PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 12, Pages 3545–3553
S 0002-9939(02)06566-8
Article electronically published on June 27, 2002

CODIMENSION OF POLYNOMIAL SUBSPACE IN $L_2(\mathbb{R}, d\mu)$ FOR DISCRETE INDETERMINATE MEASURE μ

ANDREW G. BAKAN

(Communicated by Juha M. Heinonen)

Abstract. A calculation formula is established for the codimension of the polynomial subspace in $L_2(\mathbb{R}, d\mu)$ with discrete indeterminate measure μ. We clarify how much the masspoint of the n-canonical solution of an indeterminate Hamburger moment problem differs from the masspoint of the corresponding N-extremal solution at a given point of the real axis.

1. Introduction and main result

Let $\mathcal{M}^*(\mathbb{R})$ be the set of positive Borel measures on $\mathbb{R}$ having moments of every order and infinite support,

$$\mathcal{N} := \{f \in \mathrm{Hol}\,(\mathbb{C} \setminus \mathbb{R}) \quad | \quad \mathrm{Im} f(z) \,/\, \mathrm{Im} z \quad > \quad 0 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}\};$$

$\mathfrak{P} := \mathcal{N} \cup \mathbb{R}$, $\mathfrak{P}^* := \mathfrak{P} \cup \{\infty\}$ and $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$. We write

$$\mathcal{N}_2 := \left\{ \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \Big| \ a, d, b, c \in \mathcal{E} \ ; \quad a(z)d(z) - b(z)c(z) \equiv 1 \ ; \right.$$

$$\left. \frac{a(z)t + c(z)}{b(z)t + d(z)} \in \mathcal{N} \quad \forall\, t \in \mathbb{R}^* \right\}$$

for the set of all Nevanlinna matrices, where $\mathcal{E}$ denotes the set of all entire functions real-valued on the real axis.

A measure $\mu \in \mathcal{M}^*(\mathbb{R})$ is said to be indeterminate if the set V_μ of all measures $\nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int_{\mathbb{R}} x^n \ \ d\mu(x) \quad = \quad \int_{\mathbb{R}} x^n \ \ d\nu(x) \ \ \forall\, n \geq 0,$$

contains at least one measure not coincident with μ. In that case the moment problem generated by $\mu \in \mathcal{M}^*(\mathbb{R})$ (or, more precisely, generated by moments of μ) is called an indeterminate Hamburger moment problem, and all measures from V_μ are referred to as its solutions (see [1, II, §1]). If $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n \geq 0}$ (see, for example, [1]) denote, corresponding to this indeterminate moment problem,

Received by the editors June 15, 2000.

2000 *Mathematics Subject Classification.* Primary 44A60, 30E05, 41A10, 46E30; Secondary 47A57, 47B36, 42A82.

This work was done in the framework of the INTAS research network 96-0858.

sequences of polynomials of the first and of the second kind, respectively, then by the Nevanlinna theorem, one can construct, using the formulas

$$A(z)= z \sum_{k=0}^{\infty} Q_k(0)Q_k(z), \qquad C(z)= 1 + z \sum_{k=0}^{\infty} P_k(0)Q_k(z),$$
$$B(z)= -1 + z \sum_{k=0}^{\infty} Q_k(0)P_k(z), \qquad D(z)= z \sum_{k=0}^{\infty} P_k(0)P_k(z),$$

the Nevanlinna matrix $\begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \in \mathcal{N}_2$ such that the known Nevanlinna formula

$$\int_{\mathbb{R}} \frac{d\nu_\varphi(t)}{t-z} \;=\; -\;\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R} \tag{1.1}$$

establishes the homeomorphism $\mathfrak{P}^* \ni \varphi \to \nu_\varphi \in V(\mu)$ of $\mathfrak{P}^*$ onto $V(\mu)$.

