** More generally, let $M$ be a strictly increasing Orlicz function on $[0, \infty)$ (i.e., a continuous convex function satisfying $M(0) = 0$). Then the $Z$-Orlicz space $(\mathbb{R}^k, \|\|_M)$, where

$$\|(x(1), ..., x(k))\|_M = \inf\left\{u > 0 : \sum_{i=1}^{k} M\left(\frac{|x(i)|}{u}\right) \leq 1\right\},$$

has a strictly monotone norm.

In view of Theorem 2.3 it is very natural to ask whether the fixed point property is preserved under other products of uniformly noncreasy spaces.

## References

[1] A. G. Aksoy and M. A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York/Berlin, 1990. MR **91i:**47073

[2] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. L ópez Acedo, *Measures of Noncompactness in Metric Fixed Point Theory,* Birkhäuser Verlag, Basel, 1997. MR **99e:**47070

[3] L. P. Belluce, W. A. Kirk and E. F. Steiner, *Normal structure in Banach spaces,* Pacific J. Math. **26** (1968), 433–440. MR **38:**1501

[4] W. L. Bynum, *A class of spaces lacking normal structure,* Compositio Math. **25** (1972), 233–236. MR **47:**7386

[5] T. Domínguez Benavides, *Weak uniform normal structure in direct sum spaces*, Studia Math. **103** (1992), 283–290. MR **94c:**46024

[6] R. Espínola and W. A. Kirk, *Fixed points and approximate fixed points in product spaces,* Taiwanese J. Math. **5** (2001), 405–416. MR **2002b:**54043

[7] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge Studies in Advanced Mathematics, Vol. 28, Cambridge Univ. Press, Cambridge, 1990. MR **92c:**47070

[8] M. A. Japón Pineda, *A new constant in Banach spaces and stability of the fixed point property,* in Proc. of Workshop on Fixed Point Theory (Kazimierz Dolny, 1997), Ann. Univ. Mariae Curie-Skłodowska Sect. A **51** (1997), 135–141. MR **2000d:**46015

[9] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-6. MR **32:**6436

[10] M. A. Khamsi, *On normal structure, fixed-point property and contractions of type* $\gamma$, Proc. Amer. Math. Soc. **106** (1989), 995–1001. MR **90d:**46028

[11] T. Kuczumow, *Fixed point theorems in product spaces,* Proc. Amer. Math. Soc. **108** (1990), 727–729. MR **90e:**47061

[12] T. Landes, *Permanence properties of normal structure,* Pacific J. Math. **110** (1984), 125–143. MR **86e:**46014

[13] T. Landes, *Normal structure and the sum-property,* Pacific J. Math. **123** (1986), 127–147. MR **87h:**46043

[14] K. Menger, *Untersuchungen uber allgemeine metrik*, Math. Ann. **100** (1928), 75-163.

[15] S. Prus, *Banach spaces which are uniformly noncreasy*, in Proc. 2nd World Congress of Nonlinear Analysts (Athens, 1996), ed. V. Lakshmikantham, *Nonlinear Anal.* **30** (1997), 2317-2324. MR **98m:**46017

[16] B. Sims, *Ultra-techniques in Banach Space Theory,* Queen's Papers in Pure and Applied Math., Vol. 60, Queen's University, Kingston, Ont., 1982. MR **86h:**46032

[17] B. Sims and M. A. Smyth, *On some Banach space properties sufficient for weak normal structure and their permanence properties*, Trans. Amer. Math. Soc. **351** (1999), 497–513. MR **99d:**46020

[18] K.-K. Tan and H. K. Xu, *On fixed point theorems of nonexpansive mappings in product spaces,* Proc. Amer. Math. Soc. **113** (1991), 983–989. MR **92c:**47074

[19] D. Tingley, *The normal structure of James quasireflexive space,* Bull. Austral. Math. Soc. **42** (1990) 95–100. MR **91h:**46038

Department of Mathematics, Maria Curie - Skłodowska University, 20-031 Lublin, Poland

*E-mail address*: awisnic@golem.umcs.lublin.pl