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CONVEX-COCOMPACTNESS OF KLEINIAN GROUPS AND CONFORMALLY FLAT MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract. We give a sufficient condition for a higher dimensional Kleinian group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ to be convex cocompact in terms of the critical exponent of Γ. As a consequence, we see that the fundamental group of a compact conformally flat manifold with positive scalar curvature is hyperbolic in the sense of Gromov. We give some other applications to geometry and topology of conformally flat manifolds with positive scalar curvature.

1. Introduction

Throughout this paper, a Kleinian group means an infinite discrete subgroup of the isometry group $\mathrm{Isom}(\mathbb{H}^{n+1})$ of the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$, $n \geq 2$. As is well-known, the action of $\mathrm{Isom}(\mathbb{H}^{n+1})$ extends to the boundary $S^n = \partial\mathbb{H}^{n+1}$. We denote by $\Omega(\Gamma)$ the maximal open subset of S^n on which Γ acts properly discontinuously, and call the set the *domain of discontinuity* of Γ. The quotient of $\Omega(\Gamma) \cup \mathbb{H}^{n+1}$ is a manifold or orbifold with boundary $\Omega(\Gamma)/\Gamma$. We say Γ is *convex cocompact* if $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is compact. Note that when $\Omega(\Gamma)/\Gamma$ is connected and compact, $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is compact if and only if $\mathbb{H}^{n+1}/\Gamma$ has only one end that is bounded by $\Omega(\Gamma)/\Gamma$.

The first result of this paper, Theorem 1 below, says that if the action of Γ is small in a certain sense, then Γ is convex cocompact. The smallness of the action is expressed in terms of the critical exponent. See the next section for the definitions of the critical exponent and the geometrical finiteness.

Theorem 1. *Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$, $n \geq 3$, be a Kleinian group. Denote its critical exponent by $\delta(\Gamma)$. Suppose that $\Omega(\Gamma)/\Gamma$ has a compact connected component.*

(1) *If $\delta(\Gamma) < n-1$, then $\Omega(\Gamma)$ is connected and Γ is geometrically finite.*

(2) *If $\delta(\Gamma) < n/2$, then $\Omega(\Gamma)$ is connected, and $\mathbb{H}^{n+1}/\Gamma$ has only one end that is bounded by $\Omega(\Gamma)/\Gamma$. In particular, Γ is convex cocompact.*

If Γ is convex cocompact, then $\Omega(\Gamma)/\Gamma$ is compact by definition. On the other hand, the converse is false without any additional assumption on Γ. For example,

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take a parabolic group Γ of rank n in $\mathrm{Isom}(\mathbb{H}^{n+1})$, that is, a Kleinian group isomorphic to $\mathbb{Z}^n$. Then $\Omega(\Gamma) \cong \mathbb{R}^n$ and $\Omega(\Gamma)/\Gamma$ is diffeomorphic to an n-torus T^n. Hence $\Omega(\Gamma)/\Gamma$ is compact. However, $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ has a cusp and is diffeomorphic to $T^n \times [0, \infty)$. Such a Γ has $\delta(\Gamma) = n/2$. Although we have excluded the case $n = 2$, there is a deep result in this case due to Bishop and Jones ([2]): If $n = 2$, Γ is analytically finite, and the Hausdorff dimension of the limit set is less than 2, then Γ is geometrically finite. Note that there is no appropriate notion corresponding to the analytical finiteness in higher dimension. We require $\Omega(\Gamma)/\Gamma$ to have a compact connected component instead. The proof of Theorem 1 (1) depends on the methods developed in [2] and [16], and (2) is a corollary to (1).

Theorem 1 above, a sort of extension of the theorem of Bishop and Jones to higher dimensional case, is motivated by its application to conformally flat manifolds. Recall that a manifold M with conformal structure C (or a Riemannian manifold (M, g)) is called *conformally flat* if it is locally conformally equivalent to a domain of S^n with the standard conformal structure (equivalently, to a domain of the Euclidean n-space). We say C is a *flat conformal structure* if (M, C) is conformally flat. Note that the extended action of $\mathrm{Isom}(\mathbb{H}^{n+1})$ gives rise to the conformal action on S^n with the standard conformal structure. Therefore the action of any Kleinian group Γ preserves the conformal structure on $\Omega(\Gamma)$ defined by restricting the standard conformal structure on S^n. Hence if the quotient $\Omega(\Gamma)/\Gamma$ is a manifold, then $\Omega(\Gamma)/\Gamma$ inherits the flat conformal structure on $\Omega(\Gamma)$. In this manner, Kleinian groups provide a special class of conformally flat manifolds. On the other hand, a theorem of Schoen and Yau tells us the following: If a flat conformal structure C on a compact, connected n-manifold M $(n \geq 4)$ contains a metric with positive scalar curvature, then (M, C) is conformal to $\Omega(\Gamma)/\Gamma$ for some Kleinian group Γ. Moreover, Γ is isomorphic to the fundamental group of M and $\delta(\Gamma) \leq (n-2)/2$. (See Theorem 4 for the precise statement.) Note that our Theorem 1 says such a Γ must be convex cocompact. This should be a great help for the study of conformally flat manifolds with positive scalar curvature. For example, Theorem 2 below, which is a consequence of Theorem 1 and [8, Theorem 2], says that the theorem of Schoen and Yau above is true not only for C but for every flat conformal structure lying in the connected component of the moduli space that contains C.

