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REMARKS ON GINZBURG'S BIVARIANT CHERN CLASSES

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ABSTRACT. The convolution product is an important tool in the geometric representation theory. Ginzburg constructed the bivariant Chern class operation from a certain convolution algebra of Lagrangian cycles to the convolution algebra of Borel-Moore homology. In this paper we give some remarks on the Ginzburg bivariant Chern classes.

§1. INTRODUCTION

The basic references are V. Ginzburg's survey article "Geometric methods in the representation theory of Hecke algebras and quantum groups" [G2], Chriss-Ginzburg's book [CG] and H. Nakajima's survey article "Quiver varieties and quantum affine algebras" (in Japanese) [N1] (cf. its original paper [N2]).

In [G1] Ginzburg introduced the notion of "bivariant" Chern class from the abelian group of certain Lagrangian cycles satisfying some special conditions to the Borel-Moore homology group and showed that it is "convolutive", i.e., it commutes with the convolution product. The Ginzburg bivariant Chern class is, in short, expressed as the Chern-Schwartz-MacPherson class multiplied by the pullback of the Segre (cohomology) class of the nonsingular target variety. In this paper we make some remarks on the Ginzburg bivariant Chern classes and also we give a naïve "constructible function version" of the above Ginzburg's result concerning convolution.

§2. RELATIVE CHERN-MATHER CLASS AND GINZBURG BIVARIANT CHERN CLASS

The construction or definition of the Ginzburg bivariant Chern class c^{biv} given in [G1] is not direct, but in his survey article [G2] he gives an explicit description of it. It assigns to a Lagrangian cycle, i.e., $\Lambda_Y, Y \subset X_1 \times X_2$, the *relative Chern-Mather class of the fibers of the projection* $p_Y : Y \to X_2$. The projection p_Y is the restriction of the projection $p_2 : X_1 \times X_2 \to X_2$ to the subvariety Y. Let $\nu : \widehat{Y} \to Y$ be the Nash blow-up and $\widehat{TY}$ the tautological Nash tangent bundle over $\widehat{Y}$. Then

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the above relative Chern-Mather class of the fibers of the projection $p_Y : Y \to X_2$ is

$$c^{\mathrm{biv}}(\Lambda_Y) := i_{Y*}\nu_*\Big(c(\widehat{TY} - \nu^* p_Y^* TX_2) \cap [\widehat{Y}]\Big)$$

where $i_Y : Y \to X_1 \times X_2$ is the inclusion. A more concrete expression of this is

$$\begin{aligned} c^{\mathrm{biv}}(\Lambda_Y) &= i_{Y*}\Big(\frac{1}{p_Y^* c(TX_2)} \cap c^M(Y)\Big) \\ &= p_2^* s(TX_2) \cap i_{Y*} c^M(Y). \end{aligned}$$

Here $s(TX_2)$ denotes the Segre class of the tangent bundle of the manifold X_2 and $c^M(Y)$ is the Chern-Mather class of Y.

Therefore, the Ginzburg bivariant Chern class can be simply extended to a S-variety X with S being nonsingular, i.e., any morphism $\pi : X \to S$ from a possibly singular variety X to a nonsingular variety S and defined for any constructible functions on the source variety. Thus we can define

$$c_*^\pi = \pi^* s(TS) \cap c_* : F(X) \to H_*(X, \mathbb{Q})$$

where $c_* : F(X) \to H_*(X; \mathbb{Q})$ is the usual Chern-Schwartz-MacPherson class transformation with rational coefficients [M]. In fact, furthermore, we can modify the base variety a bit more. For a local complete intersection variety S in a smooth variety M, we can take the virtual tangent bundle

$$TS := TM|_S - N_S M$$

where $N_S M$ is the normal bundle of S in M (see [F]).

