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DERIVATIONS WITH LARGE SEPARATING SUBSPACE

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Abstract. In his famous paper *The image of a derivation is contained in the radical*, Marc Thomas establishes the (commutative) Singer-Wermer conjecture, showing that derivations from a commutative Banach algebra A to itself must map into the radical. The proof goes via first showing that the separating subspace of a derivation on A must lie in the radical of A. In this paper, we exhibit discontinuous derivations on a commutative unital Fréchet algebra $\mathcal{A}$ such that the separating subspace is the whole of $\mathcal{A}$. Thus, the situation on Fréchet algebras is markedly different from that on Banach algebras.

0. Introduction

The author would like to thank those at the University of Newcastle, Australia and the Australian National University for much kindness during his visit there in December '00 to March '01. The result contained in this paper is one of the fruits of that academic visit.

The Singer-Wermer conjecture was first made as long ago as 1955 [SW]; the noncommutative case is still open. After that date the theory of automatic continuity of operators was substantially developed, see e.g. Sinclair [S] and further developments in Laursen [L]; there is an exhaustive, up-to-date exposition of the whole matter in Dales' definitive reference work [D]. In all of these references one will find a central role being played by the separating subspace.

Definition 0.1. If $T : X \to Y$ is a linear map betweeen Fréchet spaces, the **separating subspace** of T is

$$\mathcal{S}(T) = \{y \in Y : y = \lim_n Tx_n \text{ for some sequence } x_n \in X \text{ with } x_n \to 0\}.$$

The separating subspace is a closed linear subspace of Y, and the closed graph theorem tells us that it is equal to $\{0\}$ if and only if T is continuous. When T is a derivation from a Fréchet algebra $\mathcal{A}$ to itself, the separating subspace is an ideal of $\mathcal{A}$. So we shall presently show directly that derivations on Fréchet algebras can have the unit 1 in their separating subspace, and we will deduce that the separating subspace is the whole algebra. It is worthwhile mentioning that in a forthcoming paper [T2], Thomas will show, very broadly, that if derivations on a Fréchet algebra $\mathcal{A}$ map outside the (Jacobson) radical, then they must do so along the lines of the

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example in this paper. There must be a continuous algebra homomorphism from $\mathcal{A}$ into a Fréchet algebra $\mathcal{A}_0[[x_0]]$ of formal power series in an indeterminate x_0, such that x_0 lies in the closure of the coefficient algebra $\mathcal{A}_0$. It is just such an algebra that we will work with here. The coefficient algebra we shall use is $\mathbb{C}[[x_1, x_2, ...]]$, the algebra of all formal power series in countably many further indeterminates x_i, and the derivation we shall use is the natural one, $\partial/\partial x_0$. Construction of a suitable topology, however, is not "natural"; it requires the axiom of choice (see Definition 1.7), a necessary development because without AC, it is consistent to assume that *all* linear maps between Fréchet spaces are continuous.

1. Preliminary definitions

Definition 1.1. Let $\mathbb{N}_0$ denote $\mathbb{N} \cup \{0\}$. Let $\mathcal{A}^{(1)}$ be the vector space of all formal sums $\sum_{i=0}^{\infty} \lambda_i x_i$, where x_i are indeterminates and $\lambda_i \in \mathbb{C}$. We form the full, (non-commutative) algebra $\mathcal{B} = \mathbb{C}_{nc}[[x_0, x_1, \dots]]$ of all formal power series in infinitely many **non-commuting** indeterminates x_i, in which $\mathcal{A}^{(1)}$ is embedded as the linear elements. We also form the full commutative algebra $\mathcal{A} = \mathbb{C}[[x_0, x_1, \dots]]$ of all formal power series in infinitely many **commuting** indeterminates x_i. The multiplication on $\mathcal{B}$ will be written $b_1 \otimes b_2$; the commutative multiplication on $\mathcal{A}$ will be written $a_1 \cdot a_2$.

