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COHOMOLOGICAL DIMENSION OF CERTAIN ALGEBRAIC VARIETIES

K. DIVAANI-AAZAR, R. NAGHIPOUR, AND M. TOUSI

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R. For finitely generated R-modules M and N with $\operatorname{Supp} N \subseteq \operatorname{Supp} M$, it is shown that $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$. Let N be a finitely generated module over a local ring $(R, \mathfrak{m})$ such that $\operatorname{Min}_{\hat{R}} \hat{N} = \operatorname{Assh}_{\hat{R}} \hat{N}$. Using the above result and the notion of connectedness dimension, it is proved that $\operatorname{cd}(\mathfrak{a}, N) \geq \dim N - c(N/\mathfrak{a}N) - 1$. Here $c(N)$ denotes the connectedness dimension of the topological space $\operatorname{Supp} N$. Finally, as a consequence of this inequality, two previously known generalizations of Faltings' connectedness theorem are improved.

1. Introduction

Throughout, let R denote a commutative Noetherian ring (with identity) and $\mathfrak{a}$ an ideal of R. The study of the cohomological dimension and connectedness of algebraic varieties has produced some interesting results and problems in local algebra. For an R-module M, the *cohomological dimension of M with respect to $\mathfrak{a}$* is defined as

$$\operatorname{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : H^i_{\mathfrak{a}}(M) \neq 0\}.$$

The cohomological dimension has been studied by several authors; see, for example, Faltings [7], Hartshorne [9] and Huneke−Lyubeznik [11]. In particular in [7] and [11], several upper bounds for cohomological dimension were obtained. The main aim of this article is to establish lower bounds for cohomological dimension of finitely generated modules over a local ring. This is done by using the notion of connectedness dimension. For a Noetherian topological space X, the *subdimension* and *connectedness dimension* of X are defined respectively as

$$\begin{aligned} \operatorname{s}\dim X &: = \min\{\dim Z : Z \text{ is an irreducible component of } X\}, \text{and} \\ c(X) &: = \min\{\dim Z : Z \subseteq X, Z \text{ is closed and } X \backslash Z \text{ is disconnected}\}. \end{aligned}$$

For more details about these notions, we refer the reader to [3, Ch. 19]. In particular, if M is an R-module and $\operatorname{Supp} M$ is considered as a subspace of $\operatorname{Spec} R$ equipped with Zariski topology, we denote $c(\operatorname{Supp} M)$ and $\operatorname{s}\dim(\operatorname{Supp} M)$ by $c(M)$

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and $\operatorname{s dim} M$ respectively. It is clear from the definition that a Noetherian topological space X is connected if and only if $c(X) \geq 0$. Recall that the dimension of the empty space is defined to be -1.

We shall prove:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a local ring and N a finitely generated R-module.*

(i) *If R is complete, then* $\operatorname{cd}(\mathfrak{a}, N) \geq \min\{c(N), \operatorname{s dim} N - 1\} - c(N/\mathfrak{a}N)$.

(ii) *If* $\operatorname{Min}_{\hat{R}} \hat{N} = \operatorname{Assh}_{\hat{R}} \hat{N}$, *then* $\operatorname{cd}(\mathfrak{a}, N) \geq \dim N - c(N/\mathfrak{a}N) - 1$.

One of our tools for proving Theorem 1.1 is the following, which plays a key rôle in this paper.

Theorem 1.2. *Let M and N be finitely generated R-modules with* $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. *Then* $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$. *In particular,* $\operatorname{cd}(\mathfrak{a}, N) = \operatorname{cd}(\mathfrak{a}, M)$ *whenever* $\operatorname{Supp} N = \operatorname{Supp} M$.

In [10], M. Hochster and C. Huneke generalized Faltings' connectedness theorem [6]. Also in [5], P. Schenzel and the first author have proved two generalizations of Faltings' connectedness theorem. As a consequence of Theorem 1.1(ii), we remove the indecomposability condition in [10, Theorem 3.3] and [5, Theorem 4.3].

Our terminology follows that of [5]. Moreover for an R-module M, the set of minimal elements of $\operatorname{Ass}_R M$ is denoted by $\operatorname{Min}_R M$ and $\{\mathfrak{p} \in \operatorname{Ass} M : \dim R/\mathfrak{p} = \dim M\}$ by $\operatorname{Assh}_R M$.