Theorem (2.1). *For the category of S-varieties with S being a smooth variety or a local complete intersection in a smooth variety, $c_*^\pi : F(X) \to H_*(X; \mathbb{Q})$ is the unique natural transformation from the constructible functions to the homology theory satisfying the condition that for a smooth variety X*

$$c_*^\pi(\mathbb{1}_X) = c(T_\pi) \cap [X],$$

where $T_\pi := TX - \pi^* TS$ is the relative virtual tangent bundle. Namely, for any commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ S & \xrightarrow[\mathrm{id}_S]{} & S \end{array}$$

we have the following commutaive diagram:

$$\begin{array}{ccc} F(X_1) & \xrightarrow{f_*} & F(X_2) \\ {\scriptstyle c_*^{\pi_1}}\downarrow & & \downarrow{\scriptstyle c_*^{\pi_2}} \\ H_*(X_1) & \xrightarrow[f_*]{} & H_*(X_2). \end{array}$$

□

This follows from MacPherson's theorem [M] and the projection formula, and the proof of the uniqueness is the same as in [M].

Now it is easy to see that, for any vector bundle $\mathcal{E}$ given on the base variety S, we can replace the "twisting" cohomology $\pi^* s(TS)$ by the pullback of any characteristic class $c\ell(\mathcal{E})$ of the bundle $\mathcal{E}$, i.e., we can consider

$$c_*^{c\ell(\mathcal{E})} := \pi^* c\ell(\mathcal{E}) \cap c_* : F(X) \to H_*(X;\mathbb{Q}).$$

Note that in this situation the base variety S can be any variety, singular or nonsingular. Then this is the unique natural transformation satisfying the smooth condition that for a smooth variety X we have

$$c_*^{c\ell(\mathcal{E})}(\mathbb{1}_X) = \Big(\pi^* c\ell(\mathcal{E}) \cup c(TX)\Big) \cap [X].$$

The requirement of S being smooth or a local complete intersection in a smooth variety is essential in the above theorem or the set up of Ginzburg. The most naïve and interesting question is whether or not we can generalize the definition of the Ginzburg bivariant Chern class to the case when the base variety S is arbitrarily singular. Consider the following fiber squares:

$$\begin{CD}
\widehat{W} \times_S \widehat{S} @>\widehat{\nu_W}>> W \times_S \widehat{S} @>\widehat{f_W}>> \widehat{S} \\
@V{\widetilde{\nu_S}}VV @V{\widehat{\nu_S}}VV @VV{\nu_S}V \\
\widehat{W} @>>{\nu_W}> W @>>{f_W}> S,
\end{CD}$$

where $\nu_W : \widehat{W} \to W$ and $\nu_S : \widehat{S} \to S$ are the Nash blow-ups and the morphisms $\widetilde{\nu_S}, \widehat{\nu_W}, \widehat{\nu_S}$ and $\widehat{f_W}$ are the projections from the corresponding fiber products. Then it is quite natural and reasonable to define the following homology class:

$$c_*^{\pi}(\mathbb{1}_W) := i_{W*}\nu_{W*}\widetilde{\nu_S}_*\Big(c\big(\widetilde{\nu_S}^*\widehat{TW} - \widehat{\nu_W}^*\widehat{f_W}^*\widehat{TS}\big) \cap [\widehat{W} \times_S \widehat{S}]\Big),$$

which is also equal to

$$i_{W*}\widehat{\nu_S}_*\widehat{\nu_W}_*\Big(c\big(\widetilde{\nu_S}^*\widehat{TW} - \widehat{\nu_W}^*\widehat{f_W}^*\widehat{TS}\big) \cap [\widehat{W} \times_S \widehat{S}]\Big).$$

Then our obvious question is whether Theorem (2.1) holds or not with this definition of $c_*^{\pi}(\mathbb{1}_W)$ for the base variety S being arbitrarily singular. It turns out that the answer is negative, which implies that the nonsingularity of the base variety S is essential in the set-up of Ginzburg, i.e., the existence of the (virtual) tangent bundle TS is essential.