Definition 1.2. Let $\mathcal{F}$ be the collection of finite sequences of elements of $\mathbb{N}_0$, and for each $\mathbf{i} = (i_1, i_2, ..., i_n) \in \mathcal{F}$ write $\mathbf{x}^{\otimes \mathbf{i}}$ for the element $x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_n} \in \mathcal{B}$. A typical element $b \in \mathcal{B}$ will be written $\sum_{\mathbf{i} \in \mathcal{F}} b_{\mathbf{i}} \mathbf{x}^{\otimes \mathbf{i}}$. The analogous idea for $\mathcal{A}$ is to declare that elements of $\mathcal{F}$ are equivalent if one sequence is a permutation of the other, and to use the collection $\mathcal{F}_0$ of equivalence classes for the relation. We write $a \in \mathcal{A}$ as $\sum_{\mathbf{i} \in \mathcal{F}_0} b_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, where $\mathbf{x}^{\mathbf{i}} = x_{i_1} \cdot x_{i_2} \cdot ... \cdot x_{i_n}$.

It is not hard to define continuous derivations on the algebras $\mathcal{A}$ and $\mathcal{B}$; one gives them a sensible topology and uses the derivation $\partial/\partial x_0$ of formal differentiation with respect to x_0. It is our business to give $\mathcal{A}$ and $\mathcal{B}$ a not-so-sensible topology, with respect to which they are still Fréchet algebras, but such that the derivation $\partial/\partial x_0$ is no longer continuous. Indeed, with respect to our (bizarre) topology one has $x_n \to x_0$ as $n \to \infty$; so one has $x_0 - x_n \to 0$ yet $\partial/\partial x_0(x_0 - x_n) = 1$. Thus 1 is in the separating subspace for $\partial/\partial x_0$; since the separating subspace is an ideal, it is the whole algebra. Let us therefore define ourselves bizarre topologies on $\mathcal{B}$ and $\mathcal{A}$.

Definition 1.3. We note that if $\mathbf{b}_n \in \mathcal{B}$, then the infinite sum $\sum_{n=1}^{\infty} \mathbf{b}_n$ is well defined provided that the sequence $\mathbf{b}_n$ is "locally finite", i.e. for each $\mathbf{j} \in \mathcal{F}$ the set of n such that $\mathbf{b}_n$ has a nonzero coefficient in $x^{\otimes \mathbf{j}}$ is finite. We define a "locally finite" linear map $T : \mathcal{B} \to \mathcal{B}$ to be one such that

$$T(\sum_{\mathbf{i} \in \mathcal{F}} b_{\mathbf{i}} \mathbf{x}^{\otimes \mathbf{i}}) = \sum_{\mathbf{i} \in \mathcal{F}} b_{\mathbf{i}} \mathbf{t}_{\mathbf{i}}$$

for some locally finite sequence $(\mathbf{t}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{F}}$.

One may likewise define "locally finite" linear maps $\mathcal{B} \to \mathcal{A}$, $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{A}$; they are precisely the linear maps between these spaces that are continuous with respect to their natural Fréchet algebra topologies, and they have the property that $T(\sum_{n=1}^{\infty} \mathbf{b}_n) = \sum_{n=1}^{\infty} T(\mathbf{b}_n)$ for all locally finite sequences $\mathbf{b}_n$.

Definition 1.4. For each $n \in \mathbb{N}_0$, let $P_n : \mathcal{B} \to \mathcal{B}$ be the locally finite map such that $P_n(1) = 0$ and for each $\mathbf{i} = (i_1, ..., i_m) \in \mathcal{F}$, we have

$$P_n(\mathbf{x}^{\otimes \mathbf{i}}) = \begin{cases} 0, & \text{if } i_1 \neq n, \\ \otimes_{j=2}^m x_{i_j}, & \text{if } i_1 = n. \end{cases}$$

Thus P_n takes the quotient on division from the left by x_n, and discards the remainder.

Definition 1.5. Let $\pi : \mathcal{B} \to \mathcal{A}$ be the natural map that forgets that the x_i fail to commute, i.e. the locally finite map such that $\pi(\mathbf{x}^{\otimes \mathbf{i}}) = \mathbf{x}^{\mathbf{i}}$ for all $\mathbf{i}$.

