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KRULL DIMENSION OF THE ENVELOPING ALGEBRA OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra. We deduce from the work of R. Bezrukavnikov, A. Braverman and L. Positselskii that the Krull-Gabriel-Rentschler dimension of $U(\mathfrak{g})$ is equal to the dimension of a Borel subalgebra of $\mathfrak{g}$.

1. Introduction

The Krull(-Gabriel-Rentschler) dimension of a ring R was introduced in [3] and is denoted by $\operatorname{Kdim} R$. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra. It has been conjectured that $\operatorname{Kdim} U(\mathfrak{g})$ is equal to $\dim \mathfrak{b}$ where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. It is easy to see that $\operatorname{Kdim} U(\mathfrak{g}) \geqslant \dim \mathfrak{b}$; indeed, this follows from the fact $U(\mathfrak{g})$ is a free (left) module over $U(\mathfrak{b})$ and that $\operatorname{Kdim} U(\mathfrak{b}) = \dim \mathfrak{b}$ (see §2). The opposite inequality is therefore the hard part of the conjecture.

P. Smith [10] proved the conjecture for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let G be a simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$, U be a maximal unipotent subgroup of G and set $X = G/U$ (the "basic affine space"). In [7] it was shown that the conjecture would follow from $\operatorname{Kdim} \mathcal{D}(X) \leqslant \dim X$, where $\mathcal{D}(X)$ is the ring of globally defined differential operators on X (in the sense of [5]). This result was established in [7] when $\mathfrak{g}$ is a direct sum of copies of $\mathfrak{sl}(2, \mathbb{C})$, and in [8] when $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Up to now, these were the only cases known and no progress was made on the conjecture.

The difficulty in the study of $\mathcal{D}(X)$ comes from the fact that $\mathcal{D}(X) = \mathcal{D}(\overline{X})$ for some singular variety $\overline{X}$. Recently R. Bezrukavnikov, A. Braverman and L. Positselskii were able to prove, among other things, that $\mathcal{D}(X)$ is a Noetherian ring. This is deduced from the existence of a finite set $\{F_w\}_{w \in W}$ (W being the Weyl group of $\mathfrak{g}$) of automorphisms of $\mathcal{D}(X)$ such that: for every $\mathcal{D}(X)$-module $M \neq 0$, there exists a twist M^{F_w} of M such that the localization $\mathcal{O}_X \otimes_{\mathcal{O}(X)} M^{F_w}$ is nonzero. In this note we want to explain how this result easily implies that $\operatorname{Kdim} \mathcal{D}(X) \leqslant \dim X$, and, consequently, $\operatorname{Kdim} U(\mathfrak{g}) = \dim \mathfrak{b}$.

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2. Krull dimension

The definitions and general results related to Krull dimension can be found in [9, Chapter 6] and we will simply quote a few facts that we need.

Recall that the deviation of a partially ordered set (poset) $(A, \preccurlyeq)$ is defined (when it exists) as follows:

- $\operatorname{dev} \emptyset = -\infty$;
- $\operatorname{dev} A = 0$ if and only if A satisfies the descending chain condition;
- $\operatorname{dev} A = \alpha$ (some ordinal) if $\operatorname{dev} A \neq \beta$ for $\beta < \alpha$, and if $(a_i)_{i \in \mathbb{N}}$ is a descending chain in A, then there exists i_0 such that $\operatorname{dev}\{x \in A : a_i \succcurlyeq x \succcurlyeq a_{i+1}\} < \alpha$ for all $i \geqslant i_0$.

For the proof of the next lemma, see [9, 6.1.5, 6.1.6].

Lemma 2.1. (a) *Let $B \hookrightarrow A$ be a strictly increasing map of posets. Then,* $\operatorname{dev} B \leqslant \operatorname{dev} A$ *when* $\operatorname{dev} A$ *exists.*

(b) *If A satisfies the ascending chain condition, then* $\operatorname{dev} A$ *exists.* □

If R is a ring we denote by R-mod the category of finitely generated left R-modules. Let $M \in R$-mod and $\mathcal{L}(M)$ be the lattice of submodules of M. Then $(\mathcal{L}(M), \subseteq)$ is a poset; we say that the Krull dimension of M exists if $\mathcal{L}(M)$ has a deviation, in which case we set $\operatorname{Kdim}_R M = \operatorname{Kdim} M = \operatorname{dev} \mathcal{L}(M)$. By Lemma 2.1, $\operatorname{Kdim} M$ exists if R is (left) Noetherian and one has $\operatorname{Kdim} M \leqslant \operatorname{Kdim} R$ ([9, 6.2.18]).

