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BIG CELLS AND LU FACTORIZATION IN REDUCTIVE MONOIDS

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ABSTRACT. It is well known that an invertible matrix admits a factorization as a product of a lower triangular matrix L and an upper triangular matrix U if and only if all the principal minors of the matrix are non-zero. The corresponding problem for singular matrices is much more subtle. We study this problem in the general setting of a reductive monoid and obtain a solution in terms of the Bruhat-Chevalley order. In the process we obtain a decomposition of the big cell $\overline{B^-}\,\overline{B}$ of a reductive monoid, where B and B^- are opposite Borel subgroups of the unit group.

1. PRELIMINARIES

Let k be an algebraically closed field. By a *reductive monoid* M we will mean an irreducible linear algebraic monoid M defined over k such that the unit group G is reductive; cf. [4], [9]. The multiplicative monoid $M_n(k)$ of all $n \times n$ matrices over k is an example. More generally the Zariski closure in $M_n(k)$, of a reductive group in $GL_n(k)$, is a reductive monoid. Let T be a maximal torus contained in opposite Borel subgroups B, B^- of G. Let $W = N_G(T)/T$ denote the Weyl group of G and let S denote the generating set of simple reflections of W. By the Bruhat decomposition,

$$G = \bigsqcup_{w \in W} BwB. \tag{1}$$

The Bruhat-Chevalley order on W is defined as:

$$x \leq y \quad \text{if } BxB \subseteq \overline{ByB}. \tag{2}$$

As is well known, this is equivalent to x being a subword of a reduced expression $y = s_1 \cdots s_m$, $s_1, \ldots, s_m \in S$. The length $\ell(y)$ is defined to be m. It is well known that for $x, y \in W$,

$$x \leq y \Leftrightarrow xBy^{-1} \cap B^-B \neq \emptyset \Leftrightarrow yB^-x^{-1} \cap B^-B \neq \emptyset. \tag{3}$$

By (1), (3),

$$xB^- \subseteq \bigcup_{x' \leq x} B^-x'B. \tag{4}$$

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Also by induction on $\ell(y)$,

$$xBy \subseteq \bigcup_{y' \leq y} B^- xy'B. \tag{5}$$

By (4), (5)

$$xB^-By \subseteq \bigcup_{\substack{x' \leq x \\ y' \leq y}} B^- x'y'B. \tag{6}$$

For $w_1, \ldots, w_n \in W$, let

$$w_1 * \cdots * w_n = \begin{cases} w_1 \cdots w_n & \text{if } \ell(w_1 \cdots w_n) = \ell(w_1) + \cdots + \ell(w_n), \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Then for $x, y, z \in W$,

$$z \leq xy \Rightarrow z = x' * y' \text{ for some } x' \leq x,\ y' \leq y. \tag{7}$$

It is also easily seen that

$$x * z \leq y * z \Rightarrow x \leq y. \tag{8}$$

For $I \subseteq S$, let W_I denote the parabolic subgroup of W generated by I and let

$$D_I = \{x \in W | xw = x * w \text{ for all } w \in W_I\}. \tag{9}$$

If w_0, v_0 are respectively the longest elements of W and W_I, then

$$w_0 D_I v_0 = D_I. \tag{10}$$

$P = BW_IB$ and $P^- = B^-W_IB^-$ are opposite parabolic subgroups of G with common Levi subgroup $L = P \cap P^-$. $B_L = B \cap L$ and $B_L^- = B^- \cap L$ are opposite Borel subgroups of L. If $x \in W$, then by [1, Chapter 2],

$$x \in D_I \Leftrightarrow xB_Lx^{-1} \subseteq B \Leftrightarrow xB_L^-x^{-1} \subseteq B^-. \tag{11}$$

Now for monoids. The idempotent set $E(\overline{T})$ of $\overline{T}$ is a finite lattice isomorphic to the face lattice of a rational polytope. The *cross-section lattice* Λ of M is defined as:

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\} = \{e \in E(\overline{T}) \mid eB^- = eB^-e\}.$$

By [3], Λ is a cross-section of the $G \times G$-orbits of M such that for all e, $f \in \Lambda$,

$$e \leq f \Leftrightarrow GeG \subseteq \overline{GfG} \Leftrightarrow e \in MfM.$$

Here as usual, $e \leq f$ means $ef = e = fe$. For $e \in \Lambda$,

$$P = \{x \in G \mid xe = exe\}, \qquad P^- = \{x \in G \mid ex = exe\}$$

are opposite parabolic subgroups. If $L = P \cap P^-$, $B_L = B \cap L$, $B_L^- = B^- \cap L$, then

$$Be = B_Le = eB_L, \qquad eB^- = eB_L^- = B_L^-e. \tag{12}$$

By [6] the Bruhat decomposition (1) extends to M as:

$$M = \bigsqcup_{\sigma \in R} B\sigma B, \tag{13}$$

where $R = \overline{N_G(T)}/T$ is the Renner monoid. W is the unit group of R and

$$R = \bigsqcup_{e \in \Lambda} WeW.$$

Let $e \in \Lambda$. Then

$$W(e) = \{w \in W \mid we = ew\}, \qquad W_e = \{w \in W \mid we = e = ew\} \tag{14}$$

are parabolic subgroups of W. Let $W(e) = W_I$, $W_e = W_K$, I, $K \subseteq S$. Then

$$W(e) = W_e \times \widetilde{W(e)}, \tag{15}$$

where $\widetilde{W(e)} = W_{I\setminus K}$. Let

$$D(e) = D_I, \qquad D_e = D_K. \tag{16}$$

Then

$$D_e \cap W(e) = \widetilde{W(e)}, \qquad D_e = D(e)\widetilde{W(e)}. \tag{17}$$

So if $\sigma \in R$, then

$$\sigma = xwey \quad \text{for unique } e \in \Lambda,\ x \in D(e),\ w \in \widetilde{W(e)},\ y \in D(e)^{-1}. \tag{18}$$

We call this the *standard form* of σ. The order (3) on W extends naturally to R if we define:

$$\sigma \le \sigma' \quad \text{if } B\sigma B \subseteq \overline{B\sigma' B}. \tag{19}$$

If $\sigma = xwey$, $\sigma' = x'w'e'y'$ are in standard form, then by [2],

$$\sigma \le \sigma' \Leftrightarrow e \le e',\ xw \le x'w'u,\ u^{-1}y' \le y \text{ for some } u \in W(e')W_e. \tag{20}$$

If $\sigma = xwey$, $\sigma' = x'w'ey'$ in standard form, then (20) simplifies to:

$$\sigma \le \sigma' \Leftrightarrow xw \le x'w'u,\ u^{-1}y' \le y \text{ for some } u \in W(e). \tag{21}$$

For the combinatorial properties of this order on WeW, see [5]. By [7], [8], the length $\ell(\sigma)$ is defined as

$$\ell(\sigma) = \ell(xw) - \ell(v_0 y w_0), \tag{22}$$

where $\sigma = xwey$ in standard form and v_0, w_0 are as in (10). This agrees with the length function given by the order of WeW.

2. Triangular monoids and LU decomposition

In this section we study the triangular monoids $\overline{B}$ and $\overline{B}^-$. As noted in [6], the decomposition (13) leads to the decomposition:

$$\overline{B} = \bigsqcup_{\sigma \in R^+} B\sigma B \tag{23}$$

for a suitable submonoid R^+ of R. Similarly,

$$\overline{B}^- = \bigsqcup_{\sigma \in R^-} B^-\sigma B^- \tag{24}$$

for a suitable submonoid R^- of R.

Example 2.1. For $M_3(k)$, the poset $(R^+, \le)$ is given in Figure 1. It corrects an error in [6, Figure 3].