Definition 1.6. Let $\alpha : \mathcal{B} \to \mathcal{B}$ be the averaging map, the locally finite map such that

$$\alpha(x_{i_1} \otimes ... \otimes x_{i_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{i_{\sigma(1)}} \otimes ... \otimes x_{i_{\sigma(n)}}.$$

The map $\pi|_{\alpha(\mathcal{B})}$ is bijective; let $\rho : \mathcal{A} \to \alpha(\mathcal{B})$ be its inverse. ρ is a natural right inverse to π.

Definition 1.7. Let us choose, once and for all, a linear functional $\psi : \mathcal{A}^{(1)} \to \mathbb{C}$ with the property that $\psi(x_n) = 1$ for all $n \in \mathbb{N}_0$. (Note that this requires extending the sequence $(x_i)_{i=0}^{\infty}$ to a Hamel basis of $\mathcal{A}^{(1)}$, and thus involves the axiom of choice.)

Definition 1.8. Now $\mathcal{B}$ is a **graded** algebra of formal power series. Let us write $\mathcal{B}^{(n)}$ for the subspace of n-homogeneous formal power series $\sum_{\mathbf{i} \in \mathbb{N}_0^n} b_{\mathbf{i}} \mathbf{x}^{\otimes \mathbf{i}}$; and write $\mathcal{A}^{(n)} = \pi(\mathcal{B}^{(n)})$, noting that $\mathcal{B}^{(1)}$ is naturally identified with $\mathcal{A}^{(1)}$. If $\mathbf{b} \in \mathcal{B} = \oplus_{n=1}^{\infty} \mathcal{B}^{(n)}$ we write $(\mathbf{b}^{(n)})_{n=1}^{\infty}$ for the unique sequence of elements $\mathbf{b}^{(n)} \in \mathcal{B}^{(n)}$ such that $\mathbf{b} = \sum_{n=0}^{\infty} \mathbf{b}^{(n)}$.

Definition 1.9. If $\phi_1, \phi_2 : \mathcal{A}^{(1)} \to \mathbb{C}$ are linear functionals, we define the "tensor product by rows" $\phi_1 \otimes \phi_2 : \mathcal{B}^{(2)} \to \mathbb{C}$ by

$$\phi_1 \otimes \phi_2(\mathbf{a}) = \phi_1(\sum_{j=0}^{\infty} x_j \cdot \phi_2(P_j \mathbf{a})).$$

Note that since infinite sums of tensors are allowed in $\mathcal{B}^{(2)}$, this is by no means the only way of defining a tensor product of linear functionals such that $\phi_1 \otimes \phi_2(\mathbf{a}_1 \otimes \mathbf{a}_2) = \phi_1(\mathbf{a}_1)\phi_2(\mathbf{a}_2)$. Tensor products $\otimes_{i=1}^n \phi_i : \mathcal{B}^{(n)} \to \mathbb{C}$ are then defined inductively by

$$\otimes_{i=1}^n \phi_i(\mathbf{a}) = \phi_1(\sum_{j=0}^{\infty} x_j \cdot \otimes_{i=2}^n \phi_i(P_j \mathbf{a})). \tag{1.1}$$

For the purposes of the next lemma, the tensor product of *no* linear functionals is deemed to be the identity map, regarded as a linear functional on $\mathbb{C} = \mathcal{B}^{(0)}$.