Examples. 1. Let $\mathfrak{m}$ be a finite dimensional complex Lie algebra and $\mathfrak{l} \subset \mathfrak{m}$ be a subalgebra. Then, $\operatorname{Kdim} U(\mathfrak{l}) \leqslant \operatorname{Kdim} U(\mathfrak{m}) \leqslant \dim \mathfrak{m}$. When $\mathfrak{m}$ is solvable an easy induction on $\dim \mathfrak{m}$ (using Lie's Theorem) shows that $\operatorname{Kdim} U(\mathfrak{m}) = \dim \mathfrak{m}$.

2. Let $\mathcal{D}(Z)$ be the ring of differential operators on a smooth affine complex algebraic variety Z. Then $\mathcal{D}(Z)$ is Noetherian and $\operatorname{Kdim} \mathcal{D}(Z) = \dim Z$; see [9, 15.1.20, 15.3.7].

We will use the following easy result:

Lemma 2.2. *Let R_j, $j = 1, \ldots, s$, be some rings and $M_j \in R_j$-mod. Then, if* $\operatorname{Kdim} M_j$ *exists for all j, we have*

$$\operatorname{Kdim}_{\oplus_{j=1}^s R_j} \Big(\bigoplus_{j=1}^{s} M_j \Big) = \max\{\operatorname{Kdim} M_j : j = 1, \ldots, s\}.$$

Proof. The claim follows from the identification of $\mathcal{L}(\bigoplus_{j=1}^s M_j)$ with $\mathcal{L}(M_1) \times \cdots \times \mathcal{L}(M_s)$. □

3. Rings of differential operators

If Z is a complex algebraic variety we denote by $\mathcal{O}_Z$ its structural sheaf and by $\mathcal{D}_Z$ the sheaf of differential operators on Z, as defined in [5]. By taking global sections we get the following $\mathbb{C}$-algebras:

$$\mathcal{O}(Z) = \mathcal{O}_Z(Z), \qquad \mathcal{D}(Z) = \mathcal{D}_Z(Z).$$

Assume that Z is smooth and denote by $\mathcal{D}_Z$-coh the category of coherent left $\mathcal{D}_Z$-modules (see [2] for a definition). Recall [2] that when Z is affine, the functor $\mathcal{M} \to \Gamma(Z, \mathcal{M})$ yields an equivalence of categories $\mathcal{D}_Z\text{-coh} \cong \mathcal{D}(Z)\text{-mod}$.

Notation. Let $\overline{X}$ be an irreducible affine variety and X be a nonempty (dense) open subset of smooth points in $\overline{X}$. We will work under the following hypothesis:

$$\overline{X} \text{ is normal and } \operatorname{codim}_{\overline{X}}(\overline{X} \smallsetminus X) \geqslant 2.$$

In this situation one has $\mathcal{O}(X) = \mathcal{O}(\overline{X})$ and it is easy to show that this implies

$$\mathcal{D}(X) = \mathcal{D}(\overline{X});$$

see, e.g., [6, II.2, Proposition 2]. Since X is quasi-compact and open in $\overline{X}$ we can write $X = \bigcup_{i=1}^{s} U_i$, where each U_i is a principal affine open subset of $\overline{X}$, i.e., $U_i = \{x \in \overline{X} : f_i(x) \neq 0\}$ for some $f_i \in \mathcal{O}(X)$. Recall that $\{f_i^k\}_{k\in\mathbb{N}}$ is an Ore subset in $\mathcal{D}(\overline{X})$ and that

$$\mathcal{D}(U_i) = \mathcal{D}(\overline{X})[f_i^{-1}] = \mathcal{O}(X)[f_i^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X).$$

Therefore, if $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{coh}$, each restriction $\mathcal{M}_{|U_i} \in \mathcal{D}_{U_i}\text{-}\mathsf{coh}$ is determined by $\mathcal{M}(U_i) = \Gamma(U_i, \mathcal{M}) \in \mathcal{D}(U_i)\text{-}\mathsf{mod}$.