Theorem 2. *Let (M, C) be a compact, connected conformally flat manifold not covered by a torus. Suppose (M, C) satisfies either*

(1) $\dim M \geq 4$ *and the scalar curvature of g is positive for some $g \in C$, or*

(2) $\dim M \geq 7$ *and the scalar curvature of g is nonnegative for some $g \in C$.*

Let $\mathcal{M}_0$ be the connected component of the moduli space containing the element represented by C above. Then, for any $\mathcal{C} \in \mathcal{M}_0$, each representative $C' \in \mathcal{C}$ is a flat conformal structure with an injective developing map, and the image of $\pi_1(M)$ under the associated holonomy representation is convex cocompact Kleinian groups. In particular, (M, C') is conformal to $\Omega(\Gamma)/\Gamma$ for some convex cocompact Kleinian group Γ.

If M is covered by a torus, then it is well-known that Theorem 2 is true for any flat conformal structures on M except the convex cocompactness of the image of the holonomy representation. Note that C' in Theorem 2 need not contain a metric with nonnegative scalar curvature. See [13, pp. 123–124]. We can say more if we deal with a certain branched covering of $\mathcal{M}_0$. See Theorem 11 in §4. These results suggest that for a manifold M as above the space of flat conformal structures can be studied from the viewpoint of the deformation theory of Kleinian groups. See [1, Chapter 7] and [6] for some results relevant to this subject. As another consequence of Theorem 1, we see that the fundamental group of M as in Theorem 2 is hyperbolic despite the fact that the scalar curvature is nonnegative.

Theorem 3. *Let (M, C) be a compact, connected conformally flat manifold which is not covered by a torus. Suppose (M, C) satisfies either* (1) *or* (2) *in Theorem 2. Then the fundamental group of M is hyperbolic in the sense of Gromov.*

We also have a vanishing for cohomology groups for such a manifold (§4, Corollary 10).

Here we would like to point out that in order to prove Theorems 2 and 3, it suffices to show (2) in Theorem 1 for the case $\delta(\Gamma) \leq (n-2)/2$. We state this case independently as Proposition 7, and give a proof by means of geometry of $\Omega(\Gamma)/\Gamma$. The idea is as follows: We interpret the convex-cocompactness as the vanishing of the higher $\hat{A}$-genus of $\Omega(\Gamma)/\Gamma$, and then apply a theorem of J. Rosenberg (Theorem 8) which says that, for a manifold satisfying certain conditions, the higher $\hat{A}$-genus is an obstruction to the existence of a metric with positive scalar curvature. Although the proof involves the deep result by Rosenberg and is not as intuitive as that in [2], the author believes it deserves to be recorded in this paper.

To prove Proposition 7 and Theorems 2 and 3, one needs to know the relation between $\delta(\Gamma)$ and the scalar curvature of $\Omega(\Gamma)/\Gamma$. We will explain the relation and give a quick review of the background in §2. In §3, we will state and prove Proposition 7. Theorem 1 and some applications of it (or Proposition 7), including two theorems above, will be shown in §4.