2. Cohomological dimension

First of all, we collect the well known properties of the notion of cohomological dimension in a lemma. Before stating the lemma, recall that the height of an ideal $\mathfrak{a}$ with respect to an R-module M is defined as $\operatorname{ht}_M \mathfrak{a} = \min\{\dim M_{\mathfrak{p}} : \mathfrak{p} \supseteq \mathfrak{a}\}$.

Lemma 2.1. *Let $\mathfrak{a}$ denote an ideal of R. Then:*

(i) *for an R-module M,* $\operatorname{ht}_M \mathfrak{a} \leq \operatorname{cd}(\mathfrak{a}, M) \leq \dim M$,

(ii) $\operatorname{cd}(\mathfrak{a}, R) = \max\{\operatorname{cd}(\mathfrak{a}, N) : N \textit{ is an } R\textit{-module}\}$
$= \max\{i \in \mathbb{Z} : H^i_{\mathfrak{a}}(N) \neq 0 \textit{ for some } R\textit{-module } N\}$,

(iii) $\operatorname{cd}(\mathfrak{a}, R) \leq ara(\mathfrak{a})$, *where $ara(\mathfrak{a})$ denotes the arithmetic rank of $\mathfrak{a}$, and*

(iv) *if $f : R \longrightarrow R'$ is a homomorphism of commutative Noetherian rings, then* $\operatorname{cd}(\mathfrak{a}R', R') \leq \operatorname{cd}(\mathfrak{a}, R)$ *and, also for any R'-module M,* $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}R', M)$. *Furthermore if f is faithfully flat, then* $\operatorname{cd}(\mathfrak{a}R', R') = \operatorname{cd}(\mathfrak{a}, R)$.

The following is one of the main results of this paper.

Theorem 2.2. *Let $\mathfrak{a}$ denote a proper ideal of R and M, N two finitely generated R-modules such that* $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. *Then* $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$.

Proof. It is enough to show that $H^i_{\mathfrak{a}}(N) = 0$ for all i with $\operatorname{cd}(\mathfrak{a}, M) < i \leq \dim M + 1$, and all finitely generated R-module N with $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. We argue this by descending induction on i. The assertion is clear for $i = \dim M + 1$ by Grothendieck vanishing theorem. Now, suppose $i \leq \dim M$. Since $\operatorname{Supp} N \subseteq \operatorname{Supp} M$, by Gruson's theorem [12, Theorem 4.1], there is a chain

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = N,$$

such that the factors N_j/N_{j-1} are homomorphic images of a direct sum of finitely many copies of M. By using short exact sequences, we may reduce the situation to the case $k = 1$. Then there is an exact sequence

$$0 \longrightarrow L \longrightarrow M^n \longrightarrow N \longrightarrow 0$$

for some $n \in \mathbb{N}$ and some finitely generated R-module L. This induces a long exact sequence of local cohomology modules

$$\dots \longrightarrow H^i_{\mathfrak{a}}(L) \longrightarrow H^i_{\mathfrak{a}}(M^n) \longrightarrow H^i_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \dots,$$

so that, by the inductive hypothesis, $H^{i+1}_{\mathfrak{a}}(L) = 0$. Hence $H^i_{\mathfrak{a}}(N) = 0$. Thus the argument is complete by induction. □

Corollary 2.3. (i) *Let* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *be an exact sequence of finitely generated R-modules. Then* $\operatorname{cd}(\mathfrak{a}, M) = \max\{\operatorname{cd}(\mathfrak{a}, L), \operatorname{cd}(\mathfrak{a}, N)\}$.

(ii) *Let $f : R \longrightarrow S$ be a monomorphism of commutative Noetherian rings such that S is finitely generated as an R-module. Then for any proper ideal $\mathfrak{a}$ of R,* $\operatorname{cd}(\mathfrak{a}, R) = \operatorname{cd}(\mathfrak{a}S, S)$.

(iii) *If M is a finitely generated faithful R-module, then* $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, R)$.

Proof. (i) From the long exact sequence

$$\dots \longrightarrow H^i_{\mathfrak{a}}(L) \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \dots,$$

we deduce $\operatorname{cd}(\mathfrak{a}, M) \leq \max\{\operatorname{cd}(\mathfrak{a}, L), \operatorname{cd}(\mathfrak{a}, N)\}$, while Theorem 2.2 implies $\max\{\operatorname{cd}(\mathfrak{a}, L), \operatorname{cd}(\mathfrak{a}, N)\} \leq \operatorname{cd}(\mathfrak{a}, M)$. Therefore (i) holds.