Lemma 1.10. *For all* $0 \le r \le n$, $\mathbf{a} \in \mathcal{B}^{(r)}$, $\mathbf{b} \in \mathcal{B}^{(n-r)}$ *and functionals* $(\phi_i)_{i=1}^n$ *on* $\mathcal{A}^{(1)}$, *we have*

$$\otimes_{i=1}^n \phi_i(\mathbf{a} \otimes \mathbf{b}) = \Big(\otimes_{i=1}^r \phi_i(\mathbf{a})\Big) \cdot \Big(\otimes_{i=r+1}^n \phi_i(\mathbf{b})\Big).$$

The proof is by induction on r, and is fairly obvious. Let's nonetheless do the inductive step when $r \geq 1$. Then (1.1) gives us

$$\otimes_{i=1}^{n}\phi_i(\mathbf{a}\otimes\mathbf{b}) = \phi_1(\sum_{j=0}^{\infty}x_j \cdot \otimes_{i=2}^{n}\phi_i(P_j(\mathbf{a}\otimes\mathbf{b}))).$$

Since $r \geq 1$, one sees that $P_j(\mathbf{a}\otimes\mathbf{b}) = P_j(\mathbf{a})\otimes\mathbf{b}$. So

$$\begin{aligned}\otimes_{i=1}^{n}\phi_i(\mathbf{a}\otimes\mathbf{b}) &= \phi_1(\sum_{j=0}^{\infty}x_j \cdot \otimes_{i=2}^{n}\phi_i(P_j\mathbf{a}\otimes\mathbf{b}))\\ &= \phi_1(\sum_{j=0}^{\infty}x_j \cdot \otimes_{i=2}^{r}\phi_i(P_j\mathbf{a})\cdot\otimes_{i=r+1}^{n}\phi_i(\mathbf{b}))\end{aligned}$$

by induction hypothesis; one may take out a scalar factor $\otimes_{i=r+1}^{n}\phi_i(\mathbf{b})$ and obtain

$$\otimes_{i=r+1}^{n}\phi_i(\mathbf{b})\cdot\phi_1(\sum_{j=0}^{\infty}x_j \cdot \otimes_{i=2}^{r}\phi_i(P_j\mathbf{a})) = \otimes_{i=1}^{r}\phi_i(\mathbf{a})\cdot\otimes_{i=r+1}^{n}\phi_i(\mathbf{b}),$$

by (1.1) again.

Corollary 1.11. *If $(\phi_i)_{i=0}^{n}$ are linear functionals on $\mathcal{A}^{(1)}$, then for each $m \in \mathbb{N}_0$ the seminorm $\|\cdot\|_n^{(m)}$ on $\mathcal{B}$ given by*

$$\|\mathbf{a}\|_n^{(m)} = |\mathbf{a}^{(0)}| + \sum_{r=1}^{m}\sum_{\mathbf{i}\in\{0,1,\ldots,n\}^r}|\otimes_{j=1}^{r}\phi_{i_j}(\mathbf{a}^{(r)})| \tag{1.2}$$

is an algebra seminorm, satisfying $\|\mathbf{ab}\|_n^{(m)} \leq \|\mathbf{a}\|_n^{(m)}\cdot\|\mathbf{b}\|_n^{(m)}$ for all $\mathbf{a},\mathbf{b}\in\mathcal{B}$.

Note that since the order of appearance of the ϕ_{i_j} can be permuted arbitrarily in (1.2), one has $\|\alpha\|_n^{(m)} = 1$ for all n and m, where α is the averaging map. Hence, these seminorms are also algebra seminorms on $\mathcal{A}$, when $\mathcal{A}$ is identified with the linear subspace $\alpha\mathcal{B}\subset\mathcal{B}$ (the multiplication of $\mathcal{A}$ is then implemented by $(\mathbf{a},\mathbf{b})\to\alpha(\mathbf{a}\otimes\mathbf{b})$).

Definition 1.12. Let $(\phi_i)_{i=0}^{\infty}$ be linear functionals on $\mathcal{A}^{(1)}$, as follows:
(a) $\phi_0 = \psi$, the "discontinuous" linear functional of Definition 1.7.
(b) For $n > 0$, $\phi_n = x_n^*$, the coordinate functional such that $x_n^*(\sum_{i=0}^{\infty}\lambda_i x_i) = \lambda_n$.
Let τ be the locally multiplicatively convex topology on $\mathcal{B}$ of convergence in all the seminorms $\|\cdot\|_n^{(m)}$ for this sequence of linear functionals ϕ_n.