Theorem 2.2. *Let $\sigma = xwey \in R^+$ be in standard form. Then*
(i) *$\sigma \in R^+$ if and only if $xw \le y^{-1}$.*
(ii) *$\sigma \in R^-$ if and only if $wy \le x^{-1}$.*
(iii) *$\sigma \in R^-R^+$ if and only if $w = w_1w_2$ with $w_1 \le x^{-1}$ and $w_2 \le y^{-1}$.*

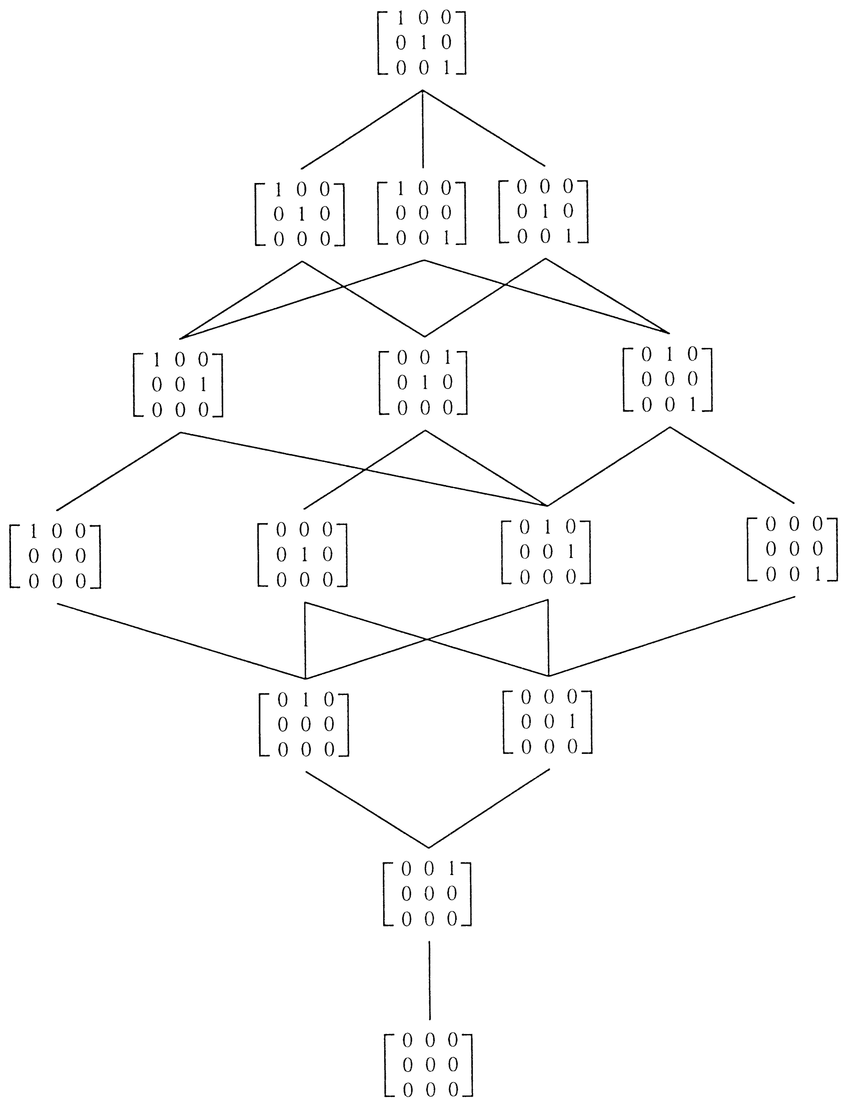

FIGURE 1.

Proof. (i) This follows from (20) since $\sigma \in R^+$ if and only if $\sigma \leq 1$.

