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ON THE AVERAGE CURVATURE OF A CONVEX CURVE IN A SURFACE OF NONPOSITIVE GAUSSIAN CURVATURE

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ABSTRACT. In this paper, the upper bound of the average curvature of a convex curve in a simply connected surface of nonpositive Gaussian curvature is obtained.

1. INTRODUCTION

It is well known that if the geodesic curvature of a curve in the hyperbolic plane H^2 is less than or equal to one, then the curve is embedded in H^2. In [1] M. Bridgeman considered the inverse question, defined the average curvature of a curve, and gave an upper bound of average curvature of a convex curve embedded in H^2. He also proved that the average curvature of the bi-infinite convex curve in H^2 is bounded above by one. It is natural to ask the following question: What is the upper bound of the average curvature of a convex curve embedded in a surface of nonpositive Gaussian curvature? In this paper we establish this upper bound.

In this paper a *surface* means a 2-dimensional complete Riemannian manifold. A curve α in a surface is called *convex* if any geodesic joining two points of α intersects α only at those two points. According to [1], the *average curvature* $K(\alpha)$ of a finite length curve α is defined by

$$K(\alpha) = \frac{\int_\alpha k_g ds}{\int_\alpha |\alpha'| ds} = \frac{\text{Total curvature along } \alpha}{\text{Length of } \alpha},$$

where k_g is the geodesic curvature of α, and s is the arc-length along α. If α is an infinite length curve, the average curvature $K(\alpha)$ is defined by

$$K(\alpha) = \limsup_{L\to\infty} \{K(\bar{\alpha}) \mid \bar{\alpha} \text{ is a subarc of } \alpha \text{ of length } L\}.$$

The main result of this paper is:

Theorem 1. *Let M be a simply connected surface whose Gaussian curvature G satisfies*

$$-1 \leq G \leq -a^2 < 0,$$

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and let α be a convex curve in M. If the length of α is L, then the average curvature $K(\alpha)$ of α satisfies

$$K(\alpha) \leq \frac{\sinh(L)}{\sinh(aL)}\sqrt{1+\left(\pi\frac{\sinh(aL)}{L\sinh(L)}\right)^2}+\frac{\pi}{L}.$$

The other results and corollaries will be discussed in Section 3 of this paper.

2. Notations and lemmas

Let M be a simply connected surface of nonpositive Gaussian curvature, and let α be a convex curve in M with endpoints x, y. Join x and y by a unit-speed geodesic γ such that $\gamma(0) = x$. Denote by θ_0 the interior angle formed by α and γ at x, and by Ω the region bounded by α and γ (see Figure 1). Let T_xM be the tangent space of M at x, let (ρ, θ) be the polar coordinate of T_xM, and let the metric in T_xM be taken as $ds^2 = d\rho^2 + \rho^2 d\theta^2$. Since the Gaussian curvature of M is nonpositive, we can express α and Ω in the following way: By the convexity of the curve α, if the orthonormal basis $\{e_1, e_2\}$ of T_xM is taken to be suitable, α can be written as

$$\alpha : \quad \exp_x(\rho(\theta)\cos\theta e_1 + \rho(\theta)\sin\theta e_2), \quad 0 \leq \theta \leq \theta_0,$$

and Ω can be written as

$$\Omega = \{\exp_x(\rho\cos\theta e_1 + \rho\sin\theta e_2) \mid 0 \leq \rho \leq \rho(\theta),\ 0 \leq \theta \leq \theta_0\},$$

where $\rho(\theta),\ 0 \leq \theta \leq \theta_0$, is a function of θ satisfying $\rho(\theta_0) = 0$. Let L be the length of α; it is easy to see that $\rho(\theta) \leq L,\ 0 \leq \theta \leq \theta_0$.

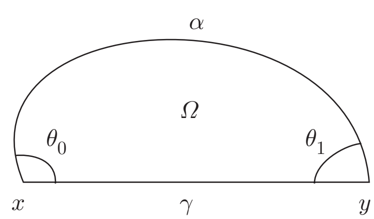

Figure 1.