To see this negative result, let us consider the following special case. Let $\nu : \widehat{S} \to S$ be the Nash blow-up and consider the diagram

$$\begin{CD}
\widehat{S} @>\nu>> S \\
@V{\nu}VV @VV{\mathrm{id}_S}V \\
S @>>{\mathrm{id}_S}> S.
\end{CD}$$

Then we ask if the following diagram commutes:

$$\begin{array}{ccc} F(\widehat{S}) & \xrightarrow{\nu_*} & F(S) \\ {\scriptstyle c_*^{\nu}}\downarrow & & \downarrow{\scriptstyle c_*^{\mathrm{id}_S}} \\ H_*(\widehat{S}) & \xrightarrow[\nu_*]{} & H_*(S). \end{array}$$

In particular, we ask if the following equality holds:

$$\nu_* c_*^{\nu}(1\!\!1_{\widehat{S}}) = c_*^{\mathrm{id}_S}(\nu_* 1\!\!1_{\widehat{S}}).$$

In fact, we can show the following:

Theorem (2.2). *Let S be a singular variety with one isolated singularity x_0 such that the Nash blow-up $\widehat{S}$ of S is nonsingular and furthermore such that the Euler-Poincaré characteristic $\chi(\nu^{-1}(x_0))$ of the fiber of the singularity x_0 is not equal to* 1. *Then we have*

$$\nu_* c_*^{\nu}(1\!\!1_{\widehat{S}}) \neq c_*^{\mathrm{id}_S}(\nu_* 1\!\!1_{\widehat{S}}).$$

Proof. To define $c_*^{\nu}(1\!\!1_{\widehat{S}})$, we need the following fiber square:

(2.2.1)
$$\begin{array}{ccc} \widehat{S}\times_S \widehat{S} & \xrightarrow{\widehat{\nu}} & \widehat{S} \\ {\scriptstyle \widehat{\nu}}\downarrow & & \downarrow{\scriptstyle \nu} \\ \widehat{S} & \xrightarrow[\nu]{} & S. \end{array}$$

By the definition we have

$$\nu_* c_*^{\nu}(1\!\!1_{\widehat{S}}) = \nu_*\Big(c\big(\widehat{\nu}^* T\widehat{S} - \widehat{\nu}^*\widehat{TS}\big)\cap[\widehat{S}\times_S \widehat{S}]\Big).$$

On the other hand, we have that

$$\nu_* 1\!\!1_{\widehat{S}} = 1\!\!1_S + \Big(\chi(\nu^{-1}(x_0)) - 1\Big)\cdot 1\!\!1_{x_0}.$$

Hence by the definition we have

$$c_*^{\mathrm{id}_S}(\nu_* 1\!\!1_{\widehat{S}}) = c_*^{\mathrm{id}_S}(1\!\!1_S) + \Big(\chi(\nu^{-1}(x_0)) - 1\Big) c_*^{\mathrm{id}_S}(1\!\!1_{x_0}).$$

Now, to define $c_*^{\mathrm{id}_S}(1\!\!1_S)$ we need the following fiber squares:

$$\begin{array}{ccccc} \widehat{S}\times_S \widehat{S} & \xrightarrow{\widehat{\nu}} & \widehat{S} & \xrightarrow{\mathrm{id}_{\widehat{S}}} & \widehat{S} \\ \downarrow{\scriptstyle \widehat{\nu}} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ \widehat{S} & \xrightarrow[\nu]{} & S & \xrightarrow[\mathrm{id}_S]{} & S, \end{array}$$

which is nothing but the above diagram (2.2.1). Therefore we get the following:

$$c_*^{\mathrm{id}_S}(1\!\!1_S) = \nu_* c_*^{\nu}(1\!\!1_{\widehat{S}}).$$

It is easy to see that $c_*^{\mathrm{id}_S}(1\!\!1_{x_0}) = 1$. Therefore, the hypothesis that $\chi(\nu^{-1}(x_0)) \neq 1$ implies that

$$\nu_* c_*^{\nu}(1\!\!1_{\widehat{S}}) \neq c_*^{\mathrm{id}_S}(\nu_* 1\!\!1_{\widehat{S}}).$$

□

A typical example of such a variety S in Theorem (2.2) is, for example, more than two lines intersecting at one point.

§3. Convolution

The notion of *convolution* (product) is an important technique ubiquitous in the geometric representation theory. Here we recall the convolution on the Borel-Moore homology theory.

In this paper the homology theory $H_*(X)$ is the Borel-Moore homology group of a locally compact Hausdorff space X, i.e., the ordinary (singular) cohomology group of the pair $(\bar{X}, \infty)$ where $\bar{X} = X \cup \infty$ is the one-point compactification of X.

