

## ON SYZYGIES OF SEGRE EMBEDDINGS

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**ABSTRACT.** We study the syzygies of the ideals of the Segre embeddings. Let  $d \in \mathbf{N}$ ,  $d \geq 3$ ; we prove that the line bundle  $\mathcal{O}(1, \dots, 1)$  on the  $P^1 \times \dots \times P^1$  ( $d$  copies) satisfies Property  $N_p$  of Green-Lazarsfeld if and only if  $p \leq 3$ . Besides we prove that if we have a projective variety not satisfying Property  $N_p$  for some  $p$ , then the product of it with any other projective variety does not satisfy Property  $N_p$ . From this we also deduce other corollaries about syzygies of Segre embeddings.

### 1. INTRODUCTION

Let  $L$  be a very ample line bundle on a smooth complex projective variety  $Y$  and let  $\varphi_L : Y \rightarrow \mathbf{P}(H^0(Y, L)^*)$  be the map associated to  $L$ . We recall the definition of Property  $N_p$  of Green-Lazarsfeld, studied for the first time by Green in [Gr1-2] (see also [G-L], [Gr3]):

*Let  $Y$  be a smooth complex projective variety and let  $L$  be a very ample line bundle on  $Y$  defining an embedding  $\varphi_L : Y \hookrightarrow \mathbf{P} = \mathbf{P}(H^0(Y, L)^*)$ ; set  $S = S(L) = \text{Sym}^* H^0(L)$ , the homogeneous coordinate ring of the projective space  $\mathbf{P}$ , and consider the graded  $S$ -module  $G = G(L) = \bigoplus_n H^0(Y, L^n)$ ; let  $E_*$*

$$0 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \dots \longrightarrow E_0 \longrightarrow G \longrightarrow 0$$

*be a minimal graded free resolution of  $G$ ; the line bundle  $L$  satisfies Property  $N_p$  ( $p \in \mathbf{N}$ ) if and only if*

$$\begin{aligned} E_0 &= S, \\ E_i &= \bigoplus S(-i-1) \quad \text{for } 1 \leq i \leq p. \end{aligned}$$

(Thus  $L$  satisfies Property  $N_0$  if and only if  $Y \subset \mathbf{P}(H^0(L)^*)$  is projectively normal, i.e.  $L$  is normally generated;  $L$  satisfies Property  $N_1$  if and only if  $L$  satisfies Property  $N_0$  and the homogeneous ideal  $I$  of  $Y \subset \mathbf{P}(H^0(L)^*)$  is generated by quadrics;  $L$  satisfies Property  $N_2$  if and only if  $L$  satisfies Property  $N_1$  and the module of syzygies among quadratic generators  $Q_i \in I$  is spanned by relations of the form  $\sum L_i Q_i = 0$ , where  $L_i$  are linear polynomials; and so on.)

Now let  $L = \mathcal{O}_{\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_d}}(a_1, \dots, a_d)$ , where  $d, a_1, \dots, a_d, n_1, \dots, n_d$  are positive integers. Among the papers on syzygies in this case we quote [B-M], [Gr1-2], [O-P], [J-P-W], [G-P], [Las], and [P-W]. In the first one, the authors examined the cases in which the resolution is “pure”, i.e. the minimal generators of each module of

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syzygies have the same degree. We quote the following results from the other papers:

Case  $d = 1$ , i.e. the case of the Veronese embedding:

**Theorem 1** (Green [Gr1-2]). *Let  $a$  be a positive integer. The line bundle  $\mathcal{O}_{\mathbf{P}^n}(a)$  satisfies Property  $N_a$ .*

**Theorem 2** (Ottaviani-Paoletti [O-P]). *If  $n \geq 2$ ,  $a \geq 3$  and the bundle  $\mathcal{O}_{\mathbf{P}^n}(a)$  satisfies Property  $N_p$ , then  $p \leq 3a - 3$ .*

**Theorem 3** (Josefiak-Pragacz-Weyman [J-P-W]). *The bundle  $\mathcal{O}_{\mathbf{P}^n}(2)$  satisfies Property  $N_p$  if and only if  $p \leq 5$  when  $n \geq 3$  and for all  $p$  when  $n = 2$ .*

(See [O-P] for a more complete bibliography.)

Case  $d = 2$ :

**Theorem 4** (Gallego-Purnapranja [G-P]). *Let  $a, b \geq 2$ . The line bundle  $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(a, b)$  satisfies Property  $N_p$  if and only if  $p \leq 2a + 2b - 3$ .*

**Theorem 5** (Lascoux-Pragacz-Weymann [Las], [P-W]). *Let  $n_1, n_2 \geq 2$ . The line bundle  $\mathcal{O}_{\mathbf{P}^{n_1} \times \mathbf{P}^{n_2}}(1, 1)$  satisfies Property  $N_p$  if and only if  $p \leq 3$ .*

Here we consider  $\mathcal{O}(1, \dots, 1)$  on  $\mathbf{P}^1 \times \dots \times \mathbf{P}^1$  ( $d$  times, for any  $d$ ). We prove (Section 2):

**Theorem 6.** *The line bundle  $\mathcal{O}(1, \dots, 1)$  on  $\mathbf{P}^1 \times \dots \times \mathbf{P}^1$  ( $d$  times) satisfies Property  $N_3$  for any  $d$ .*

Besides we prove (Section 3):

**Proposition 7.** *Let  $X$  and  $Y$  be two projective varieties and let  $L$  be a line bundle on  $X$  and  $M$  a line bundle on  $Y$ . Let  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  be the canonical projections. Suppose  $L$  and  $M$  satisfy Property  $N_1$ . Let  $p \geq 2$ . If  $L$  does not satisfy Property  $N_p$ , then  $\pi_X^*L \otimes \pi_Y^*M$  does not satisfy Property  $N_p$ , either.*

**Corollary 8.** *Let  $a_1, \dots, a_d$  be positive integers with  $a_1 \leq a_2 \leq \dots \leq a_d$ . Suppose  $k = \max\{i \mid a_i = 1\}$ . If  $k \geq 3$  the line bundle  $\mathcal{O}_{\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_d}}(a_1, \dots, a_d)$  does not satisfy Property  $N_4$  and if  $d - k \geq 2$  it does not satisfy Property  $N_{2a_{k+1} + 2a_{k+2} - 2}$ .*

In particular, from Corollary 8 and Theorem 6, we have:

**Corollary 9.** *Let  $d \geq 3$ . The line bundle  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies Property  $N_p$  if and only if  $p \leq 3$ .*

## 2. PROOF OF THEOREM 6

First we have to recall some facts on toric ideals from [St].