2. Critical exponent and scalar curvature of $\Omega(\Gamma)/\Gamma$

As we have already mentioned in §1, the action of $\mathrm{Isom}(\mathbb{H}^{n+1})$ on $\mathbb{H}^{n+1}$ extends to the boundary S^n, and the action gives rise to the conformal action on S^n with the standard conformal structure. Through this extension, we can identify $\mathrm{Isom}(\mathbb{H}^{n+1})$ with the conformal transformation group $\mathrm{Conf}(S^n)$ of S^n. Thus, given a Kleinian group Γ, we have the action of Γ on the union $\mathbb{H}^{n+1} \cup S^n$. The *limit set* $\Lambda(\Gamma)$ of Γ is defined to be the set of accumulation points of an orbit Γx, $x \in \mathbb{H}^{n+1}$. Since $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ is discrete, $\Lambda(\Gamma)$ must lie in the boundary S^n, and hence, $\Lambda(\Gamma)$ is independent of the choice of x. Note that $\Lambda(\Gamma) \neq \emptyset$ unless Γ is a finite group. As we deal only with infinite Kleinian groups in this paper (see the beginning of §1), we may assume $\Lambda(\Gamma) \neq \emptyset$. When $\Lambda(\Gamma)$ does not consist of exactly two points, $\Lambda(\Gamma)$ can be characterized as the minimal invariant closed subset with respect to the action of Γ on $\mathbb{H}^{n+1} \cup S^n$. Then the domain of discontinuity $\Omega(\Gamma)$ becomes the complement of $\Lambda(\Gamma)$ in S^n. Since the action of Γ on $\Omega(\Gamma)$ is properly discontinuous, the quotient of $\Omega(\Gamma) \cup \mathbb{H}^{n+1}$ by Γ is an orbifold with boundary. We say Γ is *convex cocompact* if the quotient $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is compact. There is a characterization of the convex-cocompactness in terms of the convex hull of $\Lambda(\Gamma)$. Let $C(\Gamma)$ be the hyperbolic convex hull of $\Lambda(\Gamma)$, that is, the minimal convex subset of $\mathbb{H}^{n+1}$ whose closure in $S^n \cup \mathbb{H}^{n+1}$ contains $\Lambda(\Gamma)$. Then Γ is convex cocompact if and only if $C(\Gamma)/\Gamma$ is compact. (This is the ordinary definiton. See [17] for the equivalence of these two definitions.) If Γ has a bound of the order of its finite subgroups and the ε-neighborhood of $C(\Gamma)/\Gamma$ in $\mathbb{H}^{n+1}/\Gamma$ has finite volume for some $\varepsilon > 0$, then Γ is called *geometrically finite*. A convex cocompact Γ is clearly finitely generated.

Therefore Γ has a torsion-free subgroup of finite index by Selberg's lemma, and in particular, has a bound of the order of its finite index subgroups. Moreover, since $C(\Gamma)/\Gamma$ is compact, the ε-neighborhood of $C(\Gamma)/\Gamma$ is also compact and has finite volume for any $\varepsilon > 0$. Thus convex cocompact groups are geometrically finite. On the other hand, convex cocompact Kleinian groups are characterized as geometrically finite groups without parabolic elements. See [4] and [1] for more about geometrically finite groups.

As we have mentioned in the preceding paragraph, the limit set is characterized as the minimal invariant closed subset with respect to the action of Γ. Therefore, roughly speaking, the size of the limit set measures the size of the action of the group. Another quantity expressing the size of the action of Γ is the *critical exponent* that is defined by

$$\delta(\Gamma) = \inf\{s > 0 \mid \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} < \infty\},$$

where $x, y \in \mathbb{H}^{n+1}$, and $d(\cdot,\cdot)$ is the hyperbolic distance on $\mathbb{H}^{n+1}$. By using the triangle inequality, it is easy to see that $\delta(\Gamma)$ does not depend on the choice of $x, y \in \mathbb{H}^{n+1}$. As one expects, there is a close relation between $\delta(\Gamma)$ and the size (the Hausdorff dimension) of $\Lambda(\Gamma)$, and the relation can be described with the help of so-called Patterson-Sullivan measures on $\Lambda(\Gamma)$. We refer the reader to [14] as a good exposition on this subject. See also [2].

In case Γ is torsion-free, then the action of Γ becomes free, and hence the quotient $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is a manifold with boundary $\Omega(\Gamma)/\Gamma$. Recall that $\Omega(\Gamma)/\Gamma$ has a natural flat conformal structure as we have explained in §1. From the viewpoint of conformal geometry of $\Omega(\Gamma)/\Gamma$, the size of the limit set is closely related to the curvature of $\Omega(\Gamma)/\Gamma$. This was first observed by Schoen and Yau in [16]. In [13] Nayatani constructed a good Γ-invariant metric on $\Omega(\Gamma)$ which is compatible with the natural conformal structure on $\Omega(\Gamma)$, and improved a result in [16]. We call the metric constructed by Nayatani in [13] a Nayatani metric, and denote the metric by g_N. In the construction of the metric, the Patterson-Sullivan measure plays an important role. This suggests that this metric enables us to study Kleinian groups by means of differential geometry of $\Omega(\Gamma)/\Gamma$. (See [7] for some results obtained in this spirit.) We only collect some results which we will use later, and omit the definition of the Nayatani metric and further explanation on results in [16] and [13].