(ii) Clearly $R^- = w_0 R^+ w_0$. Now by (10),

$$w_0 \sigma w_0 = w_0 x v_0 \cdot v_0 w v_0 \cdot e \cdot v_0 y w_0, \qquad w_0 x v_0 \in D(e),\ v_0 y w_0 \in D(e)^{-1}.$$

By (i), $\sigma \in R^-$ if and only if

$$w_0 x w v_0 = w_0 x v_0 \cdot v_0 w v_0 \leq w_0 y^{-1} v_0.$$

This is equivalent to $y^{-1} v_0 \leq x w v_0$. This is in turn equivalent by (10) to $y^{-1} w^{-1} \leq x$ since $y^{-1} v_0 = y^{-1} * w^{-1} * w v_0$ and $x w v_0 = x * w v_0$.

(iii) Suppose first that $w = w_1 w_2$ with $w_1, w_2 \in \widetilde{W(e)}$, $w_1 \leq x^{-1}$ and $w_2 \leq y^{-1}$. Then by (i), (ii), $x w_1 e \in R^-$, $w_2 e y \in R^+$. So $\sigma = (x w_1 e)(w_2 e y) \in R^- R^+$. Assume conversely that $\sigma \in R^-R^+$. Then $\sigma = \sigma^-\sigma^+$ for some $\sigma^- \in R^-$ and $\sigma^+ \in R^+$. Since $xex^{-1}\sigma = \sigma = \sigma y^{-1}ey$,

$$\sigma = (xex^{-1}\sigma^-)(\sigma^+ y^{-1}ey).$$

Hence we can assume without loss of generality that σ^+, $\sigma^- \in WeW$. Let $\sigma^- = x_1w_1ey_1$, $\sigma^+ = x_2w_2ey_2$ in standard form. Since $\sigma^-\sigma^+ = \sigma \in WeW$, $ey_1x_2e \in WeW$. So $u = y_1x_2 \in W(e)$. Hence $x_2 = y_1^{-1}u$. Since $x_2, y_1^{-1} \in D(e)$, $x_2 = y_1^{-1}$. Hence $\sigma = \sigma^-\sigma^+ = x_1w_1w_2ey_2$. So $x = x_1$, $w = w_1w_2, y = y_2$. Since $\sigma^- \in R^-$, $w_1 \leq w_1y_1 \leq x^{-1}$. Since $\sigma^+ \in R^+$, $w_2 \leq x_2w_2 \leq y^{-1}$. □

Next we obtain a decomposition of $\overline{B^-}\,\overline{B}$, the monoid analogue of Chevalley's big cell B^-B of G. We note that for $M_n(k)$, this consists of (possibly singular) matrices admitting a factorization into a product of a lower triangular matrix and an upper triangular matrix.

Theorem 2.3. *$\overline{B^-}\,\overline{B} = \bigsqcup_{\sigma\in R^-R^+} B^-\sigma B$.*

Proof. Now $B^-R^- \subseteq \overline{B^-}$ and $R^+B \subseteq \overline{B}$. Hence $B^-R^-R^+B \subseteq \overline{B^-}\,\overline{B}$. Let $\sigma = xwey \in R$ be in standard form such that $\sigma \in \overline{B^-}\,\overline{B}$. Now xex^{-1}, $yey^{-1} \in \overline{T}$ and

$$\sigma = (xex^{-1})\sigma(y^{-1}ey) \in (xex^{-1}\overline{B^-})(\overline{B}y^{-1}ey).$$

By (23), (24), there exists $\sigma_1 \in R^- \cap WeW$ and $\sigma_2 \in R^+ \cap WeW$ such that

$$\sigma \in B^-\sigma_1 B^- B \sigma_2 B. \tag{25}$$

Let $\sigma_1 = x_1w_1ey_1$ and $\sigma_2 = x_2w_2ey_2$ in standard form. By Theorem 2.2, $w_1y_1 \leq x_1^{-1}$ and $x_2w_2 \leq y_2^{-1}$. By (6),

$$w_1y_1B^-Bx_2w_2 \subseteq \bigcup_{\substack{z_1\leq w_1y_1\\ z_2\leq x_2w_2}} B^-z_1z_2B. \tag{26}$$