We claim that $\mathcal{B}$ is in fact complete under the topology τ, so, $(\mathcal{B},\tau)$ is a Fréchet algebra. Since α is a τ-continuous projection, the subspace $\mathcal{A} = \alpha(\mathcal{B}) = \ker(I-\alpha)$ is closed, so $(\mathcal{A},\tau)$ is also a Fréchet algebra. Note that if we prove our claim we have our result; for since $\phi_0(x_N - x_0) = 0$ for $N > 0$ one sees that $\|x_N - x_0\|_n^{(m)} = 0$ for all n,m with $n < N$; hence, $x_N \to x_0$ in τ. One may then use the derivation $\partial/\partial x_0$ as described before, and the separating subspace is the whole algebra, be it $\mathcal{A}$ or $\mathcal{B}$. All that remains is to prove $\mathcal{B}$ complete.

2. Proving $\mathcal{B}$ complete

Definition 2.1. Let τ_0 be the "usual" topology that makes $\mathcal{B}$ a Fréchet algebra. That is, apply the method of Corollary 1.11 to the sequence of coordinate functionals $(x_n^*)_{n=0}^\infty$, obtaining algebra seminorms $|\cdot|_n^{(m)}$,

$$|\sum a_{\mathbf{i}}\mathbf{x}^{\otimes \mathbf{i}}\Big|_n^{(m)} = |a^{(0)}| + \sum_{r=1}^{m} \sum_{\mathbf{i}\in\{0,1,\ldots,n\}^r} |a_{\mathbf{i}}|,$$

and let τ_0 be the topology of convergence with respect to all these seminorms. τ_0 is the topology of pointwise convergence of all the coefficients $a_{\mathbf{i}}$.

Definition 2.2. With an eye on (1.2), we define a linear map $\Psi : \mathcal{B} \to \mathcal{B}$ by

$$\Psi(\mathbf{a}) = a^{(0)} + \sum_{r=1}^{\infty} \sum_{\mathbf{i}\in\mathbb{N}_0^r} \mathbf{x}^{\otimes \mathbf{i}} \cdot \otimes_{j=1}^r \phi_{i_j}(\mathbf{a}^{(r)}).$$

We note that $\Psi : (\mathcal{B}, \tau) \to (\mathcal{B}, \tau_0)$ is continuous, because convergence under τ is precisely convergence of all the functionals $\otimes_{j=1}^r \phi_{i_j}(\mathbf{a}^{(r)})$ that are involved in $\Psi(\mathbf{a})$.

Theorem 2.3. *Ψ is bijective.*

Proof. Ψ evidently maps $\mathcal{B}^{(r)}$ to $\mathcal{B}^{(r)}$ for each r. It is therefore enough to show that $\Psi^{(r)} = \Psi|_{\mathcal{B}^{(r)}}$ is a bijection $\mathcal{B}^{(r)} \to \mathcal{B}^{(r)}$ for each r.

When $r = 0$ we have the identity map on $\mathbb{C}$, which is bijective. For $r > 0$ we proceed as follows.

Definition 2.4. For $0 \le k \le r$, let $\mathcal{F}^{(r,k)}$ be the set of all $\mathbf{i} = (i_1, i_2, ..., i_r) \in \mathbb{N}_0^r$ such that exactly k of the i_j are equal to zero. Let $\mathcal{B}^{(r,k)} \subset \mathcal{B}$ be the subspace consisting of all power series of form $\sum_{\mathbf{i}\in\mathcal{F}^{(r,k)}} a_{\mathbf{i}}\mathbf{x}^{\otimes \mathbf{i}}$.

Obviously $\mathcal{B}^{(r)} = \oplus_{k=0}^r \mathcal{B}^{(r,k)}$; let $\beta_{r,k}$ be the projection onto $\mathcal{B}^{(r,k)}$ parallel to the others. We claim that for each k, $\beta_{r,k}\Psi^{(r)}|_{\mathcal{B}^{(r,k)}}$ is equal to the identity map on $\mathcal{B}^{(r,k)}$; and we claim that $\Psi^{(r)}$ maps $\mathcal{B}^{(r,k)}$ into $\oplus_{l=k}^r \mathcal{B}^{(r,l)}$. Thus we shall show that the action of $\Psi^{(r)}$ is "lower triangular" and nonsingular, with respect to the decomposition $\mathcal{B}^{(r)} = \oplus_{k=0}^r \mathcal{B}^{(r,k)}$; therefore, $\Psi^{(r)}$ - and hence Ψ - is bijective.