(ii) follows by Lemma 2.1(iv) and Theorem 2.2.

(iii) Clearly $\operatorname{Supp} M = \operatorname{Spec} R$, and so the result follows by Theorem 2.2. □

Remark 2.4. (i) One can deduce Lemma 2.1(ii) from Theorem 2.2 easily, because $H^i_{\mathfrak{a}}(\cdot)$ commutes with direct limits.

(ii) Part (ii) of Corollary 2.3 is proved in [9, Proposition 2.1] by using methods of algebraic geometry.

(iii) Let M and N be two finitely generated R-modules such that $M \neq \mathfrak{a}M$ and that $\operatorname{Supp}(N/\Gamma_{\mathfrak{a}}(N)) \subseteq \operatorname{Supp}(M/\Gamma_{\mathfrak{a}}(M))$. Then $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$.

(iv) Let M and N be two finitely generated R-modules. For each $i \in \mathbb{N}_0$,

$$\max\{\operatorname{cd}(\mathfrak{a}, \operatorname{Ext}^i_R(M, N)), \operatorname{cd}(\mathfrak{a}, \operatorname{Tor}^R_i(M, N))\} \leq \min\{\operatorname{cd}(\mathfrak{a}, M), \operatorname{cd}(\mathfrak{a}, N)\}.$$

(v) In view of Corollary 2.3(iii) results concerning cohomological dimension of R with respect to an ideal $\mathfrak{a}$ can be extended to $\operatorname{cd}(\mathfrak{a}, M)$ for any finitely generated faithful R-module M. See for example [4, Theorem 2 and Remark].

We shall use the following result in the proof of Theorem 2.7.

Lemma 2.5. *Let the situation be as in Lemma* 2.1,*and let $x \in R$. Then for an R-module M,*

$$\operatorname{cd}(\mathfrak{a} + Rx, M) \leq \operatorname{cd}(\mathfrak{a}, M) + 1.$$

Proof. Let $\mathfrak{b} = \mathfrak{a} + Rx$ and $\operatorname{cd}(\mathfrak{a}, M) = r$. By [3, Proposition 8.1.2], there is a long exact sequence

$$\dots \longrightarrow H^i_{\mathfrak{b}}(M) \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M_x) \longrightarrow H^{i+1}_{\mathfrak{b}}(M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M) \longrightarrow \dots$$

where M_x is the localization of M with respect to the multiplicatively closed subset $\{x^i : i \in \mathbb{N}_0\}$ of R. Since $H^i_{\mathfrak{a}}(M) = 0$ for all $i > r$, it turns out that $H^i_{\mathfrak{a}}(M_x) \cong H^{i+1}_{\mathfrak{b}}(M)$ for all $i > r$. Thus each element of $H^i_{\mathfrak{a}}(M_x)$ is annihilated by some power of $\mathfrak{b}$. By applying the functor $H^i_{\mathfrak{a}}(\cdot)$ on the isomorphism $M_x \xrightarrow{x^n} M_x$, $n \in \mathbb{N}$, we deduce that $H^i_{\mathfrak{a}}(M_x) \xrightarrow{x^n} H^i_{\mathfrak{a}}(M_x)$ is an isomorphism. But each element of $H^i_{\mathfrak{a}}(M_x)$ is annihilated by x^n for some $n \in \mathbb{N}$. This yields that $H^i_{\mathfrak{a}}(M_x) = 0$ for all $i > r$. Therefore $H^i_{\mathfrak{b}}(M) = 0$ for all $i > r + 1$, as required. □

We recall some properties of the notions $c(N)$ and $\operatorname{s\,dim} N$ in the following lemma (see [3, Ch. 19]).

Lemma 2.6. *Let N be a finitely generated R-module. Then the following hold:*

(i) $\operatorname{s\,dim} N = \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in \operatorname{Min}_R N\}$,
(ii) $c(N) = \min\{\dim(R/(\bigcap_{\mathfrak{p}\in A} \mathfrak{p} + \bigcap_{\mathfrak{p}\in B} \mathfrak{p})) : A$ *and* B *are non-empty subsets of* $\operatorname{Min}_R N$ *such that* $A \cup B = \operatorname{Min}_R N\}$,
(iii) $c(N) \leq \operatorname{s\,dim} N$, *and*
(iv) *if* $(R, \mathfrak{m})$ *is local, then* $c(\operatorname{Supp} N \backslash \{\mathfrak{m}\}) = c(N) - 1$.