The special solutions in (1.1) corresponding to $\varphi \in \mathfrak{P}^*$ being a real constant or ∞ are called N-*extremal*. All of them are discrete measures. It is known that for each $x \in \mathbb{R}$

$$\max_{\nu \in V(\mu)} \nu(\{x\}) \;=\; \rho(x) \;:=\; \left(\sum_{n=0}^{\infty} P_n(x)^2 \right)^{-1}, \tag{1.2}$$

and this maximum is attained on only one N-extremal measure ν, depending on $x \in \mathbb{R}$ (see [1, Th.3.4.1]). More precisely, every N-extremal measure at any growth point x has a maximal mass $\rho(x)$ in the sense of (1.2), and the function $\rho(x)$ defined in (1.2) is called a *maximal weight function* of the moment problem generated by the measure μ. It is also known that the following equality holds (see, for example, [6, (2.3)]):

$$B'(x)D(x) \;-\; D'(x)B(x) \;=\; \frac{1}{\rho(x)} \quad \forall\, x \in \mathbb{R}. \tag{1.3}$$

N-extremal measures were characterized by M. Riesz in 1923. Denote by $\mathcal{P}[\mathbb{C}]$ the set of all algebraic polynomials with arbitrary complex coefficients.

Riesz's Theorem ([6])**.** *Let* $\mu \in \mathcal{M}^*(\mathbb{R})$.

1. *If* μ *is an indeterminate measure and* $\nu \in V_\mu$ *, then* $\mathcal{P}[\mathbb{C}]$ *is dense in* $L_2(\mathbb{R}, d\nu)$ *if and only if* ν *is an* N*-extremal measure.*

2. *If* μ *is a determinate measure (i.e.,* $V_\mu = \{\mu\}$*), then* $\mathcal{P}[\mathbb{C}]$ *is dense in* $L_2(\mathbb{R}, d\mu)$.

If in (1.1) $\varphi \in \mathfrak{P}^*$ is a rational function of degree n, i.e., $\varphi = \frac{p}{q}$, where p and q are polynomials without common zeros and the maximum of the degrees of p and q is equal to n, then ν_φ is called the n-*canonical measure*, and, according to (1.1),

$$\int_{\mathbb{R}} \frac{d\nu_\varphi(t)}{t-z} \;=\; -\;\frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R},\ \varphi = \frac{p}{q}. \tag{1.4}$$

That is why any n-canonical measure is also discrete with some masspoints at zeros of $B(z)p(z) - D(z)q(z)$, i.e.,

$$d\nu_\varphi(x) \;=\; \sum_{\lambda \in \Lambda_{Bp-Dq}} \nu_\lambda^\varphi \cdot \delta_\lambda(x),$$

where Λ_f denotes the set of all zeros of some entire function f, δ_λ is Dirac's measure at the point λ, and the masses ν_λ^φ are given by the corresponding residues. It is clear that if $n \geq 1$, then, according to (1.2),

$$0 \; < \; \nu_\lambda^\varphi \; < \; \rho(\lambda) \;\; \forall \, \lambda \in \Lambda_{Bp-Dq}. \tag{1.5}$$

The 0-canonical solutions and $\nu_\infty$are the same as the N-extremal measures.

It is well-known that $\nu \in V_\mu$ is n-canonical if and only if the measure

$$(1+x^2)^{-n} d\nu(x)$$

is N-extremal (see [1, Th.3.4.3]). Another characterization of n-canonical measures is given in the following result (1984) of Cassier, which generalizes Riesz's theorem.

Cassier's Theorem ([3], [2])**.** *Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be an indeterminate measure. The measure μ is n-canonical if and only if the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ is of codimension n.*

In this paper we partially answer the natural question as how much ν_λ^φ (from (1.5)) is less then $\rho(\lambda)$. Besides that, we also calculate the codimension of the closure of $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ for any indeterminate discrete measure μ.

Theorem 1. *Let*

$$d\mu(x) \;=\; \sum_{k\geq 1} \mu_k \cdot \delta_{\lambda_k}(x)$$

be any discrete indeterminate measure from the class $\mathcal{M}^*(\mathbb{R})$. Then the following statements hold.