In view of Definition 2.2, $\beta_{r,k}\Psi^{(r)}|_{\mathcal{B}^{(r,k)}}$ is equal to the identity map if and only if we have

$$\otimes_{i=1}^r \phi_{l_i}(\mathbf{a}) = a_{\mathbf{l}} \tag{2.1}$$

for each $\mathbf{l} = (l_1, ..., l_r) \in \mathcal{F}^{(r,k)}$ and $\mathbf{a} \in \mathcal{B}^{(r,k)}$; and $\Psi^{(r)}$ maps $\mathcal{B}^{(r,k)}$ into $\oplus_{l=k}^r \mathcal{B}^{(r,l)}$ if and only if

$$\otimes_{i=1}^r \phi_{l_i}(\mathbf{a}) = 0 \tag{2.2}$$

for all $m < k$, $\mathbf{l} = (l_1, ..., l_r) \in \mathcal{F}^{(r,m)}$ and $\mathbf{a} \in \mathcal{B}^{(r,k)}$.

Let us prove (2.1) and (2.2) together, by induction on r.

When $r = 1$, statement (2.1) demands that $\phi_l(\mathbf{a}) = a_l$ if either $l = 0$ and $\mathbf{a} \in \mathcal{B}^{(1,1)}$, or $l > 0$ and $\mathbf{a} \in \mathcal{B}^{(1,0)}$. For $l > 0$ the functional $\phi_l = x_l^*$ anyway, so the assertion is true when $l > 0$. But when $l = 0$, we note that $\mathcal{B}^{(1,1)}$ consists solely of multiples $a_0 x_0$; since $\phi_0(x_0) = 1$, the assertion is true for $l = 0$ also. Statement (2.2) demands that $\phi_l(\mathbf{a}) = 0$ if $l > 0$ and $\mathbf{a} \in \mathcal{B}^{(1,1)}$. But in that case, $\mathbf{a}$ is a multiple of x_0 and ϕ_l is equal to x_l^* for $l > 0$; so the assertion is again correct.

When $r > 1$ we proceed by induction. Equation (1.1) gives us

$$\otimes_{i=1}^{r}\phi_{l_i}(\mathbf{a}) = \phi_{l_1}(\sum_{j=0}^{\infty} x_j \cdot \otimes_{i=2}^{r}\phi_{l_i}(P_j\mathbf{a})). \tag{2.3}$$

To prove statement (2.1), consider the case when $\mathbf{l} \in \mathcal{F}^{(r,k)}$ and $\mathbf{a} \in \mathcal{B}^{(r,k)}$. Note that $P_j\mathbf{a}$ lies in $\mathcal{B}^{(r-1,k)}$ if $j > 0$, but in $\mathcal{B}^{(r-1,k-1)}$ if $j = 0$, because the division on the left by x_0 removes a factor x_0 from each monomial $\mathbf{x}^{\otimes \mathbf{i}}$.

If $l_1 = 0$, then $(l_2, ..., l_r) \in \mathcal{F}^{(r-1,k-1)}$ so $\otimes_{i=2}^{r}\phi_{l_i}$ will (by induction hypothesis) annihilate any vector in $\mathcal{B}^{(r-1,k)}$, but will send $\mathbf{b} \in \mathcal{B}^{(r-1,k-1)}$ to $b_{l_2,...,l_r}$. Accordingly, when $l_1 = 0$, (2.3) is equal to $\phi_0(x_0 \cdot (P_0\mathbf{a})_{l_2,...,l_r}) = (P_0\mathbf{a})_{l_2,...,l_r} = a_{0,l_2,...,l_r}$ $= a_{l_1,...,l_r}$ as required.