Theorem 2.7. *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and N a finitely generated R-module such that* $\min\{\dim N/\mathfrak{a}N, \dim N/\mathfrak{b}N\} > \dim N/(\mathfrak{a} + \mathfrak{b})N$.

(i) *If* $\operatorname{Min}_{\hat{R}} \hat{N}$ *consists of a single prime* $\mathfrak{p}$*, then*

$$\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, N) \geq \dim N - \dim N/(\mathfrak{a} + \mathfrak{b})N - 1.$$

(ii) *If R is complete, then*

$$\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, N) \geq \min\{c(N), \operatorname{s\,dim} N - 1\} - \dim N/(\mathfrak{a} + \mathfrak{b})N.$$

Proof. Let $R_1 = R/\operatorname{Ann}_R N$. Then $\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, N) = \operatorname{cd}((\mathfrak{a} \cap \mathfrak{b})R_1, R_1)$, by Lemma 2.1(iv) and Theorem 2.2. On the other hand one can easily check that $\operatorname{s\,dim} N = \operatorname{s\,dim} R_1$ and that $c(N) = c(R_1)$. Therefore we may and do assume that $N = R$. Now, by replacing $\operatorname{ara}(\mathfrak{a} \cap \mathfrak{b})$ by $\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, R)$ and using Lemma 2.5, we can process similar to the proof of [3, Proposition 19.2.7] to deduce (i). Also, in view of Lemma 2.1(i) and 2.1(iv), one can deduce (ii) by similar argument as in [3, Lemma 19.2.8]. □

Now, we are ready to state the next main theorem of this section, namely the connectedness bound for a finitely generated module over a complete local ring which is a generalization and refinement of Grothendieck's connectedness theorem (see [8, Expose XIII, Théorém 2.1]).

Theorem 2.8. *Let $\mathfrak{a}$ be a proper ideal of a complete local ring $(R, \mathfrak{m})$, and let N be a finitely generated R-module. Then*

$$\operatorname{cd}(\mathfrak{a}, N) \geq \min\{c(N), \operatorname{s\,dim} N - 1\} - c(N/\mathfrak{a}N).$$

Proof. Let $\operatorname{Min}_R(N/\mathfrak{a}N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $c := c(N/\mathfrak{a}N)$. If $n = 1$, we have $c = \dim R/\mathfrak{p}_1$ (see Lemma 2.6(ii)). Let $\mathfrak{p} \in \operatorname{Min}_R N$ be such that $\mathfrak{p} \subseteq \mathfrak{p}_1$. Then as $\operatorname{Supp} R/\mathfrak{p} \subseteq \operatorname{Supp} N$, by virtue of Lemmas 2.1(i), 2.1(iv) and Theorem 2.2,

$$\operatorname{ht} \mathfrak{p}_1/\mathfrak{p} \leq \operatorname{cd}(\mathfrak{p}_1/\mathfrak{p}, R/\mathfrak{p}) = \operatorname{cd}(\mathfrak{p}_1, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{p}_1, N).$$

Because $\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R N) = \mathfrak{p}_1$, it turns out that $\operatorname{cd}(\mathfrak{p}_1, N) = \operatorname{cd}(\mathfrak{a}, N)$.

Next, since $R/\mathfrak{p}$ is catenary, we deduce that $c = \dim R/\mathfrak{p} - \operatorname{ht} \mathfrak{p}_1/\mathfrak{p} \geq \operatorname{s\,dim} N - \operatorname{cd}(\mathfrak{a}, N)$, as desired. Accordingly, we may assume that $n > 1$. Then there exist two non-empty subsets A, B of $\operatorname{Min}_R N/\mathfrak{a}N$ for which $A \cup B = \operatorname{Min}_R N/\mathfrak{a}N$, and

$$c = \dim(R/(\bigcap_{\mathfrak{p}\in A} \mathfrak{p}) + (\bigcap_{\mathfrak{p}\in B} \mathfrak{p})).$$