(A) *If ρ is a maximal weight function of the indeterminate moment problem generated by μ, then*

$$\sum_{k\geq 1} \left(1 - \frac{\mu_k}{\rho(\lambda_k)}\right) \;=\; \operatorname{codim}_{L_2(\mathbb{R},d\mu)} \mathcal{P}[\mathbb{C}], \tag{1.6}$$

where $\operatorname{codim}_{L_2(\mathbb{R},d\mu)}\mathcal{P}[\mathbb{C}] \;\in \{0,1,2,\dots\} \cup \{+\infty\}$ *denotes the codimension of the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$.*

(B) *If μ is an n-canonical measure for some nonnegative integer n, then there exist numbers $\theta_k \in [0,1)$, $k \geq 1$, such that*

$$\begin{cases} \mu_k \;=\; (1-\theta_k)\,\rho(\lambda_k) \quad \forall k \geq 1; \\ \sum\limits_{k\geq 1} \theta_k \;=\; n. \end{cases}$$

2. Auxiliary lemma

It has been proved in [1, III, 1.1.] that $f(z) \in \mathcal{N}$ and

$$\sup_{|y|\geq 1} |y f(iy)| \; < \; \infty \tag{2.1}$$

if and only if there exists a nondecreasing function $\sigma(x)$ of bounded variation on the whole real axis such that

$$f(z) \;=\; \int_{\mathbb{R}} \frac{d\sigma(u)}{u-z} \quad \forall \, z \in \mathbb{C} \setminus \mathbb{R}.$$

In Lemma 1 below we establish a useful corollary of this statement.

Let $\varphi \in \mathcal{N}$ be a meromorphic function with the set of all its zeros $\{b_k\}_{k\geq 1} \subset \mathbb{R}$. Denote by $\{a_k\}_{k\geq 1} \subset \mathbb{R}$ all its nonzero poles. Then by a known theorem (see [5, VII, Th.2]), there exist nonnegative numbers $A_*(\varphi)$, $A_{-1}(\varphi), A_k(\varphi), k \geq 1$, and a real number $A_0(\varphi)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$\varphi(z) = A_{-1}(\varphi)z + A_0(\varphi) - \frac{A_*(\varphi)}{z} + \sum_{k\geq 1} A_k(\varphi)\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right), \tag{2.2}$$

where

$$\sum_{k\geq 1} \frac{A_k(\varphi)}{1+a_k^2} < \infty. \tag{2.3}$$

Lemma 1. *Let $\varphi \in \mathcal{N}$ be a meromorphic function with zeros $\{b_k\}_{k\geq 1} \subset \mathbb{R}$, and assume the coefficient $A_{-1}(\varphi)$ in its representation (2.2) is positive. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$ corresponding to (2.2), the representation of the function $-\frac{1}{\varphi} \in \mathcal{N}$ has the following specific form:*

$$-\frac{1}{\varphi(z)} = \sum_{k\geq 1} \frac{A_k(-\frac{1}{\varphi})}{b_k - z}, \quad \sum_{k\geq 1} A_k\left(-\frac{1}{\varphi}\right) < \infty, \tag{2.4}$$

where $A_k(-\frac{1}{\varphi}) \geq 0 \quad \forall k \geq 1$.

Proof. It is easy to verify that (2.3) implies

$$\sum_{k\geq 1} A_k(\varphi)\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right) = \sum_{k\geq 1} A_k(\varphi)\frac{z}{a_k(a_k - z)} = \bar{\bar{o}}\,(y), \tag{2.5}$$

where $z = iy$ and $|y| \to \infty$. Since $A_{-1}(\varphi) > 0$, then (2.2) and (2.5) yield

$$\varphi(iy) = A_{-1}(\varphi)iy + \bar{\bar{o}}\,(|y|), \quad |y| \to \infty,$$

and hence as $|y| \to \infty$ we have

$$-\frac{1}{\varphi(iy)} = -\frac{1}{A_{-1}(\varphi)iy + \bar{\bar{o}}\,(|y|)} = \frac{i}{yA_{-1}(\varphi)}\left(1 + \bar{\bar{o}}\,(1)\right). \tag{2.6}$$