By (25), (26), there exists $z_1 \leq w_1y_1$, $z_2 \leq x_2w_2$ such that

$$\sigma \in B^-x_1eB^-z_1z_2Bey_2B. \tag{27}$$

By (12), $Be = eB_L$ and $eB^- = B_L^-e$. Since $x_1, y_2^{-1} \in D(e)$ we see by (11) that $x_1B_L^-x_1^{-1} \subseteq B^-$ and $y_2^{-1}B_Ly_2 \subseteq B$. Thus by (27),

$$\sigma \in B^-x_1ez_1z_2ey_2B.$$

By (13), $\sigma = x_1ez_1z_2ey_2$. Hence $x = x_1$, $y = y_2$, $z_1z_2 \in W(e)$ and $we = z_1z_2e$. By (7), $w = u_1u_2$ with

$$u_1 \leq z_1 \leq w_1y_1 \leq x^{-1}, \qquad u_2 \leq z_2 \leq x_2w_2 \leq y^{-1}.$$

By Theorem 2.2, $\sigma \in R^-R^+$. This completes the proof. □

The Bruhat decomposition (1) and the Bruhat-Renner decomposition (13) lead respectively to the closure orders (2) and (19) on W and R. In the same way the decomposition in Theorem 2.3 of the monoid big cell $\overline{B^-}\,\overline{B}$ leads to the order $\preceq$ on R^-R^+ given by

$$\sigma \preceq \sigma' \quad \text{if } B^-\sigma B \subseteq \overline{B^-\sigma' B}.$$

Example 2.4. For $M_3(k)$, the order $\preceq$ on the rank 2 elements R^-R^+ is given in Figure 2.

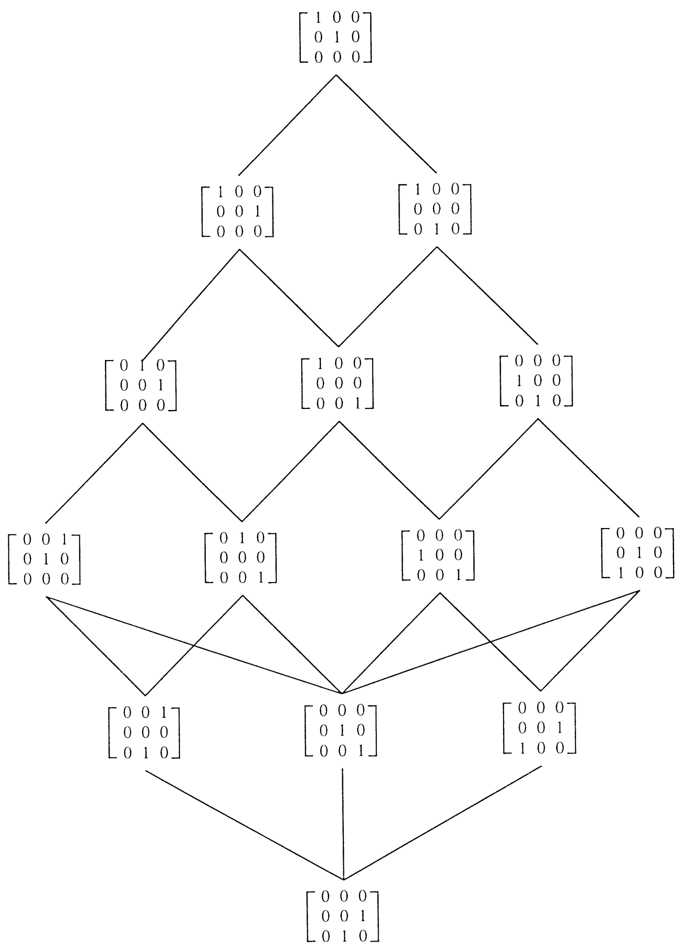

FIGURE 2.

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