Lemma 1. *If the Gaussian curvature G of M satisfies*

$$-1 \leq G \leq -a^2 < 0,$$

then the area $Area(\Omega)$ of Ω satisfies

$$\frac{1}{a}\int_0^{\theta_0}(\cosh(a\rho(\theta)) - 1)d\theta \leq Area(\Omega) \leq \int_0^{\theta_0}(\cosh(\rho(\theta)) - 1)d\theta.$$

Proof. Let $\frac{\partial}{\partial\theta}$ be the vector field in T_xM that is orthogonal to radical direction. If $p = \exp_x(\rho\cos\theta e_1 + \rho\sin\theta e_2)$ is a point in Ω, then

$$\Gamma_0(t) = \exp_x t(\cos\theta e_1 + \sin\theta e_2), \quad 0 \leq t \leq \rho,$$

is the unit-speed geodesic connecting x to p. From [4] there exists a Jacobi field

$U(t)$ along the geodesic $\Gamma_0(t)$ such that:

(1) $U(t)$ is orthogonal to $\Gamma_0'(t)$.
(2) $U(0) = 0$, $U'(0) = \frac{1}{\rho}\frac{\partial}{\partial\theta}(\rho,\theta)$. Hence from $\left|\frac{\partial}{\partial\theta}(\rho,\theta)\right| = \rho$, we have $|U'(0)| = 1$.
(3) $d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta) = U(\rho)$.

Since $U(t)$ is a Jacobi field along $\Gamma_0(t)$, it is well known that there exists a function $f(t)$ on $[0,\rho]$ such that $|f(t)| = |U(t)|$ and which satisfies

$$\begin{cases} f''(t) + G(t)f(t) = 0, \\ f(0) = 0, \quad f'(0) = 1, \end{cases}$$

where $G(t)$ is the Gaussian curvature of M at $\Gamma_0(t)$.

Consider the equations

$$\begin{cases} f_1''(t) - a^2 f_1(t) = 0, \\ f_1(0) = 0, \quad f_1'(0) = 1, \end{cases} \quad \text{and} \quad \begin{cases} f_2''(t) - f_2(t) = 0, \\ f_2(0) = 0, \quad f_2'(0) = 1. \end{cases}$$

Since $-1 \le G(t) \le -a^2$, by the Sturm comparison theorem ([3]) we deduce

$$\sinh(at) \le f(t) \le \sinh(t), \quad 0 \le t \le \rho.$$

Therefore

$$\sinh(a\rho) \le \left| d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta) \right| = |U(\rho)| = |f(\rho)| \le \sinh(\rho).$$

Since the Gaussian curvature of M is nonpositive, according to [2], the Riemannian metric on M can be written as

$$ds^2 = d\rho^2 + \left| d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta) \right|^2 d\theta^2.$$

Hence $Area(\Omega) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \left| d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta) \right| d\rho$. From the above inequality we have

$$\begin{aligned} &\frac{1}{a}\int_0^{\theta_0} (\cosh(a\rho(\theta)) - 1) d\theta = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh(a\rho) d\rho \le Area(\Omega) \\ &\le \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh(\rho) d\rho = \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta. \end{aligned} \qquad \square$$

Lemma 2. *If the Gaussian curvature G of M satisfies $G \le -a^2 < 0$, then the length of α satisfies*

$$\int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(\rho(\theta))}\, d\theta \le L \frac{\sinh(L)}{\sinh(aL)}.$$

Proof. From the proof of Lemma 1 we have $\left|d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right| \geq \sinh(a\rho(\theta))$. Notice that the function $\frac{\sinh(at)}{\sinh(t)}$ is monotonically decreasing when $t \geq 0$, and $\rho(\theta) \leq L$. From the equation of α and the representation of the metric ds^2 in Lemma 1 we deduce that

$$\begin{aligned} L &= \int_0^{\theta_0} \sqrt{\rho'(\theta)^2 + \left|d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right|^2}\, d\theta \\ &\geq \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(a\rho(\theta))}\, d\theta \\ &= \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \frac{\sinh^2(a\rho(\theta))}{\sinh^2(\rho(\theta))}\sinh^2(\rho(\theta))}\, d\theta \\ &\geq \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \frac{\sinh^2(aL)}{\sinh^2(L)}\sinh^2(\rho(\theta))}\, d\theta \\ &\geq \frac{\sinh(aL)}{\sinh(L)} \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(\rho(\theta))}\, d\theta. \end{aligned}$$

□

The following lemma is an application of the classical elementary Lobachevskii (plane hyperbolic) geometry; we omit the proof.

Lemma 3. *Let C be a circle in a surface of constant Gaussian curvature $-b^2 < 0$. If the circumference of C is L, then the area $A(L)$ of the disk bounded by C is*

$$A(L) = \frac{L}{b}\sqrt{1 + \left(\frac{2\pi}{L}\right)^2} - \frac{2\pi}{b}.$$

Hence $A(L)$ is a monotonically increasing function of L.

3. Proofs of the theorems

We denote by $Area(\cdot)$ the area of a region, and by $Length(\cdot)$ the length of a curve.