If on the other hand $l_1 > 0$, then ϕ_{l_1} is just $x_{l_1}^*$, so (2.3) is equal to $\otimes_{i=2}^{r}\phi_{l_i}(P_{l_1}\mathbf{a})$. In this case, $(l_2, ..., l_r) \in \mathcal{F}^{(r-1,k)}$ and $P_{l_1}\mathbf{a} \in \mathcal{B}^{(r-1,k)}$ so by induction hypothesis, (2.3) is equal to $(P_{l_1}\mathbf{a})_{l_2,...,l_r} = a_{l_1,...,l_r}$ as before.

This deals with statement (2.1). For (2.2), we consider $\mathbf{a} \in \mathcal{B}^{(r,k)}$ as before, but now use sequences $\mathbf{l} \in \mathcal{F}^{(r,m)}$ for some $m < k$.

If $l_1 = 0$, then $(l_2, ..., l_r) \in \mathcal{F}^{(r-1,m-1)}$; since each $P_j\mathbf{a}$ is either in $\mathcal{B}^{(r-1,k)}$ or $\mathcal{B}^{(r-1,k-1)}$, by induction hypothesis the linear map $\otimes_{i=2}^{r}\phi_{l_i}$ annihilates all of the vectors $P_j\mathbf{a}$, so (2.3) is equal to zero as required.

If $l_1 > 0$, then $\phi_{l_1} = x_{l_1}^*$ so (2.3) is equal to $\otimes_{i=2}^{r}\phi_{l_i}(P_{l_1}\mathbf{a})$. Since $(l_2, ..., l_r) \in \mathcal{F}^{(r-1,m)}$ and $P_{l_1}\mathbf{a} \in \mathcal{B}^{(r-1,k)}$, by induction hypothesis this is zero also. Thus (2.1) and (2.2) are proved, and the proof of Theorem 2.3 is concluded. □

Theorem 2.5. *$(\mathcal{B}, \tau)$ is complete with respect to the defining seminorms $\|\cdot\|_n^{(m)}$. The derivation $\partial/\partial x_0 : (\mathcal{B}, \tau) \to (\mathcal{B}, \tau)$ is discontinuous, and its separating subspace is all of $\mathcal{B}$. The derivation $\partial/\partial x_0 : \mathcal{A} \to \mathcal{A}$ is also discontinuous, and its separating subspace is all of $\mathcal{A}$.*

Proof. Let $(u_i)_{i=1}^{\infty}$ be a $\|\cdot\|_n^{(m)}$ Cauchy sequence for all n, m. Then $\Psi(u_i)$ is a Cauchy sequence in the usual seminorms $|\cdot|_n^{(m)}$. It is well known that $\mathcal{B}$ is complete under the usual seminorms - we just have the topology of pointwise convergence of all the coefficients - so let v be the τ_0-limit of the vectors $\Psi(u_n)$. The τ-limit of the u_i is just $\Psi^{-1}(v)$. So $(\mathcal{B}, \tau)$ is complete under its seminorms $\|\cdot\|_n^{(m)}$.

The argument that shows the rest of the assertions, given $\mathcal{B}$ complete, has already been given. □

3. Conclusions

The above example confirms the impression that derivations can behave quite badly on Fréchet algebras, which impression is initially given simply by the fact that $\partial/\partial x_0$ maps outside the radical, on any formal power series algebra $\mathcal{A}_0[[x_0]]$. However, this paper and Thomas' forthcoming paper [T2] together show that while the derivation can map outside the radical, and even its separating subspace can fail to lie inside the radical, yet in some sense the formal power series algebra is the one and only way in which that can happen. In that sense, there is a parallel with the much harder *Michael problem*: we do not know whether discontinuous characters can exist on commutative Fréchet algebras, but if they do, they must exist on certain algebras of entire analytic functions (see Dixon and Esterle [DE]).

Slowly, a structure theory for Fréchet algebras, as distinct from Banach algebras, is beginning to emerge.

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