For any closed subsets X and X' in a smooth manifold M, we have the cup product

$$\cup : H^p(M, M \setminus X) \otimes H^q(M, M \setminus X') \to H^{p+q}(M, M \setminus (X \cap X')),$$

which implies, by the Alexander duality isomorphism $H^\bullet(M, M \setminus A) \cong H_{\dim M - \bullet}(A)$, the following intersection product:

$$\cdot : H_i(X) \otimes H_j(X') \to H_{i+j-\dim M}(X \cap X').$$

Let M_1, M_2, M_3 be smooth oriented manifolds. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the canonical projections. Let $Z \subset M_1 \times M_2$ and $Z' \subset M_2 \times M_3$ be closed subsets and assume that the restricted map

$$p_{13} : p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \to M_1 \times M_3$$

is proper. Then its image is denoted by the $Z \circ Z'$, i.e., the composite of the two correspondences Z and Z' (see Fulton's book [F]). With this set-up, the convolution

$$\bigstar : H_i(Z) \otimes H_j(Z') \to H_{i+j-\dim M}(Z \circ Z')$$

is defined by

$$\alpha \bigstar \beta := p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta). \tag{3.1}$$

In particular, when $M_1 = M_2 = M_3 = M$, for any closed subvariety $Z \subset M \times M$, we have $Z \circ Z = Z$ and therefore $H_*(Z)$ is a convolution algebra.

As one can see in the above construction of convolution, as long as the operations of product, pullback and pushforward are available on certain algebraic objects defined on (topological) spaces, one can always define a convolution product. For example, the convolution of constructible functions is the obvious one defined in the same way as (3.1), using the usual product, pullback and pushforward of constructible functions.

For varieties X, Y, we set

$$\widetilde{\mathbb{F}}(X \times Y) := \mathbb{1}_X \times F(Y) = \{\mathbb{1}_X \times \alpha | \alpha \in F(Y)\}.$$

Let $\pi : X \times Y \to X$ be the projection to the first factor. Then we can see that for a nonsingular variety X we have

$$c_*^\pi(\mathbb{1}_X \times \alpha) = [X] \times c_*(\alpha).$$

This observation follows from the multiplicativity of Chern-Schwartz-MacPherson class, $c_*(\gamma \times \delta) = c_*(\gamma) \times c_*(\delta)$ ([K] and also see [KY]). Indeed,

$$\begin{aligned} c_*^{\pi}(\mathbb{1}_X \times \alpha) &= \pi^* s(TX) \cap c_*(\mathbb{1}_X \times \alpha) \\ &= \big(s(TX) \times 1\big) \cap \big(c_*(\mathbb{1}_X) \times c_*(\alpha)\big) \\ &= \big(s(TX) \times 1\big) \cap \big((c(TX) \cap [X]) \times c_*(\alpha)\big) \\ &= \Big(s(TX) \cap \big(c(TX) \cap [X]\big)\Big) \times (1 \cap c_*(\alpha)) \\ &= [X] \times c_*(\alpha). \end{aligned}$$

Or, we can simply take this as the definition of the homomorphism $c_*^{\pi} : \widetilde{\mathbb{F}}(X \times Y) \to H_*(X \times Y)$ with $\pi : X \times Y \to X$ being the projection. The following theorem is a naïve "constructible function version" of Ginzburg's theorem [G2, Theorem 6.7]:

Theorem (3.2). *Let X be a nonsingular variety and Y be an arbitrary variety. Then the homomorphism $c_*^{\mathrm{biv}} : \widetilde{\mathbb{F}}(X \times Y) \to H_*(X \times Y)$ defined by $c_*^{\mathrm{biv}}(\mathbb{1}_X \times \alpha) = [X] \times c_*(\alpha)$ is convolutive; i.e., the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{\mathbb{F}}(X_1 \times X_2) \otimes \widetilde{\mathbb{F}}(X_2 \times X_3) & \xrightarrow{\ \bigstar\ } & \widetilde{\mathbb{F}}(X_1 \times X_3) \\ {\scriptstyle c_*^{\mathrm{biv}} \otimes c_*^{\mathrm{biv}}} \downarrow & & \downarrow {\scriptstyle c_*^{\mathrm{biv}}} \\ H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\ \bigstar\ ]{} & H_*(X_1 \times X_3). \end{array}$$