The next lemma is well known, and we include a proof for completeness.

Lemma 3.1. *Let $\mathcal{L}(\mathcal{M})$ be the lattice of $\mathcal{D}_X$-submodules of $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{coh}$. Then $\mathcal{L}(\mathcal{M})$ satisfies the ascending chain condition.*

Proof. Let $(\mathcal{M}_j)_{j\in\mathbb{N}}$ be an ascending chain of $\mathcal{D}_X$-submodules of $\mathcal{M}_0 = \mathcal{M}$. Set $\mathcal{M}_{j,i} = \mathcal{M}_j(U_i)$ for $i = 1, \ldots, s$ and $j \in \mathbb{N}$. Since the functor $\Gamma(U_i, -)$ is left exact, $(\mathcal{M}_{j,i})_{j\in\mathbb{N}}$ is an ascending chain of submodules in the finitely generated $\mathcal{D}(U_i)$-module $\mathcal{M}(U_i)$. Therefore, there exists $j(i) \in \mathbb{N}$ such that $\mathcal{M}_{j,i} = \mathcal{M}_{j(i),i}$ for all $j \geqslant j(i)$. Set $j_0 = \max\{j(i) : i = 1, \ldots, s\}$; then, since $X = \bigcup_{i=1}^{s} U_i$, we get that $\mathcal{M}_j = \mathcal{M}_{j_0}$ for all $j \geqslant j_0$. □

The previous lemma and §2 enable us to define the Krull dimension of $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{coh}$ by

$$\operatorname{Kdim} \mathcal{M} = \operatorname{dev} \mathcal{L}(\mathcal{M}).$$

Proposition 3.2. *Let $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{coh}$. Then,*

$$\operatorname{Kdim} \mathcal{M} \leqslant \max\{\operatorname{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \ldots, s\} \leqslant \dim X.$$

Proof. Observe that $M = \bigoplus_{i=1}^{s} \mathcal{M}(U_i)$ is a finitely generated module over the ring $R = \bigoplus_{i=1}^{s} \mathcal{D}(U_i)$. As $\Gamma(U_i, -)$ is left exact and $X = \bigcup_{i=1}^{s} U_i$, the map $\mathcal{N} \to \bigoplus_{i=1}^{s} \mathcal{N}(U_i)$ yields a strictly increasing map from $\mathcal{L}(\mathcal{M})$ to $\mathcal{L}(M)$. Thus, by definition and Lemma 2.2, we obtain

$$\operatorname{Kdim} \mathcal{M} \leqslant \operatorname{Kdim} M = \max\{\operatorname{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \ldots, s\}.$$

Since $\operatorname{Kdim} \mathcal{D}(U_i) = \dim U_i = \dim X$ for all i (cf. §2, Example 2), the assertion is proved. □

Recall that we have a localization functor $L : \mathcal{D}(X)\text{-}\mathsf{mod} \to \mathcal{D}_X\text{-}\mathsf{coh}$ defined by

$$L(M) = \mathcal{D}_X \otimes_{\mathcal{D}(X)} M.$$

Lemma 3.3. *The functor L is exact.*

Proof. Let $V = \overline{X}_f = \{x \in \overline{X}; f(x) \neq 0\}$, $f \in \mathcal{O}(X)$, be a principal open subset of $\overline{X}$ contained in X. We have already noticed that $\mathcal{D}_X(V) = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X)$, hence

$$\Gamma(V, L(M)) = \mathcal{D}_X(V) \otimes_{\mathcal{D}(X)} M = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} M.$$

The lemma then follows from the exactness of the localization functor $M \to \mathcal{O}_X \otimes_{\mathcal{O}(X)} M$ on the category $\mathcal{O}(X)$-mod. □

Suppose that $\tau \in \operatorname{Aut} \mathcal{D}(X)$ is an automorphism of the algebra $\mathcal{D}(X)$. If $M \in \mathcal{D}(X)$-mod we denote by $M^\tau \in \mathcal{D}(X)$-mod the module defined by: $M^\tau = M$ as an abelian group and $a.v = \tau(a)v$ for all $a \in \mathcal{D}(X)$, $v \in M$. We now make the supplementary hypothesis:

(H) *There exist* $\tau_1, \dots, \tau_p \in \operatorname{Aut} \mathcal{D}(X)$ *such that, for every* $0 \neq M \in \mathcal{D}(X)$-mod, $L(M^{\tau_j}) \neq 0$ *for some* $j \in \{1, \dots, p\}$.