Theorem 4 ([16, Proposition 3.3])**.** *Let (M, C) be a compact, connected conformally flat manifold satisfying either*

(1) $n = \dim M \geq 4$ *and the scalar curvature of g is positive for some $g \in C$, or*

(2) $n = \dim M \geq 7$ *and the scalar curvature of g is nonnegative for some $g \in C$.*

Then there exists a Kleinian group Γ such that (M, g) is conformally equivalent to $\Omega(\Gamma)/\Gamma$. Moreover, if M is not covered by a torus, we have $\delta(\Gamma) = d(M, C) \leq (n-2)/2$, where $d(M, C)$ is the Schoen-Yau invariant of (M, C).

The Schoen-Yau invariant $d(M, C)$ is defined in terms of the Green's function of the conformal Laplacian on the holonomy covering of (M, g) for any $g \in C$. And in [16] Schoen and Yau proved that if $d(M, C) < (n-2)^2/n$, then the developing map of (M, C) is injective. What we have presented above is one of the consequences of this result. See [16, p. 54] and [7, Definition 3.3] for the precise definition of the Schoen-Yau invariant.

Remark 1. In [16], they claim that Theorem 4 holds under a much weaker assumption, instead of (1) and (2), that $\dim M \geq 3$ and the scalar curvature of g is nonnegative. However, as far as the author knows, no complete proof is available yet.

Theorem 5 ([13, Theorem 3.3])**.** *Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$, $n \geq 3$, be a torsion-free Kleinian group with $\Omega(\Gamma)/\Gamma$ compact. Assume that $\Omega(\Gamma)/\Gamma$ is not covered by a torus. Then the scalar curvature of g_N is positive* (*resp. zero, resp. negative*) *if and only if $\delta(\Gamma) < (n-2)/2$* (*resp.* $= (n-2)/2$, *resp.* $> (n-2)/2$).

We have the following consequence of Theorem 5.

Lemma 6. *Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$, $n \geq 3$, be a torsion-free Kleinian group with $\delta(\Gamma) \leq (n-2)/2$. Suppose that $\Omega(\Gamma)/\Gamma$ is compact. Then $\Omega(\Gamma)/\Gamma$ admits a* (*not necessarily conformally flat*) *metric with positive scalar curvature.*

Proof. If $\delta(\Gamma) < (n-2)/2$, the Nayatani metric g_N for Γ has positive scalar curvature by Theorem 5. Suppose $\delta(\Gamma) = (n-2)/2$. Then the scalar curvature of g_N is identically zero. Assume $\Omega(\Gamma)/\Gamma$ does not carry a metric with positive scalar curvature. Then, by a result in [3] (see also [11, Theorem 1]), the Ricci curvature of g_N is zero. When $n = 3$, this means g_N is flat. When $n \geq 4$, because of the conformal flatness, g_N is again flat (see [12]). Therefore $\Omega(\Gamma)/\Gamma$ must be covered by a flat torus. In other words, Γ contains a subgroup Γ_0 of finite index which is isomorphic to $\mathbb{Z}^n$. Such a Γ_0 is called a parabolic subgroup of rank n, and has $\delta(\Gamma_0) = n/2$, as we have mentioned in §1. By definition, $\delta(\Gamma) \geq \delta(\Gamma_0) = n/2$. This contradicts our assumption $\delta(\Gamma) \leq (n-2)/2$. ☐

3. Convex-cocompactness via geometry of $\Omega(\Gamma)/\Gamma$

This section is devoted to the proof of Proposition 7 below.

Proposition 7. *Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$, $n \geq 3$, be a Kleinian group. Denote its critical exponent by $\delta(\Gamma)$. Suppose that $\Omega(\Gamma)/\Gamma$ has a compact connected component and that $\delta(\Gamma) \leq (n-2)/2$. Then $\Omega(\Gamma)$ is connected, and $\mathbb{H}^{n+1}/\Gamma$ has only one end that is bounded by $\Omega(\Gamma)/\Gamma$. In particular, Γ is convex cocompact.*