The asymptotic representation (2.6) indicates that the function $-1/\varphi \in \mathcal{N}$ satisfies condition (2.1): $\sup_{|y|\geq 1} |y|\,|-1/\varphi(iy)| < \infty$. Applying the fact from [1, III, 1.1] mentioned at the beginning of this section, we obtain the existence of nonnegative numbers $A_k(-\frac{1}{\varphi})$, $k \geq 1$, such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ the relations (2.4) are true. Lemma 1 is proved. ☐

3. Proof of Theorem 1

3.1. Let us consider some positive integer $n \geq 1$ and any n-canonical measure $\mu \in \mathcal{M}^*(\mathbb{R})$. According to (1.4), for the indeterminate moment problem generated by μ, we have two polynomials p, q, $\max\{\deg p, \deg q\} = n$, and two entire functions $U(z), V(z)$ such that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t-z} = -\frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} =: \frac{U(z)}{V(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1}$$

But then, for some $\alpha \in [0, 2\pi]$,

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} = \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \begin{pmatrix} g(z) \\ h(z) \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned}\begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} &= \begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \\ \begin{pmatrix} g(z) \\ h(z) \end{pmatrix} &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}.\end{aligned} \tag{3.3}$$

It follows from known properties of the class $\mathcal{N}$ and Nevanlinna matrices (see [5, VII, Th.2], p. 412, (1); p. 414, Theorem) that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathcal{N}_2, \quad \frac{g}{h} \in \mathcal{N}\ . \tag{3.4}$$

Moreover, one can easily derive that for almost all $\alpha \in [0, 2\pi]$, the following relations hold:

$$\begin{cases} \deg h \ = \ \deg\left(p(z)\sin\alpha + q(z)\cos\alpha\right) \ = \ \max\{\deg p,\ \deg q\} \ = \ n; \\ 0 \notin \Lambda_\varphi\,,\ \Lambda_{\varphi_1} \cap \Lambda_{\varphi_2} \ = \ \emptyset \ \ \forall \varphi, \varphi_1 \ \neq \ \varphi_2 \in \{U, V, a, b, c, d, g, h\}\ . \end{cases} \tag{3.5}$$

Equalities (1.3) and (3.1) can be rewritten as follows:

$$d'(x)b(x) \ - \ b'(x)d(x) \ = \ \frac{1}{\rho(x)} \ \ \forall\, x \in \ \mathbb{R}, \tag{3.6}$$

$$\frac{U(z)}{V(z)} \ = \int\limits_{\mathbb{R}} \frac{d\mu(t)}{t-z} \ = \ \frac{a(z)g(z) + c(z)h(z)}{b(z)g(z) + d(z)h(z)} \quad \forall\, z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.7}$$

Everywhere below we assume that the number $\alpha \in [0, 2\pi]$, introduced in (3.3), satisfies (3.5).

3.2. It is easy to see from (1.2) that, for every $a \in \mathbb{R}$, the maximal weight function corresponding to the shifted measure $d\mu_a(x) := d\mu(x-a)$ equals $\rho(x-a)$ and that $\operatorname{codim}_{L_2(\mathbb{R}, d\mu_a)}\mathcal{P}[\mathbb{C}] = \operatorname{codim}_{L_2(\mathbb{R}, d\mu)}\mathcal{P}[\mathbb{C}]$. That is why, without loss of generality, we assume that $0 \notin \{\lambda_k\}_{k\geq 1}$ and $V(0) \ = \ 1$. According to (3.7), $U(z)\,/\,V(z) = \sum_{k\geq 1} \mu_k\,/\,(\lambda_k - z)$, and therefore

$$U(\lambda_k) \ = \ - \ \mu_k\ V'(\lambda_k) \quad \forall\ k \geq \ 1. \tag{3.8}$$