Let  $k \in \mathbf{N}$ . Let  $A = \{a_1, \dots, a_n\}$  be a subset of  $\mathbf{Z}^k$ . The toric ideal  $\mathcal{I}_A$  is defined as the ideal in  $\mathbf{C}[x_1, \dots, x_n]$  generated as vector space by the binomials

$$x_1^{u_1} \dots x_n^{u_n} - x_1^{v_1} \dots x_n^{v_n}$$

for  $(u_1, \dots, u_n), (v_1, \dots, v_n) \in \mathbf{N}^n$ , with  $\sum_{i=1, \dots, n} u_i a_i = \sum_{i=1, \dots, n} v_i a_i$ .

We have that  $\mathcal{I}_A$  is homogeneous if and only if  $\exists \omega \in \mathbf{Q}^k$  s.t.  $\omega \cdot a_i = 1 \forall i = 1, \dots, n$ ; the rings  $\mathbf{C}[x_1, \dots, x_n]$  and  $\mathbf{C}[x_1, \dots, x_n]/\mathcal{I}_A$  are multigraded by  $\mathbf{N}A$  via  $\deg x_i = a_i$ ; the element  $x_1^{u_1} \dots x_n^{u_n}$  has multidegree  $b = \sum_i u_i a_i \in \mathbf{N}A$  and degree  $\sum_i u_i = b \cdot \omega$ ; we define  $\deg b = b \cdot \omega$ .

For each  $b \in \mathbf{NA}$ , let  $\Delta_b$  be the simplicial complex on the set  $\{1, \dots, n\}$  defined as follows:

$$\Delta_b = \{F \subset \{1, \dots, n\} : b - \sum_{i \in F} a_i \in \mathbf{NA}\}$$

(thus, by identifying  $\{1, \dots, n\}$  with  $A$ , we have:

$$\Delta_b = \bigcup_{k \in \mathbf{N}, a_{i_1}, \dots, a_{i_k} \in A, a_{i_1} + \dots + a_{i_k} = b} \langle a_{i_1}, \dots, a_{i_k} \rangle,$$

where  $\langle a_{i_1}, \dots, a_{i_k} \rangle$  is the simplex generated by  $a_{i_1}, \dots, a_{i_k}$ ).

The following theorem studies the syzygies of the ideal  $\mathcal{I}_A$ ; it was proved by Campillo and Marijuan for  $k = 1$  in [C-M] and by Campillo and Pison for general  $k$  and  $j = 0$  in [C-P]; the following more general statement is due to Sturmfels (Theorem 12.12 p. 120 in [St]).

**Theorem 10** (see [St] and also [C-M], [C-P]). *Let  $A = \{a_1, \dots, a_n\}$  be a subset of  $\mathbf{N}^k$  and  $\mathcal{I}_A$  be the associated toric ideal. Let  $0 \rightarrow E_n \rightarrow \dots \rightarrow E_1 \rightarrow E_0 \rightarrow G \rightarrow 0$  be a minimal free resolution of  $G = \mathbf{C}[x_1, \dots, x_n]/\mathcal{I}_A$  on  $\mathbf{C}[x_1, \dots, x_n]$ . Each of the generators of  $E_j$  has a unique multidegree. The number of the generators of multidegree  $b \in \mathbf{NA}$  of  $E_{j+1}$  equals the rank of the  $j$ -th reduced homology group  $\tilde{H}_j(\Delta_b, \mathbf{C})$  of the simplicial complex  $\Delta_b$ .*

*Notation 11.* If  $\alpha$  is a chain in a topological space,  $sp(\alpha)$  will denote the support of  $\alpha$ , i.e. the union of the supports of the simplexes  $\sigma_i$  s.t.  $\alpha = \sum_i c_i \sigma_i$ ,  $c_i \in \mathbf{Z}$ . If  $X$  is a simplicial complex,  $sk^i(X)$  will denote the  $i$ -skeleton of  $X$ .

*Proof of Theorem 6.* If we take  $A = A_d = \{(1, \epsilon_1, \dots, \epsilon_d) | \epsilon_i \in \{0, 1\}\}$ , we have that  $\mathcal{I}_{A_d}$  is the ideal of the Segre embedding of  $\mathbf{P}^1 \times \dots \times \mathbf{P}^1$  ( $d$  times), i.e. the ideal of the embedding of  $\mathbf{P}^1 \times \dots \times \mathbf{P}^1$  ( $d$  times) by the line bundle  $\mathcal{O}(1, \dots, 1)$ . In fact the ideal of the Segre embedding is generated as vector space by the homogeneous equations of the form

$$\prod_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} x_{\epsilon_1, \dots, \epsilon_d}^{v_{\epsilon_1, \dots, \epsilon_d}} - \prod_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} x_{\epsilon_1, \dots, \epsilon_d}^{u_{\epsilon_1, \dots, \epsilon_d}} = 0$$

with  $u_{\epsilon_1, \dots, \epsilon_d}, v_{\epsilon_1, \dots, \epsilon_d} \in \mathbf{N}$  and

$$\sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} v_{\epsilon_1, \dots, \epsilon_d}(\epsilon_1, \dots, \epsilon_d) = \sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} u_{\epsilon_1, \dots, \epsilon_d}(\epsilon_1, \dots, \epsilon_d).$$

The last condition is equivalent to

$$\sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} v_{\epsilon_1, \dots, \epsilon_d}(1, \epsilon_1, \dots, \epsilon_d) = \sum_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} u_{\epsilon_1, \dots, \epsilon_d}(1, \epsilon_1, \dots, \epsilon_d)$$

(the equality of the first coordinate gives the homogeneity).