Proof. By our assumption, $\Omega(\Gamma)/\Gamma$ is a (possibly disconnected) orbifold and has a compact component M_0. Let Ω_0 be a connected component of $\Omega(\Gamma)$ which covers M_0. Denote by $\Gamma_0 \subset \Gamma$ the component subgroup of Ω_0, that is, the stabilizer of Ω_0. Clearly, Ω_0/Γ_0 coincides with M_0. Since M_0 is compact, Γ_0 is finitely generated and contains a torsion-free subgroup $\hat{\Gamma}_0$ of finite index by Selberg's lemma. Note that $\delta(\hat{\Gamma}_0) \leq \delta(\Gamma) \leq (n-2)/2 < n-1$ and that $\hat{M}_0 = \Omega_0/\hat{\Gamma}_0$ is a compact manifold. Applying [7, Proposition 3.2] to $\hat{M}_0 = \Omega_0/\hat{\Gamma}_0$, we see that the Hausdorff dimension of $S^n \setminus \Omega(\hat{\Gamma}_0)$ is not greater than $\delta(\hat{\Gamma}_0) < n-1$ and that $\Omega_0 = \Omega(\hat{\Gamma}_0)$. In particular, the complement of $\Omega(\hat{\Gamma}_0) = \Omega_0$ has no interior points. Hence $\Omega_0 = \Omega(\Gamma)$ and $\Omega(\Gamma)$ is connected. Since Γ_0 is the stabilizer of Ω_0, $\Gamma_0 = \Gamma$. Thus $M_0 = \Omega(\Gamma)/\Gamma$. Since $\Gamma = \Gamma_0$ is finitely generated, again by Selberg's lemma, Γ contains a torsion-free subgroup of finite index (e.g., $\hat{\Gamma}_0$). Note that it is obvious by definition that Γ is convex cocompact if and only if some (or any) finite index subgroup of Γ is convex cocompact. Thus, in what follows, we may assume Γ is torsion-free and, furthermore, contained in the identity component of $\mathrm{Isom}(\mathbb{H}^{n+1})$.

Next we show that the higher $\hat{A}$-genus of $\Omega(\Gamma)/\Gamma$ vanishes. Recall that the $\hat{A}$-class of a manifold M is defined as

$$\hat{A}(M) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots ,$$

where 1 is the generator of $H^0(M;\mathbb{Z})$ and $p_i \in H^{4i}(M;\mathbb{Q})$ is the ith Pontrjagin class. Denote the classifying space of $\pi_1(M)$ by $B(\pi_1(M))$. The higher $\hat{A}$-class of a manifold M with respect to $u \in H^*(B(\pi_1(M));\mathbb{Q})$ is defined by

$$\hat{A}_u(M) = \hat{A}(M) \cup f^*u,$$

where $f : M \longrightarrow B(\pi_1(M))$ is a classifying map. The number obtained by evaluating $\hat{A}_u(M)$ on the fundamental class $[M]$, which we write as $\langle \hat{A}_u(M), [M]\rangle$, is called the higher $\hat{A}$-genus of M with respect to u.

The higher $\hat{A}$-genus is known to be an obstruction to the existence of a metric with positive scalar curvature as the following theorem due to J. Rosenberg asserts.

Theorem 8 ([15, Theorem 3.5])**.** *Let M be a compact, connected, and oriented manifold satisfying the following three conditions:*

(1) *The universal covering $\tilde{M}$ of M has a spin structure.*
(2) *The strong Novikov conjecture holds for $\pi_1(M)$.*
(3) *M admits a metric with positive scalar curvature.*

Then the higher $\hat{A}$-genus of M vanishes for any $u \in H^*(B(\pi_1(M));\mathbb{Q})$.

Let us check that these three conditions hold for $\Omega(\Gamma)/\Gamma$. By the assumption $\delta(\Gamma) \leq (n-2)/2$ and [7, Proposition 3.2], the Hausdorff dimension of $\Lambda(\Gamma)$ is not greater than $(n-2)/2$. Therefore $\Omega(\Gamma)$ is simply connected and $\pi_1(\Omega(\Gamma)/\Gamma) \cong \Gamma$. Now the universal covering of $\Omega(\Gamma)/\Gamma$ is $\Omega(\Gamma) = S^n \setminus \Lambda(\Gamma)$. Since it is an open submanifold of $\mathbb{R}^n = S^n \setminus \{\text{a point}\}$, its tangent bundle is trivial. Thus $\Omega(\Gamma)$ admits a spin structure. Assuming Γ is a discrete subgroup of the identity component of the Lie group $\mathrm{Isom}(\mathbb{H}^{n+1})$, we see that Γ satisfies the second condition by [10] (see also [15, p. 202]). The third condition follows from Lemma 6. Thus, by Theorem 8, the higher $\hat{A}$-genus of $\Omega(\Gamma)/\Gamma$ vanishes.