Besides that, it follows from (3.2) and (3.4) that

$$\begin{cases} g(z) = & U(z)d(z) - V(z)c(z), \\ h(z) = & V(z)a(z) - U(z)b(z), \end{cases}$$

from which one can easily obtain

$$\begin{cases} g(\lambda_k) = U(\lambda_k)d(\lambda_k) \overset{(3.8)}{=} - \ \mu_k d(\lambda_k)V'(\lambda_k), \\ h(\lambda_k) = -U(\lambda_k)b(\lambda_k) \overset{(3.8)}{=} \ \mu_k b(\lambda_k)V'(\lambda_k). \end{cases} \tag{3.9}$$

But the inclusions (3.4), together with known properties of Nevanlinna matrices (see [7]), mean that

$$\frac{V}{bh} = \frac{bg + dh}{bh} = \frac{g}{h} + \frac{d}{b} \in \mathcal{N}\ ,$$

and so,

$$-\frac{bh}{V} \ = \ - \ \frac{1}{\dfrac{g}{h} + \dfrac{d}{b}} \ \in \ \mathcal{N}\ .$$

Therefore, by a well-known theorem [5, VII, Th.2], there exist $V_0 \in \mathbb{R}$ and $V_{-1}, V_k \geq 0 \ \forall\ k \geq 1$ such that

$$-\frac{h(z)b(z)}{V(z)} = V_{-1}z + V_0 + \sum_{k\geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k \quad \forall\ z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.10}$$

3.3. We prove now that in the expansion (3.10), $V_{-1} = 0$. Assume the contrary: $A_{-1}\left(-\frac{bh}{V}\right) \equiv V_{-1} > 0$ (here and everywhere below we use the notation from (2.2) for all functions $\varphi \in \mathcal{N}$). Then (2.4) gives

$$\frac{V}{bh} = \frac{g}{h} + \frac{d}{b} = \sum_{k\geq 1} \frac{A_k(\frac{V}{bh})}{\beta_k - z}, \quad \sum_{k\geq 1} A_k\left(\frac{V}{bh}\right) < \infty, \tag{3.11}$$

where $\{\beta_k\}_{k\geq 1}$ are all the zeros of $b(z)h(z)$. Denoting by $\{c_k\}_{k\geq 1}$ all the zeros of the entire function $b(z)$, we conclude by (3.11) that, if $\beta_k = c_m$ for some positive integers m and k, then

$$A_k\left(\frac{V}{bh}\right) = -\frac{d(c_m)}{b'(c_m)}$$

and, hence,

$$\sum_{k\geq 1} -\frac{d(c_k)}{b'(c_k)} < \infty.$$

But on the other hand, using equality (3.6) we have

$$\infty > \sum_{k\geq 1} -\frac{d(c_k)}{b'(c_k)} = \sum_{k\geq 1} \frac{-d(c_k)b'(c_k)}{b'(c_k)^2} = \sum_{k\geq 1} \frac{1}{\rho(c_k)b'(c_k)^2}. \tag{3.12}$$

Due to (3.4) and (3.7), the function $b(z)$ is an element of the Nevanlinna matrix corresponding to the indeterminate moment problem generated by the measure μ. That's why the measure $\sum_{k\geq 1} \rho(c_k)\delta_{c_k}(x)$ is N-extremal (see (1.1)). But now inequality (3.12) gives a contradiction with the second necessary Hamburger condition for N-extremal measures ([4, p. 516, (8.24)], [1, IV, Addenda and exercises, 2, Th. 1, (7)]). This contradiction proves the required equality $V_{-1} = 0$ in (3.10). Thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$ we can rewrite (3.10) as follows:

$$-\frac{h(z)b(z)}{V(z)} = V_0 + \sum_{k\geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k, \quad V_0 \in \mathbb{R}, \quad V_k \geq 0 \ \ \forall k \geq 1. \tag{3.13}$$

3.4. Differentiating equality (3.13), we get

$$-\frac{(h(z)b(z))'V(z) - h(z)b(z)V'(z)}{V(z)^2} = -\left(\frac{h(z)b(z)}{V(z)}\right)' = \sum_{k\geq 1} \frac{V_k}{(\lambda_k - z)^2}$$

and

$$\sum_{k\geq 1} V_k \left(\frac{V(z)}{\lambda_k - z}\right)^2 = h(z)b(z)V'(z) - h'(z)b(z)V(z) - h(z)b'(z)V(z)\,. \tag{3.14}$$