In this case  $\omega = \omega_d = (1, 0, \dots, 0)$  (0 repeated  $d$  times) and  $n = 2^d$ .

Let  $b \in \mathbf{NA}_d$ ; we have that  $\deg b = (b \cdot \omega) = k$  if and only if  $b$  is the sum of  $k$  (not necessarily distinct) elements of  $A_d$ . By identifying the set  $\{1, \dots, 2^d\}$  with  $A_d$ , we have that, if  $k = \deg b$ ,  $\Delta_b = \bigcup_{a_{i_1}, \dots, a_{i_k} \in A_d, a_{i_1} + \dots + a_{i_k} = b} \langle a_{i_1}, \dots, a_{i_k} \rangle$ ; we say that  $\langle a_{i_1}, \dots, a_{i_k} \rangle$  is a degenerate  $(k - 1)$ -simplex if  $\exists l, m \in \{1, \dots, k\}$  with  $l \neq m$  s.t.  $a_{i_l} = a_{i_m}$ ; thus  $\Delta_b$  is equal to the union of the (possibly degenerate)  $(k - 1)$ -simplexes  $S$  with vertices in  $A_d$  such that the sum of the vertices (with multiplicities) of  $S$  is  $b$ .

By Theorem 10, in order to prove that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies  $N_2$ , we have to prove that  $H_1(\Delta_b) = 0$  for each  $b \in \mathbf{NA}_d$  with  $\deg b \geq 4$ . Analogously in order to prove that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies  $N_3$ , we have to prove that  $H_2(\Delta_b) = 0$  for each  $b \in \mathbf{NA}_d$  with  $\deg b \geq 5$ .

The proof is by induction on  $d$ . Observe that any  $b' \in \mathbf{NA}_{d+1}$  with  $\deg b' = k$  is equal to  $\binom{b}{\varepsilon}$  for some  $b \in \mathbf{NA}_d$  with  $\deg b = k$  and for some  $\varepsilon \in \{0, 1, \dots, k\}$ . Then, in order to prove  $N_2$  we suppose (by induction) that  $H_1(\Delta_b) = 0 \forall b \in \mathbf{NA}_d$  with  $\deg b = k, k \geq 4$ , and we show that  $H_1(\Delta_{\binom{b}{\varepsilon}}) = 0$  for  $\varepsilon \in \{0, \dots, k\}$  and in order to prove  $N_3$  we suppose (by induction) that  $H_2(\Delta_b) = 0 \forall b \in \mathbf{NA}_d$  with  $\deg b = k, k \geq 5$ , and we show that  $H_2(\Delta_{\binom{b}{\varepsilon}}) = 0$  for  $\varepsilon \in \{0, \dots, k\}$ .

Observe that, if  $\varepsilon \in \{0, k\}$  ( $k := \deg b$ ), then obviously  $\Delta_{\binom{b}{\varepsilon}}$  and  $\Delta_b$  are isomorphic; besides  $\Delta_{\binom{b}{k-\varepsilon}}$  is isomorphic to  $\Delta_{\binom{b}{\varepsilon}} \forall \varepsilon \in \{0, \dots, k\}$  (the isomorphism is given by substituting 0 with 1 and 1 with 0 in the last coordinate). Thus we may consider only the cases  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ .

First we need some preliminary notation and lemmas.

*Notation 12.* Let  $S = \langle a_1, \dots, a_k \rangle$  be a (possibly degenerate)  $k-1$ -simplex,  $a_i \in A_d$ . Let  $\varepsilon \in \{0, \dots, k\}$ . We denote

$$S'_\varepsilon = \bigcup_{\substack{(\chi_1, \dots, \chi_k) \text{ s.t. } \chi_j \in \{0,1\} \text{ for } j=1, \dots, k \\ \text{and exactly } \varepsilon \text{ of } \chi_1, \dots, \chi_k \text{ are equal to } 1}} \left\langle \binom{a_1}{\chi_1}, \dots, \binom{a_k}{\chi_k} \right\rangle.$$

**Example 13.** Let  $S = \langle a_1, a_2, a_3, a_4 \rangle$  be a (possibly degenerate) tetrahedron,  $a_i \in A_d$ . The set  $S'_1$  is the union of the four (possibly degenerate) tetrahedrons

$$\begin{aligned} &\left\langle \binom{a_1}{1}, \binom{a_2}{0}, \binom{a_3}{0}, \binom{a_4}{0} \right\rangle, && \left\langle \binom{a_1}{0}, \binom{a_2}{1}, \binom{a_3}{0}, \binom{a_4}{0} \right\rangle, \\ &\left\langle \binom{a_1}{0}, \binom{a_2}{0}, \binom{a_3}{1}, \binom{a_4}{0} \right\rangle, && \left\langle \binom{a_1}{0}, \binom{a_2}{0}, \binom{a_3}{0}, \binom{a_4}{1} \right\rangle. \end{aligned}$$

Thus  $S'_1$  can be obtained from  $S$  by “constructing a tetrahedron on every one of the four faces of  $S$ ” and considering the union of these four tetrahedrons. The set  $S'_2$  is the union of the following six (possibly degenerate) tetrahedrons:

$$\begin{aligned} &\left\langle \binom{a_1}{0}, \binom{a_2}{0}, \binom{a_3}{1}, \binom{a_4}{1} \right\rangle, && \left\langle \binom{a_1}{0}, \binom{a_2}{1}, \binom{a_3}{0}, \binom{a_4}{1} \right\rangle, \\ &\left\langle \binom{a_1}{0}, \binom{a_2}{1}, \binom{a_3}{1}, \binom{a_4}{0} \right\rangle, && \left\langle \binom{a_1}{1}, \binom{a_2}{0}, \binom{a_3}{0}, \binom{a_4}{1} \right\rangle, \\ &\left\langle \binom{a_1}{1}, \binom{a_2}{0}, \binom{a_3}{1}, \binom{a_4}{0} \right\rangle, && \left\langle \binom{a_1}{1}, \binom{a_2}{1}, \binom{a_3}{0}, \binom{a_4}{0} \right\rangle. \end{aligned}$$

Then  $S'_2$  can be obtained from  $S$  by “constructing a tetrahedron on every one of the six edges of  $S$ ” and considering the union of these six tetrahedrons (see Figure 1, representing  $S'_\varepsilon$  in the case  $S$  is not degenerate).