Proof of Theorem 1. Take a fixed point O in the hyperbolic plane H^2, and an orthonormal basis $\{\bar{e}_1, \bar{e}_2\}$ of $T_O H^2$. Let α_H be the curve in H^2,

$$\alpha_H: \quad \exp_O(\rho(\theta)\cos\theta\bar{e}_1 + \rho(\theta)\sin\theta\bar{e}_2), \quad 0 \leq \theta \leq \theta_0,$$

where θ_0 and $\rho(\theta)$ were defined in Section 2. Set

$$\Omega_H = \{\exp_O(\rho\cos\theta\bar{e}_1 + \rho\sin\theta\bar{e}_2) \mid 0 \leq \rho \leq \rho(\theta),\ 0 \leq \theta \leq \theta_0\}$$

and

$$\gamma_H: \quad \exp_O(\rho\bar{e}_1), \quad 0 \leq \rho \leq \rho(0),$$

so γ_H is the unit-speed geodesic connecting the endpoints of α_H, and Ω_H is the region bounded by α_H and γ_H.

Furthermore, set

$$\bar{\alpha}_H: \quad \exp_O(\rho(|\theta|)\cos\theta\bar{e}_1 + \rho(|\theta|)\sin\theta\bar{e}_2), \quad -\theta_0 \leq \theta \leq 0,$$

and

$$\bar{\Omega}_H = \{\exp_O(\rho\cos\theta\bar{e}_1 + \rho\sin\theta\bar{e}_2) \ | \ 0 \le \rho \le \rho(|\theta|), \ -\theta_0 \le \theta \le 0\}.$$

Graphically, the curve $\bar{\alpha}_H$ and region $\bar{\Omega}_H$ are the reflection of α and Ω about γ_H in H^2 respectively. Notice that since $\left|d\exp_O \frac{\partial}{\partial\theta}(\rho,\theta)\right| = \sinh\rho$ in H^2, we have

$$Area(\Omega_H) = Area(\bar{\Omega}_H) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh\rho d\rho = \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1)d\theta$$

and

$$Length(\alpha_H) = Length(\bar{\alpha}_H) = \int_0^{\theta_0} \sqrt{{\rho'}^2(\theta) + \sinh^2(\rho(\theta))}d\theta.$$

Let the interior angle formed by α and γ at y be θ_1 (see Figure 1). Then by the Gauss-Bonnet theorem we have

$$\int_\Omega GdV + \int_{\alpha\cup\gamma} k_g ds + \pi - \theta_0 + \pi - \theta_1 = 2\pi\chi(\Omega). \tag{1}$$

Since γ is a geodesic, $k_g = 0$ on γ. Obviously $\chi(\Omega) = 1$, hence

$$\int_\Omega GdV + \int_\alpha k_g ds = \theta_0 + \theta_1.$$

By the assumption, $G \ge -1$, so

$$Area(\Omega) \ge \int_\alpha k_g ds - (\theta_0 + \theta_1);$$

hence by Lemma 1 we have

$$\int_0^{\theta_0} (\cosh(\rho(\theta)) - 1)d\theta \ge \int_\alpha k_g ds - (\theta_0 + \theta_1). \tag{2}$$

Assume that $K(\alpha) > \frac{\sinh(L)}{\sinh(aL)}\sqrt{1 + (\pi\frac{\sinh(aL)}{L\sinh(L)})^2} + \frac{\pi}{L}$. Since $L \cdot K(\alpha) = \int_\alpha k_g ds$, we have

$$\begin{aligned} Area(\Omega_H \cup \bar{\Omega}_H) &= 2\int_0^{\theta_0} (\cosh(\rho(\theta)) - 1)d\theta \ge 2\left(\int_\alpha k_g ds - (\theta_0 + \theta_1)\right) \\ &= 2\left(L \cdot K(\alpha) - (\theta_0 + \theta_1)\right) \\ &> 2L\frac{\sinh(L)}{\sinh(aL)}\sqrt{1 + \left(\pi\frac{\sinh(aL)}{L\sinh(L)}\right)^2} + 2\pi - 2(\theta_0 + \theta_1). \end{aligned}$$

Denote by $A(S)$ the area of the disk bounded by a circle of circumference S in H^2. Since $\theta_0, \theta_1 \le 2\pi$, from Lemma 3 we have

$$Area(\Omega_H \cup \bar{\Omega}_H) > A\left(2L\frac{\sinh(L)}{\sinh(aL)}\right). \tag{3}$$