Denote by $\eta_1,\ \eta_2,\ \dots, \eta_n$ all zeros of the polynomial $h(z)$ and substitute $z=\eta_m$ in (3.14):

$$\sum_{k\geq 1} V_k \left(\frac{V(\eta_m)}{\lambda_k-\eta_m}\right)^2 \ = -\ h'(\eta_m)b(\eta_m)V(\eta_m) \ \ \forall\ 1\leq m\leq n.$$

In addition to these equalities, the equality $V=bg+dh$ implies $V(\eta_m)=b(\eta_m)g(\eta_m)$, and, therefore,

$$-h'(\eta_m)b(\eta_m) \ = \ \sum_{k\geq 1} V_k \frac{V(\eta_m)}{(\lambda_k-\eta_m)^2} \ = \ \left(\sum_{k\geq 1}\frac{V_k}{(\lambda_k-\eta_m)^2}\right) b(\eta_m)g(\eta_m),$$

from which we get

$$1 \ = \ \sum_{k\geq 1} V_k \left(-\ \frac{g(\eta_m)}{h'(\eta_m)}\right)\frac{1}{(\lambda_k-\eta_m)^2} \quad \forall\ 1\leq m\leq n. \tag{3.15}$$

Under our condition on the number α, differentiation of the obvious equality

$$\frac{g(z)}{h(z)} \ = \ C_0 \ + \ \sum_{m=1}^{n}\frac{g(\eta_m)}{h'(\eta_m)}\frac{1}{(z-\eta_m)}$$

gives

$$\left(\frac{g(z)}{h(z)}\right)' \ = \ \sum_{m=1}^{n}\left(-\ \frac{g(\eta_m)}{h'(\eta_m)}\right)\frac{1}{(z-\eta_m)^2}. \tag{3.16}$$

Thus, summing (3.15) over all m and taking (3.16) into account, we have

$$n \ = \ \sum_{k\geq 1} V_k \left(\frac{g}{h}\right)'(\lambda_k). \tag{3.17}$$

To finish the proof of our theorem, it remains only to recount the terms in the right side of (3.17).

3.5. Equality (3.13) shows that

$$V_k \ = \ \frac{h(\lambda_k)b(\lambda_k)}{V'(\lambda_k)} \ \ \forall k\geq 1,$$

which, together with (3.9), indicates that

$$V_k \ = \ \frac{h(\lambda_k)^2}{\mu_k V'(\lambda_k)^2} \ \ \forall k\geq 1,$$

and therefore

$$V_k \left(\frac{g}{h}\right)'(\lambda_k) \ = \ \frac{1}{\mu_k V'(\lambda_k)^2}\left(g'(\lambda_k)h(\lambda_k) \ - \ g(\lambda_k)h'(\lambda_k)\right) \ \ \forall k\geq 1. \tag{3.18}$$

For the sake of convenience, we denote

$$\left\|\frac{F}{G}\right\|(z) \ := \ F'(z)G(z) \ - \ F(z)G'(z)$$

for any two entire functions $F(z), G(z)$. That is why equalities (3.18) can be rewritten as follows:

$$V_k \left(\frac{g}{h}\right)'(\lambda_k) \ = \ \frac{\left\|\frac{g}{h}\right\|(\lambda_k)}{\mu_k V'(\lambda_k)^2} \ \ \forall k\geq 1. \tag{3.19}$$

3.6. Now we will find an acceptable expression for $\left\|\frac{g}{h}\right\|(\lambda_k)$ from (3.19). Differentiating the equality

$$\frac{V}{bh} \;=\; \frac{g}{h} \;+\; \frac{d}{b},$$

we get

$$V'(z)b(z)h(z) \;-\; V(z)(bh)'(z) \;=\; b(z)^2\left\|\frac{g}{h}\right\|(z) \;+\; h(z)^2\left\|\frac{d}{b}\right\|(z).$$