Let  $b \in \mathbf{NA}_d$  with  $\deg b = k$  and  $\varepsilon \in \{0, \dots, k\}$ . Obviously

$$\Delta_{\binom{b}{\varepsilon}} = \bigcup_{S = \langle a_1, \dots, a_k \rangle \text{ with } a_1 + \dots + a_k = b} S'_\varepsilon.$$

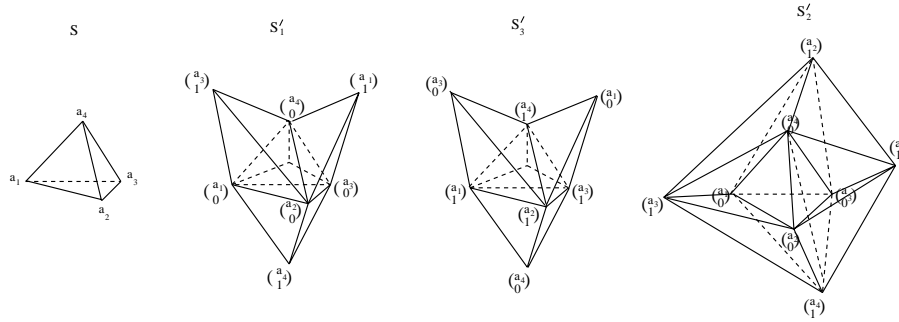


FIGURE 1.

Note to Figure 1. In the representation of  $S'_2$ , for the sake of simplicity, we do not represent the tetrahedrons  $\langle \binom{a_1}{0}, \binom{a_2}{1}, \binom{a_3}{0}, \binom{a_4}{1} \rangle$  and  $\langle \binom{a_1}{1}, \binom{a_2}{0}, \binom{a_3}{1}, \binom{a_4}{0} \rangle$ .

Notation 14. Let  $b \in \mathbf{NA}_d$  with  $\deg b = k$ . For  $l \in \mathbf{N}$ ,  $0 \leq l \leq k - 1$ , let

$$F^l(\Delta_b) = \bigcup_{a_1, \dots, a_k \in A_d \text{ s.t. } a_1 + \dots + a_k = b} \bigcup_{i_0, \dots, i_l \in \{1, \dots, k\}} \left\langle \binom{a_{i_0}}{0}, \dots, \binom{a_{i_l}}{0} \right\rangle.$$

Observe that  $F^l(\Delta_b) \subseteq \Delta_{\binom{b}{\varepsilon}}$  iff  $k - \varepsilon \geq l + 1$ .

The idea of the proof is to consider an  $l$ -cycle (for  $l = 1, 2$ ) in  $\Delta_{\binom{b}{\varepsilon}}$  and to show that it is homologous to an  $l$ -cycle in  $F^l(\Delta_b)$  and then to show that it is homologous to 0 by using that  $H_l(\Delta_b) = 0$ .

Remark 15. Let  $S = \langle a_1, \dots, a_k \rangle$ ,  $a_i \in A_d$ . If  $k \geq 4$  and  $1 \leq \varepsilon \leq k - 2$ , the set  $S'_\varepsilon$  contains the cone with vertex  $\binom{a_l}{1}$  on the border of  $\langle \binom{a_{i_1}}{0}, \binom{a_{i_2}}{0}, \binom{a_{i_3}}{0} \rangle$  for any  $i_1, i_2, i_3, l \in \{1, \dots, k\}$ , with  $l \neq i_j$  for  $j = 1, 2, 3$ . This is true in particular if  $k \geq 4$  and  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ .

If  $k \geq 5$  and  $1 \leq \varepsilon \leq k - 3$ , the set  $S'_\varepsilon$  contains the cone with vertex  $\binom{a_l}{1}$  on the border of  $\langle \binom{a_{i_1}}{0}, \dots, \binom{a_{i_4}}{0} \rangle$  for any  $i_1, \dots, i_4, l \in \{1, \dots, k\}$  with  $l \neq i_j$  for  $j = 1, 2, 3, 4$ . This is true in particular if  $k \geq 5$  and  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ .

Definition 16. For any  $c \in \mathbf{NA}_d$  with  $\deg c = s$  and  $\varepsilon \in \{1, \dots, s\}$ , we define  $R_{c,\varepsilon}$  to be the following set:

$$\bigcup_{\substack{\alpha_1, \dots, \alpha_s \in A_d \\ \text{s.t. } \alpha_1 + \dots + \alpha_s = c}} \bigcup_{\substack{i_1, \dots, i_{s-1} \in \{1, \dots, s\} \\ i_l \neq i_m}} \left\langle \binom{\alpha_{i_1}}{1}, \dots, \binom{\alpha_{i_{\varepsilon-1}}}{1}, \binom{\alpha_{i_\varepsilon}}{0}, \dots, \binom{\alpha_{i_{s-1}}}{0} \right\rangle.$$

Lemma 17. Let  $c \in \mathbf{NA}_d$  with  $\deg c = s$ . We have that  $\tilde{H}_i(\Delta_{\binom{c}{\varepsilon-1}}) = 0$  implies  $\tilde{H}_i(R_{c,\varepsilon}) = 0$  if we are in one of the following cases: a)  $i = 0$ ,  $s \geq 3$ ,  $\varepsilon \in \{1, \dots, \lfloor \frac{s+1}{2} \rfloor\}$ ; b)  $i = 1$ ,  $s \geq 4$ ,  $\varepsilon \in \{1, \dots, \lfloor \frac{s+1}{2} \rfloor\}$ .