Proof. By the definition of the convolution of constructible functions it follows that for constructible functions $\mathbb{1}_{X_1} \times \alpha \in \widetilde{\mathbb{F}}(X_1 \times X_2)$, $\mathbb{1}_{X_2} \times \beta \in \widetilde{\mathbb{F}}(X_2 \times X_3)$, we have

$$(\mathbb{1}_{X_1} \times \alpha) \bigstar (\mathbb{1}_{X_2} \times \beta) = \Big(\int_{X_2} \alpha\Big)(\mathbb{1}_{X_1} \times \beta),$$

where $\int_{X_2} \alpha$ denotes the Euler-Poincaré characteristic of the constructible function α, $\chi(\alpha)$, in other words, it is nothing but the degree of the 0-dimensional component of the Chern-Schwartz-MacPherson class $c_*(\alpha)$ of the constructible function α. Therefore we get

$$c_*^{\mathrm{biv}}\Big((\mathbb{1}_{X_1} \times \alpha) \bigstar (\mathbb{1}_{X_2} \times \beta)\Big) = \Big(\int_{X_2} \alpha\Big)([X_1] \times c_*(\beta)).$$

Let $\pi_{13}^{123} : X_1 \times X_2 \times X_3 \to X_1 \times X_3$ be the projection and let $\mathcal{D}_X : H^*(X) \to H_*(X)$ be the Poincaré duality isomorphism. Then, by the definition of convolution of Borel-Moore homology we get

$$\begin{aligned} & c_*^{\mathrm{biv}}(\mathbb{1}_{X_1} \times \alpha) \bigstar c_*^{\mathrm{biv}}(\mathbb{1}_{X_2} \times \beta) \\ &= \big([X_1] \times c_*(\alpha)\big) \bigstar \big([X_2] \times c_*(\beta)\big) \\ &= \big(\pi_{13}^{123}\big)_* \Big(\big((1 \times \mathcal{D}_{X_2}^{-1}(c_*(\alpha)) \times 1) \cup (1 \times 1 \times \mathcal{D}_{X_3}^{-1}(c_*(\beta)))\big) \cap [X_1 \times X_2 \times X_3]\Big) \\ &= \big(\pi_{13}^{123}\big)_* \left(\left(\big(1 \times \mathcal{D}_{X_2}^{-1}(c_*(\alpha)) \times 1 \cup \mathcal{D}_{X_3}^{-1}(c_*(\beta))\right) \cap ([X_1] \times [X_2] \times [X_3]) \right) \\ &= \big(\pi_{13}^{123}\big)_* ([X_1] \times c_*(\alpha) \times c_*(\beta)) \\ &= \Big(\int_{X_2} \alpha\Big)([X_1] \times c_*(\beta)). \end{aligned}$$

Thus the above diagram is commutative. □

Motivated by Theorem (3.2), in [Y2] we proved the following theorem:

Theorem (3.3). *For nonsingular varieties X_1, X_2, X_3, the homomorphism c_*^{biv} : $\mathbb{F}(X \times Y \xrightarrow{\pi} X) \to H_*(X \times Y)$ defined by $c_*^{\mathrm{biv}} := c_*^{\pi}$ is convolutive; i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{F}(X_1 \times X_2 \to X_1) \otimes \mathbb{F}(X_2 \times X_3 \to X_2) & \xrightarrow{\star} & \mathbb{F}(X_1 \times X_3 \to X_1) \\ {\scriptstyle c_*^{\mathrm{biv}} \otimes c_*^{\mathrm{biv}}} \downarrow & & \downarrow {\scriptstyle c_*^{\mathrm{biv}}} \\ H_*(X_1 \times X_2) \otimes H_*(X_2 \times X_3) & \xrightarrow[\star]{} & H_*(X_1 \times X_3). \end{array}$$

Here $\mathbb{F}(X \times Y \to X)$ is the Fulton-MacPherson bivariant group of constructible functions (see [B] *and* [FM]*).*

The key ingredient of the proof of the above theorem is Brasselet's bivariant Chern class theorem [B, III, Théorème] (see [Y2] for the details). For some other properties of the Ginzburg bivariant Chern classes, see [Y1], [Y2], [Y3].

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