Setting $z = \lambda_k$ here, we obtain

$$V'(\lambda_k)b(\lambda_k)h(\lambda_k) \;=\; b(\lambda_k)^2\left\|\frac{g}{h}\right\|(\lambda_k) \;+\; h(\lambda_k)^2\left\|\frac{d}{b}\right\|(\lambda_k). \tag{3.20}$$

Replacement of $h(\lambda_k)$ here by its expression from (3.9) gives

$$\mu_k V'(\lambda_k)^2 b(\lambda_k)^2 \;=\; b(\lambda_k)^2\left\|\frac{g}{h}\right\|(\lambda_k) \;+\; \mu_k^2 V'(\lambda_k)^2 b(\lambda_k)^2\left\|\frac{d}{b}\right\|(\lambda_k),$$

or

$$\mu_k V'(\lambda_k)^2 \;=\; \left\|\frac{g}{h}\right\|(\lambda_k) \;+\; \mu_k^2 V'(\lambda_k)^2\left\|\frac{d}{b}\right\|(\lambda_k).$$

That is why

$$\left\|\frac{g}{h}\right\|(\lambda_k) \;=\; \mu_k V'(\lambda_k)^2\left(1-\mu_k\left\|\frac{d}{b}\right\|(\lambda_k)\right) \quad \forall k \geq 1. \tag{3.21}$$

Substituting (3.21) in (3.19) and taking into account the equality $\left\|\frac{d}{b}\right\|(\lambda_k) = \frac{1}{\rho(\lambda_k)}$ evoked by (3.6), we establish the desired relation (1.6) for any n-canonical measure μ with a positive integer n such that

$$n \;=\; \sum_{k\geq 1}\left(1-\mu_k\left\|\frac{d}{b}\right\|(\lambda_k)\right) \;=\; \sum_{k\geq 1}\left(1-\frac{\mu_k}{\rho(\lambda_k)}\right).$$

With the help of an integral representation of the functions from $\mathcal{N}$ (see [1, III, §1, (3)]) and (1.1), it is possible to approximate any non-canonical but discrete measure from V_μ by canonical measures with their orders n increasing to infinity, and, due to equality (1.6) established for canonical measures, to get for such a measure a convergence to infinity of the series in the left side of (1.6). Finally, statement (B) of the theorem represents a simple reformulation of (A) with the help of Cassier's theorem. For $n = 0$ statements (A) and (B) are evident, and this completes the proof of Theorem 1.

Acknowledgment

The author thanks Professors Christian Berg, Matts Esse'n, and Mikhail Sodin for useful discussions, and Professor Thomas Craven for help with the English grammar.

References

1. N. I. Akhiezer, *The classical moment problem*, Oliver and Boyd, Edinburgh, 1965. MR **32:**1518
2. H. Buchwalter and G.Cassier, *Mesures canoniques dans le probleme classique des moments*, Ann. Inst. Fourier **34** (1984), 45–52. MR **86a:**44011
3. G. Cassier, *Mesures canoniques dans le probleme classique des moments*, C.R.Acad.Sci., Serie I, **296** (1984), 717–719. MR **84m:**44019
4. H. L. Hamburger, *Hermitian transformations of deficiency index (1,1), jacobi matrices and undetermined moment problems*, Amer. J. of Math. **LXVI** (1944), 489–522. MR **6:**130d
5. B. Ja. Levin, *Distribution of zeros of entire functions*, AMS, Providence, RI, 1964. MR **28:**217
6. M. Riesz, *Sur le probleme des moments et le theoreme de Parseval correspondant*, Acta Litt.Ac.Sci Szeged **1** (1923), 209–225.
7. M. Sodin, *A remark to the definition of Nevanlinna matrices*, Mat. Fiz., Anal., Geom. **3** (1996), no. 3/4, 412–422. MR **99e:**30020

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereschenkovskaja 3, Kyiv 01601, Ukraine

E-mail address: andrew@bakan.kiev.ua