Proof. Observe that  $R_{c,\varepsilon} \subseteq \Delta_{\binom{c}{\varepsilon-1}}$ . Since  $\tilde{H}_i(\Delta_{\binom{c}{\varepsilon-1}}) = 0$ , we have

$$\tilde{H}_i(sk^{i+1}(\Delta_{\binom{c}{\varepsilon-1}})) = 0.$$

Obviously  $sk^{i+1}(R_{c,\varepsilon}) \subseteq sk^{i+1}(\Delta_{\binom{c}{\varepsilon-1}})$ . We want to show  $\tilde{H}_i(sk^{i+1}(R_{c,\varepsilon})) = 0$ . Let  $\beta$  be an  $i$ -cycle in  $sk^{i+1}(R_{c,\varepsilon})$ . Since  $\tilde{H}_i(sk^{i+1}(\Delta_{\binom{c}{\varepsilon-1}})) = 0$ , there exists an

$(i + 1)$ -chain  $\eta$  in  $sk^{i+1}(\Delta_{(\frac{c}{\varepsilon-1})})$  s.t.  $\partial\eta = \beta$ . Suppose  $sp(\eta) = \bigcup_j F_j$ , where  $F_j$  are  $(i + 1)$ -simplexes in  $sk^{i+1}(\Delta_{(\frac{c}{\varepsilon-1})})$ ; consider now an  $(i + 1)$ -chain  $\psi$  in  $sk^{i+1}(R_{c,\varepsilon})$  whose support is  $\bigcup_j \hat{F}_j$ , where  $\hat{F}_j = F_j$  if  $F_j \subseteq sk^{i+1}(R_{c,\varepsilon})$  and  $\hat{F}_j$  is a cone on the border of  $F_j$  if  $F_j \not\subseteq sk^{i+1}(R_{c,\varepsilon})$ , in such way that  $\partial\psi = \beta$  (observe that in our cases such cones exist, in fact:  $R_{c,\varepsilon}$  is the union of the (possibly degenerate)  $(s - 2)$ -simplexes “obtained from the (possibly degenerate)  $(s - 1)$ -simplexes of  $\Delta_{(\frac{c}{\varepsilon-1})}$  by taking off a vertex whose last coordinate is 0”; in the case  $i = 0$  one can check that the 1-simplexes s.t. the last coordinate of a vertex and the last coordinate of the other vertex are 1, 1 or 1, 0 are contained in  $R_{c,\varepsilon}$ , while for a 1-simplex  $F$  s.t. the last coordinate of each vertex is 0, there exists a cone,  $\hat{F}$ , on the border of  $F$  with  $\hat{F} \subseteq R_{c,\varepsilon}$ , since  $s \geq 3$ ; analogously case b)). Thus we proved  $\tilde{H}_i(sk^{i+1}(R_{c,\varepsilon})) = 0$ . Thus  $\tilde{H}_i(R_{c,\varepsilon}) = 0$ .  $\square$

**Proof that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  satisfies property  $N_2$ .**

**Lemma 18.** *Let  $b \in \mathbf{N}A_d$ ,  $\deg b = k$ ,  $k \geq 4$  and  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ . Every 1-cycle  $\gamma$  in  $\Delta_{(\frac{b}{\varepsilon})}$  is homologous to a 1-cycle in  $F^1(\Delta_b)$  (which is  $\subseteq \Delta_{(\frac{b}{\varepsilon})}$  since  $k - \varepsilon \geq 2$ ).*

*Proof.* Obviously we can suppose  $sp(\gamma) \subseteq sk^1(\Delta_{(\frac{b}{\varepsilon})})$ . The proof is by induction on the cardinality of  $(sp(\gamma) \cap sk^0(\Delta_{(\frac{b}{\varepsilon})})) - F^1(\Delta_b)$ , i.e. we will prove that  $\gamma$  is homologous to a 1-cycle  $\tilde{\gamma}$  s.t.

$$\#((sp(\tilde{\gamma}) \cap sk^0(\Delta_{(\frac{b}{\varepsilon})})) - F^1(\Delta_b)) < \#((sp(\gamma) \cap sk^0(\Delta_{(\frac{b}{\varepsilon})})) - F^1(\Delta_b)).$$

First we remark that if  $P, \binom{a}{1} \in sk^0(\Delta_{(\frac{b}{\varepsilon})})$  and  $\langle P, \binom{a}{1} \rangle \subseteq \Delta_{(\frac{b}{\varepsilon})}$ , then  $P \in R_{b-a,\varepsilon}$  (\*).

In fact,  $\langle P, \binom{a}{1} \rangle \subseteq \Delta_{(\frac{b}{\varepsilon})}$ , then  $P \in \Delta_{(\frac{b-a}{\varepsilon-1})}$ ; we recall that  $R_{b-a,\varepsilon}$  is

$$\bigcup_{\substack{\alpha_1, \dots, \alpha_{k-1} \in A_d \\ \text{s.t. } \alpha_1 + \dots + \alpha_{k-1} = b-a}} \bigcup_{\substack{i_1, \dots, i_{k-2} \in \{1, \dots, k-1\} \\ i_l \neq i_m}} \left\langle \binom{\alpha_{i_1}}{1}, \dots, \binom{\alpha_{i_{\varepsilon-1}}}{1}, \binom{\alpha_{i_\varepsilon}}{0}, \dots, \binom{\alpha_{i_{k-2}}}{0} \right\rangle,$$

i.e.  $R_{b-a,\varepsilon}$  is the union of the (possibly degenerate)  $(k - 3)$ -simplexes “obtained from the (possibly degenerate)  $(k - 2)$ -simplexes of  $\Delta_{(\frac{b-a}{\varepsilon-1})}$  by taking off a vertex whose last coordinate is 0”; then, if the last coordinate of  $P$  is 1, we may conclude at once that  $P \in R_{b-a,\varepsilon}$ ; also if the last coordinate of  $P$  is 0, we may conclude that  $P \in R_{b-a,\varepsilon}$ , because the number of the vertices whose last coordinate is 0 in a (possibly degenerate)  $(k - 2)$ -simplex of  $\Delta_{(\frac{b-a}{\varepsilon-1})}$  is  $k - 1 - (\varepsilon - 1) \geq 2$ .