Moreover, we may assume that $A \cap B = \emptyset$. Put $\mathfrak{b} := \bigcap_{\mathfrak{p}\in A} \mathfrak{p}$ and $\mathfrak{c} := \bigcap_{\mathfrak{p}\in B} \mathfrak{p}$. Then $\dim N/\mathfrak{b}N > c$, $\dim N/\mathfrak{c}N > c$ and $\mathfrak{b} \cap \mathfrak{c} = \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R N)$. Therefore the proof finishes by Theorem 2.7(ii). □

Corollary 2.9. *Let the situation be as in Theorem* 2.8. *Then* $\operatorname{cd}(\mathfrak{a}, N) \geq c(N) - c(N/\mathfrak{a}N) - 1$. *Moreover if* $|\operatorname{Min}_R N| > 1$, *then the inequality is strict.*

Proof. The assertion is clear by Theorem 2.8, because, by Lemma 2.6(iii), $c(N) \leq \operatorname{s}\dim N$, with strict inequality if $|\operatorname{Min}_R N| > 1$. □

3. Connectedness theorem

In [10], M. Hochster and C. Huneke have extended Faltings' original connectedness theorem [6] as follows. Let $(R, \mathfrak{m})$ be an equidimensional complete local ring of dimension d, and $\mathfrak{a}$ a proper ideal of R. If $H^d_{\mathfrak{m}}(R)$ is indecomposable, then the punctured spectrum of $R/\mathfrak{a}$ is connected provided $\operatorname{cd}(\mathfrak{a}, R) \leq d-2$. Next this result is generalized to finitely generated modules in [5]. In this section, our objective is to remove the indecomposability assumption. To this end, we give a refinement of Theorem 2.8 in Theorem 3.4. Before we do this, we bring some definitions and lemmas.

Definition. Let $(R, \mathfrak{m})$ be a d-dimensional local ring. A finitely generated R-module K is called the *canonical module* of R, if $K \otimes_R \hat{R} \cong \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E(R/\mathfrak{m}))$.

Proposition 3.1. *Let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *be prime ideals of a finite dimensional Noetherian ring* R *such that* $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ *for all* $1 \leq i \neq j \leq n$. *Suppose that* R *is* (S_2) *and that* $R_{\mathfrak{p}}$ *possesses a canonical module for all* $\mathfrak{p} \in \operatorname{Spec} R$. *Also, assume that for each prime ideal* $\mathfrak{p}$ *of* R, $\dim R = \dim R/\mathfrak{p} + \operatorname{ht}\mathfrak{p}$. *Set* $\mathfrak{a} := \bigcap_{i=1}^m \mathfrak{p}_i$ *and* $\mathfrak{b} = \bigcap_{i=m+1}^n \mathfrak{p}_i$ *for some* $1 \leq m < n$. *Then*

$$\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, R) \geq \dim R - \dim R/(\mathfrak{a} + \mathfrak{b}) - 1.$$

Proof. Let $\mathfrak{q}$ be a prime ideal of R containing $\mathfrak{a} + \mathfrak{b}$ such that $\dim R/(\mathfrak{a} + \mathfrak{b}) = \dim R/\mathfrak{q}$. Our assumption on $\mathfrak{p}_i$'s implies that the ideals $\mathfrak{a}R_{\mathfrak{q}}$ and $\mathfrak{b}R_{\mathfrak{q}}$ are not $\mathfrak{q}R_{\mathfrak{q}}-$primary. Now the claim follows immediately from Lemma 2.1(iv) and the following lemma. □

Lemma 3.2. *Let* $(R, \mathfrak{m})$ *be a* (S_2) *local ring which possesses a canonical module. Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two non-*$\mathfrak{m}$*-primary ideals of* R *such that* $\mathfrak{a} + \mathfrak{b}$ *is* $\mathfrak{m}$*-primary. Then*

$$\operatorname{cd}(\mathfrak{a} \cap \mathfrak{b}, R) \geq \dim R - 1.$$

Proof. Assume that the contrary is true. Then the Mayer-Vietoris sequence (see e.g. [3, 3.2.3]) yields the isomorphism

$$H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{a}+\mathfrak{b}}(R) \cong H^d_{\mathfrak{a}}(R) \oplus H^d_{\mathfrak{b}}(R).$$