In what follows, we show that the vanishing of the higher $\hat{A}$-genus implies the convex-cocompactness, and complete the proof of Proposition 7. Recall that $\pi_1(\Omega(\Gamma)/\Gamma) \cong \Gamma$. Then we can take $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ as a classifying space, and the inclusion $i : \Omega(\Gamma)/\Gamma \hookrightarrow (\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ turns out to be a classifying map. Since we are assuming that $\Omega(\Gamma)/\Gamma$ is orientable, $H_n(\Omega(\Gamma)/\Gamma;\mathbb{Z}) \cong \mathbb{Z}$ and it is generated by the homology class $[\Omega(\Gamma)/\Gamma]$ represented by $\Omega(\Gamma)/\Gamma$ with a given orientation. For any $u \in H^n((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;\mathbb{Q})$, we have

$$\hat{A}_u([\Omega(\Gamma)/\Gamma]) = \bigl(1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots\bigr) \cup i^*u = i^*u,$$

since $i^*u \in H^n(\Omega(\Gamma)/\Gamma;\mathbb{Q})$ and $\dim \Omega(\Gamma)/\Gamma = n$. The vanishing of the higher $\hat{A}$-genus means that

$$0 = \langle i^*u, [\Omega(\Gamma)/\Gamma]\rangle = \langle u, i_*[\Omega(\Gamma)/\Gamma]\rangle$$

for any $u \in H^n((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;\mathbb{Q})$. Thus the map $i_* : H_n(\Omega(\Gamma)/\Gamma;\mathbb{Z}) \longrightarrow H_n((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;\mathbb{Z})$ is not injective, and the image of i_* is a torsion subgroup of $H_n((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;\mathbb{Z})$. Since $\mathbb{H}^{n+1}/\Gamma$ is a noncompact manifold, for any Abelian group G, $H_{n+1}((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;G) \cong H_{n+1}(\mathbb{H}^{n+1}/\Gamma;G) = 0$. Hence, by the universal coefficient theorem, $H_n((\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma;\mathbb{Z})$ has no torsion. Thus the image of i_* is trivial. This means $i(\Omega(\Gamma)/\Gamma)$ bounds an $(n+1)$-chain, and the chain must be $(\Omega(\Gamma)\cup\mathbb{H}^{n+1})/\Gamma$ with the suitable orientation. Therefore $(\Omega(\Gamma)\cup\mathbb{H}^{n+1})/\Gamma$ is compact. (Another way to see this is, though essentially the same, to use the Meyer-Vietoris sequence for homology groups as in the proof of [7, Proposition 4.9].) This completes the proof of Proposition 7. □

Remark 2. Suppose that $\Omega(\Gamma)$ is connected and simply connected, and that Γ is convex cocompact. Then $\Omega(\Gamma)/\Gamma$ bounds the $(n+1)$-chain $(\Omega(\Gamma)\cup\mathbb{H}^{n+1})/\Gamma$. Thus $i_* : H_n(\Omega(\Gamma)/\Gamma;\mathbb{Z}) \longrightarrow H_n((\Omega(\Gamma)\cup\mathbb{H}^{n+1})/\Gamma;\mathbb{Z})$ is trivial. Note that we have seen that when $\Omega(\Gamma)$ is connected, the triviality of i_* is equivalent to the convex-cocompactness of Γ. It is well-known that the Pontrjagin classes of a manifold is written in terms of the Weyl curvature of any metric on it. On the other hand, conformally flat manifolds are characterized by the vanishing of the Weyl curvature (see [12]). Thus, assuming that Γ is convex cocompact, we see that the higher $\hat{A}$-genus of $\Omega(\Gamma)/\Gamma$ with respect to $u \in H^*((\Omega(\Gamma)\cup\mathbb{H}^{n+1})/\Gamma;\mathbb{Q})$ becomes

$$\langle i^*u, [\Omega(\Gamma)/\Gamma]\rangle = \langle u, i_*[\Omega(\Gamma)/\Gamma]\rangle = 0$$

for any u. Therefore, when $\Omega(\Gamma)$ is connected and simply connected, the vanishing of higher $\hat{A}$-genus of $\Omega(\Gamma)/\Gamma$ is equivalent to the convex-cocompactness of Γ.

Remark 3. Because of the vanishing of Pontrjagin classes, most characteristic classes of conformally flat manifolds vanish. This suggests that the classical Atiyah-Singer index theorem tells us almost nothing about conformally flat manifolds. However, as we have seen above, higher characteristic classes could be nontrivial, and they carry some information of the fundamental group (and this is all that higher characteristic classes know because of the vanishing of Pontrjagin classes). The author believes that there should be some other applications of index theorems to Kleinian group theory via geometry of conformally flat manifolds.

