

A NON-RIBBON PLUMBING OF FIBERED RIBBON KNOTS

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ABSTRACT. A closer look at an example introduced by Livingston and Melvin and later studied by Miyazaki shows that a plumbing of two fibered ribbon knots (along their fiber surfaces) may be algebraically slice yet not ribbon.

Trivially, the connected sum (i.e., 2-gonal Murasugi sum) of ribbon knots is ribbon. Non-trivially [6, 1], any Murasugi sum of fibered knots (along their fiber surfaces) is fibered. In light of these facts, perhaps the following is a bit surprising.

Theorem. *There exist fibered ribbon knots K_1, K_2 and an algebraically slice plumbing (4-gonal Murasugi sum) $K_1 * K_2$, along fiber surfaces, which is not ribbon.*

The proof uses a slight embellishment of a result from [4]. Following [4], for any knot K , and relatively prime integers m, n with $m \geq 1$, let $K\{m, n\}$ denote any simple closed curve on the boundary $\partial N(K)$ of a tubular neighborhood $N(K)$ of K in S^3 such that $K\{m, n\}$ represents m times the class of K in $H_1(N(K); \mathbb{Z})$ and has linking number n with K . For instance, if O denotes an unknot, then $O\{m, n\}$ is the (fibered) torus knot of type (m, n) . Abbreviate $K\{m, n\}\{p, q\}$ to $K\{m, n; p, q\}$. (N.B.: there is no universally accepted standard notation for such iterated cable knots. In particular, instead of $O\{m, n; p, q\}$, Livingston and Melvin [2] write $(q, p; n, m)$, and Miyazaki [3] writes $(p, q; m, n)$.)

Proposition. *For any K , $K\{m, n\} = K\{m, \operatorname{sgn} n\} * O\{m, n\}$ is a $2m$ -gonal Murasugi sum, along a suitable Seifert surface $F_{(m, \operatorname{sgn} n)}$ for $K\{m, \operatorname{sgn} n\}$ and a fiber surface $D_{(m, n)}$ for $O\{m, n\}$; for fibered K , $F_{(m, \operatorname{sgn} n)}$ is a fiber surface.*

Proof. The case $m = 1$ is trivial. Take $m > 1$, $\operatorname{sgn} n = \pm 1$. Let $F \subset S^3$ be a Seifert surface for K . Construct a Seifert surface $F_{(m, \pm 1)}$ from m parallel copies of F , each adjacent pair of copies joined, in order, by a 1-handle with a half-turn of sign \pm . If F is a fiber surface, then so is $F_{(m, \pm 1)}$ (see [6]); in any case $\partial F_{(m, \pm 1)} = K\{m, \pm 1\}$. The proof is finished by contemplating an appropriate figure (see Figure 1). \square

Proof of the Theorem. Livingston and Melvin [2] drew attention to the connected sum of iterated torus knots

$$K := O\{2, 3; 2, 13\} \# O\{2, 15\} \# O\{2, -3; 2, -15\} \# O\{2, -13\}.$$

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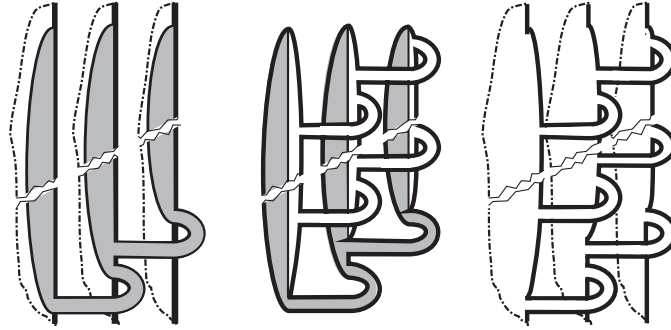


FIGURE 1. A 3-patch N_1 on $F_{(3,1)}$; a 3-patch N_2 on the (braided) fiber surface $D_{(3,n)}$; $F_{(3,n)}$ as a 6-gonal Murasugi sum $F_{(3,1)} *_h D_{(3,n)}$ along the obvious diffeomorphism $h : N_1 \rightarrow N_2$.

(Their motivation was a vague question [5] about relations among the concordance classes of the knots associated to complex plane curve singularities, and they gave an answer to one form of the question by observing that K is algebraically slice.) Miyazaki [3] showed that K is not ribbon. By the proposition, and the indifference of connected sums (of knots) to the location of the summation,

$$\begin{aligned} K &= (O\{2, 3; 2, 1\} * O\{2, 13\}) \# O\{2, 15\} \\ &\quad \# (O\{2, -3; 2, -1\} * O\{2, -15\}) \# O\{2, -13\} \\ &= ((O\{2, 3; 2, 1\} \# O\{2, -3; 2, -1\}) * (O\{2, 13\} \# O\{2, -13\})) \\ &\quad * (O\{2, 15\} \# O\{2, -15\}) \end{aligned}$$

(where all the Murasugi sums are 4-gonal and along fiber surfaces). The connected sum of a knot and its mirror image is ribbon, so $R_1 := O\{2, 3; 2, 1\} \# O\{2, -3; 2, -1\}$, $R_2 := O\{2, 13\} \# O\{2, -13\}$, and $R_3 := O\{2, 15\} \# O\{2, -15\}$ are ribbon. If $R_1 * R_2$ is ribbon, let $K_1 := R_1 * R_2$, $K_2 := R_3$; if not, let $K_1 := R_1$, $K_2 := R_2$. In either case, K_1 and K_2 are ribbon but $K_1 * K_2$ is not ribbon. On the other hand, since (as mentioned in [4]) the Murasugi sum in the proposition is obviously “direct” (i.e., the Seifert linking between any cycle on $F_{(m, \text{sgn } n)}$ and any cycle on $D_{(m, n)}$, taken in either order, is 0), $K_1 * K_2$ is algebraically slice, for clearly a Murasugi direct sum is algebraically concordant to the connected sum of the same summands. \square

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