By Lemma 2,

$$\begin{aligned} Length(\alpha_H \cup \bar{\alpha}_H) &= 2Length(\alpha_H) \\ &\le 2\int_0^{\theta_0} \sqrt{{\rho'}^2(\theta) + \sinh^2(\rho(\theta))}d\theta \\ &\le 2L\frac{\sinh(L)}{\sinh(aL)}. \end{aligned} \tag{4}$$

From the isoperimetric inequality, $Area(\Omega_H \cup \bar{\Omega}_H)$ is less than or equal to

$$A(Length(\alpha_H \cup \bar{\alpha}_H)),$$

and from (4) we have $A(Length(\alpha_H \cup \bar{\alpha}_H)) < A\left(2L\frac{\sinh(L)}{\sinh(aL)}\right)$. Hence

$$Area(\Omega_H \cup \bar{\Omega}_H) < A\left(2L\frac{\sinh(L)}{\sinh(aL)}\right),$$

which contradicts (3). □

If α is a convex curve of length L in the hyperbolic plane H^2, then for any a with $-1 \leq -a^2$, Theorem 1 holds. Letting $a \to 1$, we deduce that

Corollary 1. *If α is a convex curve of length L in the hyperbolic plane H^2, then*

$$K(\alpha) \leq \sqrt{1 + \left(\frac{\pi}{L}\right)^2} + \frac{\pi}{L}.$$

Furthermore, if α is a convex curve of infinite length, then $K(\alpha) \leq 1$.

This corollary is Theorem 1 and Corollary 1 in [1].

Theorem 2. *Let M be a simply connected surface whose Gaussian curvature G satisfies*

$$-1 \leq -b^2 \leq G \leq 0,$$

and let α be a convex curve in M. If the length of α is L, then the geodesic curvature along α satisfies

$$\int_\alpha k_g ds \leq \sinh(bL)\sqrt{1 + \left(\frac{\pi}{\sinh(bL)}\right)^2} + \pi.$$

Proof. Since the idea of the proof of this theorem is similar to Theorem 1, we only give a sketch. In this proof, we use the same notations as in Section 2 and this section.

By the same method as Lemma 1 and Lemma 2, we can deduce that the area of Ω satisfies

$$Area(\Omega) \leq \frac{1}{b}\int_0^{\theta_0} (\cosh(b\rho(\theta)) - 1)d\theta, \tag{5}$$

and the length of α satisfies

$$\int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(b\rho(\theta))}d\theta \leq \sinh(bL).$$

From the equality (1) we can deduce that

$$b^2 Area(\Omega) \geq \int_\alpha k_g ds - \theta_0 + \theta_1,$$

so from (5) and $b^2 < 1$ we have

$$\int_0^{\theta_0} (\cosh(b\rho(\theta)) - 1)d\theta \geq b\int_0^{\theta_0} (\cosh(b\rho(\theta)) - 1)d\theta \geq \int_\alpha k_g ds - (\theta_0 + \theta_1);$$

hence

$$\frac{1}{b}\int_0^{\theta_0} (\cosh(b\rho(\theta)) - 1)d\theta \geq \frac{1}{b}\int_\alpha k_g ds - \frac{1}{b}(\theta_0 + \theta_1),$$

which is similar to (2). By the same discussion as that of Theorem 1, replacing H^2 by the surface of constant Gaussian curvature $-b^2$, and using Lemma 3 we can derive the conclusion. □

From Theorem 2, letting $b \to 0$, we deduce the following interesting corollary.

Corollary 2. *The curvature k of the convex curve α in the Euclidean plane R^2 satisfies*

$$\int_\alpha k ds \leq 2\pi.$$

Remark 1. From the proof of Theorems 1 and 2 it is easy to see that if M is a simply connected surface whose Gaussian curvature G satisfies

$$-1 \leq -b^2 \leq G \leq -a^2 < 0,$$

and α is a convex curve of length L in M, then

$$K(\alpha) \leq \frac{\sinh(bL)}{\sinh(aL)}\sqrt{1+\left(\pi\frac{\sinh(aL)}{L\sinh(bL)}\right)^2} + \frac{\pi}{L}.$$

References

1. M.Bridgeman, *Average curvature of convex curves in H^2*, Proc. Amer. Math. Soc., **126**(1998), 221–224. MR **98c:**52012
2. W.Klingenberg, *A course in differential geometry*, Springer-Verlag, New York, 1978. MR **57:**13702
3. W.Klingenberg, *D.Gromoll and W.Meyer, Riemannsche geometrie in Grossen*, Springer-Verlag, Berlin, 1968. MR **37:**4751
4. H.Wu, C.L.Shen and Y.L.Yu, *Introduction of Riemannian geometry (Chinese)*, Beijing University Press, Beijing, 1989.

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