4. Proof of theorems and some applications

First we give some applications of Proposition 7 including Theorems 2 and 3. Our starting point is the following, which is obvious from Proposition 7 and Theorem 4.

Lemma 9. *Let (M,C) be a compact, connected conformally flat manifold which is not covered by a torus. Suppose either*

(1) $n = \dim M \geq 4$ *and the scalar curvature of g is positive for some $g \in C$, or*

(2) $n = \dim M \geq 7$ *and the scalar curvature of g is nonnegative for some $g \in C$.*

Then there exists a convex cocompact Kleinian group Γ such that (M,C) is conformally equivalent to $\Omega(\Gamma)/\Gamma$. Moreover Γ is isomorphic to $\pi_1(M)$ and $\delta(\Gamma) = d(M,C) \leq (n-2)/2$, where $d(M,C)$ denotes the Schoen-Yau invariant of (M,C).

First we give the proof of Theorem 3.

Proof of Theorem 3. By Lemma 9, $(M,C) \cong \Omega(\Gamma)/\Gamma$, where Γ is convex cocompact and is isomorphic to $\pi_1(M)$. Let $C(\Gamma)$ be the hyperbolic convex hull of $\Lambda(\Gamma)$ which is the minimal convex subset of $\mathbb{H}^{n+1}$ whose closure in $S^n \cup \mathbb{H}^{n+1}$ contains $\Lambda(\Gamma)$. As we have mentioned in §2, the convex-cocompactness is equivalent to the compactness of $C(\Gamma)/\Gamma$. Thus, if Γ is convex cocompact, then it is quasi-isometric to $C(\Gamma)$. Since $C(\Gamma)$ is complete hyperbolic space in the sense of Gromov, Γ is a hyperbolic group. □

Our next result is on cohomology groups of a compact conformally flat manifold with positive scalar curvature. This tells us that the topology of such a manifold is determined by its fundamental group at least at the (co)homology level.

Proposition 10. *Let (M, C) be a compact, connected, and oriented conformally flat manifold which is not covered by a torus. Suppose that $\pi_1(M)$ is torsion-free and that either*

(1) *$n = \dim M \geq 4$ and the scalar curvature of g is positive for some $g \in C$, or*

(2) *$n = \dim M \geq 7$ and the scalar curvature of g is nonnegative for some $g \in C$.*

Denote the Schoen-Yau invariant of (M, C) and the fundamental group of M by $d(M, C)$ and π respectively. Then for any $\mathbb{Z}\pi$-module $\mathcal{S}$, we have

(a) *$\mathrm{cd}\pi \leq d(M, C) + 1 = \delta(\Gamma) + 1$, where $\mathrm{cd}\pi$ is the cohomological dimension of π,*

(b) *for $p < n - d(M, C) - 1$, $H^p(M; \mathcal{S})$ is isomorphic to $H^p(\pi; \mathcal{S})$,*

(c) *for $p < n - d(M, C) - 1$, $H^{n-p}(M; \mathcal{S})$ is isomorphic to $H_p(\pi; \mathcal{S})$, and*

(d) *for $d(M, C) + 1 < p < n - d(M, C) - 1$, $H^p(M; \mathcal{S}) = 0$.*

Here we regard $\mathcal{S}$ as a local system on M in order to define $H^p(M; \mathcal{S})$.

Proof. Under our assumption, (M, C) is conformally equivalent to $\Omega(\Gamma)/\Gamma$ for some convex cocompact Γ and $d(M, C) = \delta(\Gamma)$ by Lemma 9. Then the result follows from [7, Proposition 4.13], [7, Theorem 4.15] and [9, Lemma 5.3]. □

Next we consider spaces of flat conformal structures. Our results are consequences of Lemma 9 and the results in [8]. See also [1, Chapter 7] for some results relevant to that in [8].