We then define $\lambda(M) \in \mathcal{D}_X$-coh for $M \in \mathcal{D}(X)$-mod by setting

$$\lambda(M) = \bigoplus_{j=1}^{p} L(M^{\tau_j}).$$

Theorem 3.4. *One has* $\operatorname{Kdim} M \leqslant \operatorname{Kdim} \lambda(M)$ *for all* $M \in \mathcal{D}(X)$-mod. *In particular,*

$$\operatorname{Kdim} \mathcal{D}(X) \leqslant \dim X.$$

Proof. The hypothesis (H) ensures that $N \to \lambda(N)$ is a strictly increasing map from $\mathcal{L}(M)$ to $\mathcal{L}(\lambda(\mathcal{M}))$. Thus, using Proposition 3.2,

$$\operatorname{Kdim} M = \operatorname{dev} \mathcal{L}(M) \leqslant \operatorname{dev} \mathcal{L}(\lambda(\mathcal{M})) = \operatorname{Kdim} \lambda(M) \leqslant \dim X,$$

as required. □

The properties of the map $\lambda : \mathcal{L}(M) \to \mathcal{L}(\lambda(\mathcal{M}))$ imply that $M \in \mathcal{D}(X)$-mod is Noetherian; cf. [1, Theorem 1.3].

4. The Krull dimension of $U(\mathfrak{g})$

Let G be a simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Let U be a maximal unipotent subgroup of G and set $X = G/U$.

Theorem 4.1. *The quasi-affine variety* X *satisfies the hypotheses of* §3 (*in particular the hypothesis* (H)).

Proof. It is a classical fact that X can be embedded in a normal affine variety $\overline{X}$ such that $\operatorname{codim}_{\overline{X}}(\overline{X} \smallsetminus X) \geqslant 2$. This can be shown as follows. Let $\varpi_1, \dots, \varpi_\ell$ be the fundamental dominant weights of $\mathfrak{g}$; denote by $E(\varpi_j)$, $j = 1, \dots, \ell$, a simple G-module with highest weight ϖ_j and set $E = \bigoplus_{j=1}^{\ell} E(\varpi_j)$. If $v_j \in E(\varpi_j)$ is a highest weight vector, the orbit $G.(v_1 \oplus \cdots \oplus v_\ell) \subset E$ is isomorphic to X and its closure $\overline{X}$ (in E) has the required properties, see [4] and [11].

Thanks to [1], each element w of the Weyl group of $\mathfrak{g}$ yields an automorphism $F_w \in \operatorname{Aut} \mathcal{D}(X)$. By [1, Theorem 3.8], for every nonzero $M \in \mathcal{D}(X)$-mod there exists w such that $L(M^{F_w}) \neq 0$. Thus X satisfies the hypothesis (H). □

Observe that $\dim X$ is the dimension of a Borel subalgebra of $\mathfrak{g}$.

Corollary 4.2. *One has*

$$\operatorname{Kdim} U(\mathfrak{g}) = \operatorname{Kdim} \mathcal{D}(X) = \dim X.$$

Proof. By Theorem 3.4 we have $\operatorname{Kdim} \mathcal{D}(X) \leqslant \dim X$. From [7, Proposition 3.2] we know that $\operatorname{Kdim} U(\mathfrak{g}) \leqslant \operatorname{Kdim} \mathcal{D}(X)$, thus $\operatorname{Kdim} U(\mathfrak{g}) \leqslant \operatorname{Kdim} \mathcal{D}(X) \leqslant \dim X$. The result then follows from $\dim X \leqslant \operatorname{Kdim} U(\mathfrak{g})$ (see §2, Example 1). □

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