Now let  $\binom{a}{1} \in (sp(\gamma) \cap sk^0(\Delta_{(\frac{b}{\varepsilon})})) - F^1(\Delta_b)$ , ( $a \in A_d$ ). Intuitively we want to modify slightly  $\gamma$  to obtain a homologous 1-cycle  $\tilde{\gamma}$  passing through  $\binom{a}{0}$  instead of  $\binom{a}{1}$ .

Let  $\mathcal{S}_{\binom{a}{1}}$  be the set of all 1-simplexes  $\tau$  of  $\gamma$  s.t.  $sp(\tau) \ni \binom{a}{1}$ . We can suppose that every 1-simplex  $\tau \in \mathcal{S}_{\binom{a}{1}}$  be s.t.  $sp(\tau)$  be a 1-simplex of  $\Delta_{(\frac{b}{\varepsilon})}$  and  $\tau : [0, 1] \rightarrow \Delta_{(\frac{b}{\varepsilon})}$  be injective.

For  $\tau \in \mathcal{S}_{\binom{a}{1}}$ , let  $\tau' : [0, 1] \rightarrow \Delta_{(\frac{b}{\varepsilon})}$  be a 1-simplex s.t.  $\tau'(\epsilon) = \tau(\epsilon)$  if  $\tau(\epsilon) \neq \binom{a}{1}$  and  $\tau'(\epsilon) = \binom{a}{0}$  if  $\tau(\epsilon) = \binom{a}{1}$ , for  $\epsilon \in \{0, 1\}$ . Let  $\alpha$  be the 1 cycle  $\sum_{\tau \in \mathcal{S}_{\binom{a}{1}}} (-\tau + \tau')$ .

By remark (\*), we have that  $sp(\alpha) \subseteq C$ , where  $C$  is the union of the two cones  $\langle \binom{a}{1}, R_{b-a,\varepsilon} \rangle$  and  $\langle \binom{a}{0}, R_{b-a,\varepsilon} \rangle$ . Observe that  $C \subseteq \Delta_{\binom{b}{\varepsilon}}$ . We state that  $H_1(C) = 0$ : by Theorem 10, since  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies Property  $N_1 \forall d$ , we have  $\tilde{H}_0(\Delta_g) = 0 \forall g \in \mathbf{NA}_d$  with  $\deg g \geq 3, \forall d$  (this can easily be proved directly without using that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  satisfies Property  $N_1$ ); then  $\tilde{H}_0(\Delta_{\binom{b-a}{\varepsilon-1}}) = 0$ ; thus  $\tilde{H}_0(R_{b-a,\varepsilon}) = 0$  by Lemma 17; we have  $\tilde{H}_i(C) = \tilde{H}_{i-1}(R_{b-a,\varepsilon})$ ; thus  $H_1(C) = \tilde{H}_0(R_{b-a,\varepsilon}) = 0$ . Thus we have that  $\alpha$  is homologous to 0. Thus  $\gamma$  is homologous to  $\gamma + \alpha$ .

Obviously the support of  $\tilde{\gamma} := \gamma + \alpha$  can be obtained from  $sp(\gamma)$  by substituting  $sp(\tau)$  with  $sp(\tau') \forall \tau \in \mathcal{S}_{\binom{a}{1}}$ . Then

$$\#((sp(\tilde{\gamma}) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^1(\Delta_b)) < \#((sp(\gamma) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^1(\Delta_b));$$

thus we conclude the proof of Lemma 18. □

In order to prove that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies  $N_2$  for any  $d$ , we suppose (by induction) that  $H_1(\Delta_b) = 0 \forall b \in \mathbf{NA}_d$  with  $\deg b = k, k \geq 4$ , and we show that  $H_1(\Delta_{\binom{b}{\varepsilon}}) = 0$  for  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ .

Cases  $\varepsilon \leq k - 3$ . We know that every 1-cycle  $\gamma$  in  $\Delta_{\binom{b}{\varepsilon}}$  is homologous to a 1-cycle in  $F^1(\Delta_b)$  by Lemma 18. Thus, since  $F^2(\Delta_b) \subseteq \Delta_{\binom{b}{\varepsilon}}$  and  $H_1(F^2(\Delta_b)) = 0$  (because, by induction hypothesis,  $H_1(\Delta_b) = 0$ ), we have that  $H_1(\Delta_{\binom{b}{\varepsilon}}) = 0$ .

Cases  $\varepsilon > k - 3$ . These cases are slightly more difficult. By Lemma 18 every 1-cycle  $\gamma$  in  $\Delta_{\binom{b}{\varepsilon}}$  is homologous to a 1-cycle  $\gamma'$  in  $F^1(\Delta_b)$ . But in these cases we have not the inclusion  $F^2(\Delta_b) \subseteq \Delta_{\binom{b}{\varepsilon}}$ , thus we have to conclude the proof in another way.

Since  $H_1(F^2(\Delta_b)) = 0$ , there exists a 2-chain  $\mu$  in  $F^2(\Delta_b)$  s.t.  $\partial\mu = \gamma'$ . Let  $sp(\mu) = \bigcup_i F_i, F_i$  triangles in  $F^2(\Delta_b)$ . Consider a 2-chain  $\psi$  in  $\Delta_{\binom{b}{\varepsilon}}$  whose support is  $\bigcup_i \hat{F}_i$ , where  $\hat{F}_i$  is a cone  $\subseteq \Delta_{\binom{b}{\varepsilon}}$  on the border of  $F_i$  (there exists by Remark 15), in such way that  $\partial\psi = \gamma'$ ; thus  $[\gamma'] = 0$  in  $H_1(\Delta_{\binom{b}{\varepsilon}})$ , thus  $[\gamma] = 0$  in  $H_1(\Delta_{\binom{b}{\varepsilon}})$ . Thus  $H_1(\Delta_{\binom{b}{\varepsilon}}) = 0$ .