The module $H^d_{\mathfrak{m}}(R)$ is indecomposable by [2, Remark 1.4] and so either $H^d_{\mathfrak{a}}(R) = 0$ or $H^d_{\mathfrak{b}}(R) = 0$. Suppose $H^d_{\mathfrak{b}}(R) = 0$; then $H^d_{\mathfrak{m}}(R) \cong H^d_{\mathfrak{a}}(R)$. It follows from [2, Proposition 1.2 and Lemma 1.1] that $\operatorname{Assh}\hat{R} = \operatorname{Ass}\hat{R}$. By virtue of [3, Ex. 8.2.6], once applied to $\mathfrak{m}$ and a second time applied to $\mathfrak{a}$, it follows that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Ass}\hat{R}$. This leads that $\mathfrak{a}$ is $\mathfrak{m}$-primary, which is a contradiction. □

Lemma 3.3. *Let R be a Noetherian ring such that R is (S_2) and that $R_{\mathfrak{p}}$ has a canonical module for all $\mathfrak{p} \in \operatorname{Spec} R$. Assume that dim R is finite and that for each $\mathfrak{p} \in \operatorname{Spec} R$, $\dim R = \dim R/\mathfrak{p} + \operatorname{ht} \mathfrak{p}$. Then for each proper ideal $\mathfrak{a}$ of R,*

$$\operatorname{cd}(\mathfrak{a}, R) \geq \dim R - c(R/\mathfrak{a}) - 1.$$

Proof. Without loss of generality we can and do assume that $\mathfrak{a} = \operatorname{Rad}(\mathfrak{a})$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the distinct minimal primes of $\mathfrak{a}$, and let $c := c(R/\mathfrak{a})$. If $n = 1$, we have $\mathfrak{a} = \mathfrak{p}_1$ and $c = \dim R/\mathfrak{p}_1$. Hence

$$c = \dim R - \operatorname{ht} \mathfrak{p}_1 \geq \dim R - \operatorname{cd}(\mathfrak{p}_1, R).$$

Consider now the case where $n > 1$. By Lemma 2.6(ii), there exist two disjoint non-empty subsets A, B of $\{1, \ldots, n\}$ for which $A \cup B = \{1, \ldots, n\}$ and $c = \dim(R/(\bigcap_{i \in A} \mathfrak{p}_i) + (\bigcap_{j \in B} \mathfrak{p}_j))$. Set $\mathfrak{b} = \bigcap_{i \in A} \mathfrak{p}_i$ and $\mathfrak{c} = \bigcap_{j \in B} \mathfrak{p}_j$. Then $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $1 \leq i, j \leq n$, and $\mathfrak{b} \cap \mathfrak{c} = \mathfrak{a}$. We can now use Proposition 3.1 to complete the proof. □

Theorem 3.4. *Let $\mathfrak{a}$ be a proper ideal of a local ring $(R, \mathfrak{m})$ and let N be a finitely generated R-module such that $\operatorname{Min}_{\hat{R}} \hat{N} = \operatorname{Assh}_{\hat{R}} \hat{N}$. Then*

$$\operatorname{cd}(\mathfrak{a}, N) \geq \dim N - c(N/\mathfrak{a}N) - 1.$$

Proof. Let $R_1 = R/\operatorname{Ann}_R N$. Then $c(N/\mathfrak{a}N) = c(R_1/\mathfrak{a}R_1)$ and $\operatorname{cd}(\mathfrak{a}, N) = \operatorname{cd}(\mathfrak{a}R_1, R_1)$ by Lemma 2.1(iv) and Theorem 2.2. On the other hand $\operatorname{Min} \hat{R}_1 = \operatorname{Assh} \hat{R}_1$. Thus it is sufficient to prove the claim for the ring R itself. Since $c(R/\mathfrak{a}) \geq c(\hat{R}/\mathfrak{a}\hat{R})$ by [3, Lemma 19.3.1], we can assume that R is complete. Since $\operatorname{s dim} R = \dim R$, in view of Theorem 2.8 it is enough to show that $c(R) \geq \dim R - 1$. Let $J = \bigcap \mathfrak{q}$, where $\mathfrak{q}$ runs through all the primary components of the zero ideal of R such that $\dim R/\mathfrak{q} = \dim R$. It is clear that $\dim R/J = \dim R$. Also, since $\operatorname{Min} R = \operatorname{Assh} R$, it follows from Lemma 2.6(ii) that $c(R/J) = c(R)$. Thus by replacing R with R/J, we may assume that $\operatorname{Assh} R = \operatorname{Ass} R$. By [1, 1.11 and Theorem 3.2], there exists a commutative Noetherian semi-local ring S and a monomorphism $\varphi : R \longrightarrow S$ such that:

(i) S is finitely generated as an R-module,
(ii) S is (S_2),
(iii) $S_{\mathfrak{p}}$ has a canonical module for all $\mathfrak{p} \in \operatorname{Spec} S$, and
(iv) every maximal chain of prime ideals in S is of length $\dim S$.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the distinct minimal prime ideals of R. Then there exist two non-empty subsets A, B of $\{1, \ldots, n\}$ for which $A \cup B = \{1, \ldots, n\}$ and

$$c(R) = \dim(R/(\bigcap_{i \in A} \mathfrak{p}_i) + (\bigcap_{j \in B} \mathfrak{p}_j)).$$

Since by condition (i), S is integral over R, it follows that $\dim R = \dim S$ and that for each $1 \leq i \leq n$ there exists $\mathfrak{q}_i \in \operatorname{Spec} S$ such that $\varphi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$. For a given prime ideal $\mathfrak{q}$ of S, we show that $\mathfrak{q} \in \operatorname{Min} S$ if and only if $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) \in \operatorname{Min} R$. To this end, first note that the ring $S/\mathfrak{q}$ is integral over the ring $R/\mathfrak{p}$, and so $\dim S/\mathfrak{q} = \dim R/\mathfrak{p}$. Since $\operatorname{Ass} R = \operatorname{Assh} R$, it turns out that $\mathfrak{p} \in \operatorname{Min} R$ if and only if $\dim R/\mathfrak{p} = \dim R$. On the other hand (iv) implies that $\mathfrak{q} \in \operatorname{Min} S$ if and only if

$\dim S/\mathfrak{q} = \dim S$. Therefore the claim is immediate. Put

$$A' = \{\mathfrak{q} \in \mathrm{Min} S : \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}_i \text{ for some } i \in A\}$$

and $B' = \{\mathfrak{q} \in \mathrm{Min} S : \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}_j \text{ for some } j \in B\}$. So, we have

$$\begin{aligned} c(R) &\geq \dim R/\varphi^{-1}((\bigcap_{\mathfrak{q} \in A'} \mathfrak{q}) + (\bigcap_{\mathfrak{q} \in B'} \mathfrak{q})) \\ &= \dim S/(\bigcap_{\mathfrak{q} \in A'} \mathfrak{q} + \bigcap_{\mathfrak{q} \in B'} \mathfrak{q}) \geq c(S). \end{aligned}$$

Therefore the result follows by Lemma 3.3. Note that by (iv), for each prime ideal $\mathfrak{p}$ of S, $\dim S = \dim S/\mathfrak{p} + \mathrm{ht}\, \mathfrak{p}$. □

Now we are prepared to present the main result of this section which is a generalization of [10, Theorem 3.3] and of [5, Corollary 4.2 and Theorem 4.3].

Corollary 3.5. *Let $\mathfrak{a}$ be a proper ideal of a local ring $(R, \mathfrak{m})$. Let N be a d-dimensional finitely generated R-module such that* $\mathrm{Assh}_{\hat{R}} \hat{N} = \mathit{Min}_{\hat{R}} \hat{N}$. *Then* $\mathrm{Supp}\, N/\mathfrak{a}N \backslash \{\mathfrak{m}\}$ *is connected provided* $\mathrm{cd}(\mathfrak{a}, N) \leq d - 2$.

Proof. By Lemma 2.6(iv), $c(\mathrm{Supp}(N/\mathfrak{a}N)\backslash\{\mathfrak{m}\}) = c(N/\mathfrak{a}N) - 1$. Hence by Theorem 3.4, $c(\mathrm{Supp}(N/\mathfrak{a}N)\backslash\{\mathfrak{m}\}) \geq \dim N - \mathrm{cd}(\mathfrak{a}, N) - 2$. Thus

$$c(\mathrm{Supp}(N/\mathfrak{a}N)\backslash\{\mathfrak{m}\}) \geq 0,$$

and so $\mathrm{Supp}(N/\mathfrak{a}N)\backslash\{\mathfrak{m}\}$ is connected, as desired. □

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Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Az-Zahra University, Tehran, Iran

E-mail address: kdivaani@ipm.ir

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, University of Tabriz, Tabriz, Iran

E-mail address: naghipour@tabrizu.ac.ir

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Shahid Beheshti University, Tehran, Iran

E-mail address: mtousi@vax.ipm.ac.ir