Proof of Theorem 2. As we have mentioned following Theorem 4, the developing map of (M, g) is injective. On the other hand, Lemma 9 and [7, Proposition 4.13] show that the virtual cohomological dimension of $\pi_1(M)$ is less than $\delta(\Gamma) \leq (n-2)/2 < n$. Thus the two conditions in [8, Theorem 2] are satisfied. This completes the proof. □

Let M be a compact and connected n-manifold admitting a flat conformal structure. Denote by $\mathrm{Diff}_C(M)$ the group of diffeomorphisms which have a lift to the universal covering $\tilde{M}$ of M commuting with the covering transformations. Define $\mathcal{T}_C(M)$ as the quotient of the space of developing maps by the action of $\mathrm{Diff}_C(M)$, and equip $\mathcal{T}_C(M)$ with the quotient topology. This space $\mathcal{T}_C(M)$ is a branched covering space of the moduli space of flat conformal structures, and can be regarded as the Teichmüller space of flat conformal structures on M. (See [8] for the details.) Denote by $R(\pi_1(M), \mathrm{Conf}(S^n))$ the quotient of $\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ by the conjugate action of $\mathrm{Conf}(S^n)$. Then the map that assigns the holonomy representation to a developing map induces a map $\mathrm{hol}'' : \mathcal{T}_C(M) \longrightarrow R(\pi_1(M), \mathrm{Conf}(S^n))$. This map is known to be continuous and open. For a manifold satisfying the assumptions in Theorem 2, we get the following embedding from hol''. The proof is the same as that of Theorem 2 except the use of [8, Theorem 3] instead of [8, Theorem 2].

Theorem 11. *Let M be as in Theorem 2. Denote by $\mathcal{T}_0$ the connected component of $\mathcal{T}_C(M)$ corresponding to $\mathcal{M}_0$ in Theorem 2. Then the map hol'' turns out to be an embedding of $\mathcal{T}_0$ onto an open subset of $R(\pi_1(M), \mathrm{Conf}(S^n))$.*

We close this section with the proof of Theorem 1. Noting the first paragraph of the proof of Proposition 7, one sees that the proof of [2, Theorem 5.2] is valid in our case. We only give a sketch of the proof.

Proof of Theorem 1. (1) By the first paragraph in the proof of Proposition 7 in §3, $\Omega(\Gamma)$ is connected, $\Omega(\Gamma)/\Gamma$ is compact, and Γ is finitely generated. It follows from the definition that Γ is geometrically finite if and only if some (any) finite index subgroup of Γ is geometrically finite. Thus we may assume Γ has no torsion as before. Suppose the quotient of the convex hull $C(\Gamma)/\Gamma \subset \mathbb{H}^{n+1}/\Gamma$ has no interior points. Then $C(\Gamma)$ lies in a proper totally geodesic subspace of $\mathbb{H}^{n+1}$, and Γ is convex cocompact by the compactness of $\Omega(\Gamma)/\Gamma$ and [7, Lemma 2.2]. Thus we have only to consider the case when $C(\Gamma)/\Gamma$ has interior points. In this case, the nearest point retraction from $\Omega(\Gamma)$ to $C(\Gamma)$ is a surjection onto the boundary of $C(\Gamma)$. This map induces a surjection $\Omega(\Gamma)/\Gamma \longrightarrow \partial C(\Gamma)/\Gamma$ ($\partial C(\Gamma)/\Gamma$ is denoted by $\partial C(M)$ in [2]). Hence $\partial C(\Gamma)/\Gamma$ is compact. On the other hand, we can construct the Lipschitz graph over $\Omega(\Gamma)$ as in [2, Lemma 3.2]. By the compactness of $\Omega(\Gamma)/\Gamma$ and $\partial C(\Gamma)/\Gamma$, it is almost obvious that [2, Lemma 3.4] and [2, Lemma 3.6] is valid for our Γ. Also note that the estimate of the heat kernel [2, Theorem 4.1] due to Davies [5] is valid for any dimension. Keeping these in mind, we easily check that the proof of [2, Theorem 5.2] (see [2, pp. 25–27]) works in our case; we see that if our Γ is not geometrically finite, then the Hausdorff dimension of $\Lambda(\Gamma)$ equals n. On the other hand, by the first paragraph of the proof of Proposition 7, the Hausdorff dimension of $\Lambda(\Gamma)$ does not exceed $\delta(\Gamma) < n-1$. Thus Γ must be geometrically finite.

(2) For a geometrically finite group, $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is the union of a compact set and a finite number of cusps which come from parabolic subgroups (see [4]). Suppose Γ has a parabolic subgroup. Let Γ_0 be a maximal one. Since $\Omega(\Gamma)/\Gamma$ is compact, it is easy to see that Γ_0 must have a subgroup isomorphic to $\mathbb{Z}^n$. Then $\delta(\Gamma) \geq \delta(\Gamma_0) \geq n/2$. Therefore Γ has no parabolic subgroups by our assumption. Thus, in our case, $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is compact, namely, Γ is convex cocompact. This completes the proof. □

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