**Proof that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  satisfies property  $N_3$ .**

**Lemma 19.** *Let  $b \in \mathbf{NA}_d$  with  $\deg b = k$  and  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ . If  $k \geq 5$ , every 2-cycle  $\mu$  in  $\Delta_{\binom{b}{\varepsilon}}$  is homologous to a 2-cycle in  $F^2(\Delta_b)$  (which is  $\subseteq \Delta_{\binom{b}{\varepsilon}}$  since  $k - \varepsilon \geq 3$ ).*

*Proof.* Obviously we can suppose that  $sp(\mu) \subseteq sk^2(\Delta_{\binom{b}{\varepsilon}})$ . The proof is by induction on the cardinality of  $(sp(\mu) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^2(\Delta_b)$ , i.e. we will prove that  $\mu$  is homologous to a 2-cycle  $\tilde{\mu}$  s.t.  $\#((sp(\tilde{\mu}) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^2(\Delta_b)) < \#((sp(\mu) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^2(\Delta_b))$ .

We remark that if  $P, Q, \binom{a}{1} \in sk^0(\Delta_{\binom{b}{\varepsilon}})$  and  $\langle P, Q, \binom{a}{1} \rangle \subseteq \Delta_{\binom{b}{\varepsilon}}$ , then  $\langle P, Q \rangle \subseteq R_{b-a,\varepsilon}$ . (\*)

In fact  $\langle P, Q, \binom{a}{1} \rangle \subseteq \Delta_{\binom{b}{\varepsilon}}$ ; then  $\langle P, Q \rangle \subseteq \Delta_{\binom{b-a}{\varepsilon-1}}$ . Since  $R_{b-a,\varepsilon}$  is the union of the (possibly degenerate)  $(k-3)$ -simplexes “obtained from the (possibly degenerate)  $(k-2)$ -simplexes of  $\Delta_{\binom{b-a}{\varepsilon-1}}$  by taking off a vertex whose last coordinate is 0” and

since the number of the vertices whose last coordinate is 0 in a (possibly degenerate)  $(k - 2)$ -simplex of  $\Delta_{\binom{b-a}{\varepsilon-1}}$  is  $k - 1 - (\varepsilon - 1) \geq 3$ , we have  $\langle P, Q \rangle \subseteq R_{b-a,\varepsilon}$ .

Let  $\mathcal{S}_{\binom{a}{1}}$  be the set of all 2-simplexes  $\tau : \langle (0, 0), (0, 1), (1, 0) \rangle \rightarrow \Delta_{\binom{b}{\varepsilon}}$  of  $\mu$  s.t.  $sp(\tau) \ni \binom{a}{1}$ . We can suppose that every 2-simplex  $\tau \in \mathcal{S}_{\binom{a}{1}}$  be s.t.  $sp(\tau)$  be a 2-simplex of  $\Delta_{\binom{b}{\varepsilon}}$ ,  $\tau(\epsilon) \in sk^0(\Delta_{\binom{b}{\varepsilon}})$  for  $\epsilon \in \{(0, 0), (0, 1), (1, 0)\}$  and  $\tau : \langle (0, 0), (0, 1), (1, 0) \rangle \rightarrow \Delta_{\binom{b}{\varepsilon}}$  be injective.

For  $\tau \in \mathcal{S}_{\binom{a}{1}}$  let  $\tau' : \langle (0, 0), (0, 1), (1, 0) \rangle \rightarrow \Delta_{\binom{b}{\varepsilon}}$  be a 2-simplex s.t.  $\tau'(\epsilon) = \tau(\epsilon)$  if  $\tau(\epsilon) \neq \binom{a}{1}$  and  $\tau'(\epsilon) = \binom{a}{1}$  if  $\tau(\epsilon) = \binom{a}{1}$  for  $\epsilon \in \{(0, 0), (0, 1), (1, 0)\}$ . Let  $\alpha$  be the 2-cycle  $\sum_{\tau \in \mathcal{S}_{\binom{a}{1}}} (-\tau + \tau')$ .

By remark (\*), we have that  $sp(\alpha) \subseteq C$ , where  $C$  is the union of the two cones  $\langle \binom{a}{1}, R_{b-a,\varepsilon} \rangle$  and  $\langle \binom{a}{0}, R_{b-a,\varepsilon} \rangle$ . Observe that  $C \subseteq \Delta_{\binom{b}{\varepsilon}}$ . We state that  $H_2(C) = 0$ : we have already proved that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  satisfies Property  $N_2$  i.e.  $H_1(\Delta_g) = 0 \forall g$  with  $\deg g \geq 4$ ; thus  $H_1(\Delta_{\binom{b-a}{\varepsilon-1}}) = 0$ ; then  $H_1(R_{b-a,\varepsilon}) = 0$  by Lemma 17; we have  $\tilde{H}_i(C) = \tilde{H}_{i-1}(R_{b-a,\varepsilon})$ ; thus  $H_2(C) = H_1(R_{b-a,\varepsilon}) = 0$ . Thus we have that  $\alpha$  is homologous to 0. Thus  $\mu$  is homologous to  $\mu + \alpha$ .

The support of the cycle  $\tilde{\mu} := \mu + \alpha$  can be obtained from  $sp(\mu)$  by substituting  $sp(\tau)$  with  $sp(\tau') \forall \tau \in \mathcal{S}_{\binom{a}{1}}$ . Then

$$\#((sp(\tilde{\mu}) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^2(\Delta_b)) < \#((sp(\mu) \cap sk^0(\Delta_{\binom{b}{\varepsilon}})) - F^2(\Delta_b));$$

thus we conclude the proof of Lemma 19. □

In order to prove that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) satisfies  $N_3$  for any  $d$ , we suppose (by induction) that  $H_2(\Delta_b) = 0 \forall b \in \mathbf{NA}_d$  with  $\deg b = k, k \geq 5$  and we show that  $H_2(\Delta_{\binom{b}{\varepsilon}}) = 0$  for  $\varepsilon \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ .

Cases  $\varepsilon \leq k - 4$ . We have that every 2-cycle  $\mu$  in  $\Delta_{\binom{b}{\varepsilon}}$  is homologous to a 2-cycle  $\tilde{\mu}$  in  $F^2(\Delta_b)$  by Lemma 19. Since  $H_2(F^3(\Delta_b)) = 0$  (because, by induction hypothesis,  $H_2(\Delta_b) = 0$ ), we have that  $[\tilde{\mu}] = 0$  in  $H_2(F^3(\Delta_b)) = 0$ . Since in these cases  $F^3(\Delta_b) \subseteq \Delta_{\binom{b}{\varepsilon}}$ , we may conclude that  $[\mu] = [\tilde{\mu}] = 0$  in  $H_2(\Delta_{\binom{b}{\varepsilon}})$ , thus  $H_2(\Delta_{\binom{b}{\varepsilon}}) = 0$ .

Cases  $\varepsilon > k - 4$ . We have that every 2-cycle  $\mu$  in  $\Delta_{\binom{b}{\varepsilon}}$  is homologous to a 2-cycle  $\tilde{\mu}$  in  $F^2(\Delta_b)$  by Lemma 19. Since  $H_2(F^3(\Delta_b)) = 0$  (because  $H_2(\Delta_b) = 0$ ), we have that  $[\tilde{\mu}] = 0$  in  $H_2(F^3(\Delta_b)) = 0$ . But in these cases we have not the inclusion  $F^3(\Delta_b) \subseteq \Delta_{\binom{b}{\varepsilon}}$ , thus we may not conclude at once. Since  $H_2(F^3(\Delta_b)) = 0$ , there exists a 3-chain  $\nu$  in  $F^3(\Delta_b)$  s.t.  $\partial\nu = \tilde{\mu}$ . We have that  $sp(\nu) = \bigcup_i F_i, F_i$  tetrahedrons in  $F^3(\Delta_b)$ . Consider a 3-chain  $\psi$  in  $\Delta_{\binom{b}{\varepsilon}}$  whose support is  $\bigcup_i \hat{F}_i$ , where  $\hat{F}_i$  is a cone  $\subseteq \Delta_{\binom{b}{\varepsilon}}$  on the border of  $F_i$  (there exists by Remark 15), in such a way that  $\partial\psi = \tilde{\mu}$ ; thus  $[\tilde{\mu}] = 0$  in  $H_2(\Delta_{\binom{b}{\varepsilon}})$ , thus  $[\mu] = 0$  in  $H_2(\Delta_{\binom{b}{\varepsilon}})$ . Then we may conclude that  $H_2(\Delta_{\binom{b}{\varepsilon}}) = 0$ .

This completes the proof of Theorem 6. □

### 3. PROOF OF PROPOSITION 7

Let  $X$  and  $Y$  be two projective varieties and  $L$  a very ample line bundle on  $X$  and  $M$  a very ample line bundle on  $Y$ . Let  $\{\sigma_0, \dots, \sigma_k\}$  be a basis of  $H^0(X, L)$  and



let  $\{s_0, \dots, s_l\}$  be a basis of  $H^0(Y, M)$ ; we can suppose  $\exists \bar{y} \in Y$  s.t.  $s_0(\bar{y}) \neq 0$ ,  $s_j(\bar{y}) = 0$  for  $j \neq 0$ ; let  $t_{i,j}$  be the coordinates corresponding to  $\{\sigma_i \otimes s_j\}_{i,j}$  of the embedding of  $X \times Y$  by  $\pi_X^* L \otimes \pi_Y^* M$  (where  $\pi$  is the projection on  $\cdot$ ) and let  $t_i$  be the coordinates corresponding to  $\{\sigma_0, \dots, \sigma_k\}$  of the embedding of  $X$  by  $L$ .

*Remark 20.* By setting  $t_{i,j} = 0$  for  $j \neq 0$  in an equation of  $X \times Y$  and then taking off the last index (a 0) of each variable, we get an equation of  $X$  (to prove this, use  $\bar{y}$ ).

*Remark 21.* Let  $M$  be a graded module on  $\mathbf{C}[x_1, \dots, x_n]$  with a minimal set of generators of degree  $s$ ; then a subset of elements of degree  $s$  of  $M$  can be extended to a minimal set of generators if and only if these elements are linearly independent on  $\mathbf{C}$ .

*Proof of Proposition 7.* Suppose  $L$  satisfies Property  $N_{p-1}$  but not  $N_p$ . We want to show  $\pi_X^* L \otimes \pi_Y^* M$  does not satisfy Property  $N_p$ ; we can suppose  $\pi_X^* L \otimes \pi_Y^* M$  satisfies Property  $N_{p-1}$ . Let  $l_m$  and  $q_m$  be the ranks of the  $m$ -module of a minimal free graded resolution respectively of  $G(L)$  and of  $G(\pi_X^* L \otimes \pi_Y^* M)$ . Let  $\{g_j^m\}_{j=1, \dots, l_m}$  be a minimal set of generators of the  $m$ -module  $E_m$  of a minimal resolution of  $G(L)$ ,

$$\dots \rightarrow E_m \rightarrow E_{m-1} \rightarrow \dots \rightarrow E_0 \rightarrow G(L) \rightarrow 0.$$

Since  $L$  satisfies Property  $N_{p-1}$  but not  $N_p$ , there exists a syzygy  $S$  of  $(g_1^{p-1}, \dots, g_{l_{p-1}}^{p-1})$ , s.t.  $S$  is not generated by linear syzygies of  $(g_1^{p-1}, \dots, g_{l_{p-1}}^{p-1})$ . Add a 0 to the indices of the variables appearing in  $S$  and call  $\tilde{S}$  the so-obtained vector of polynomials; let  $\tilde{S}' = (\tilde{S}, 0, \dots, 0)$  with 0 repeated  $q_{p-1} - l_{p-1}$  times.

Obviously by adding a 0 to the indices of each variable appearing in the equations of  $X$ , we get equations of  $X \times Y$  and by adding a 0 to the indices of every variable appearing in the syzygies of  $X$  we get syzygies of  $X \times Y$ .

Add a 0 to the indices of the variables appearing in  $g_j^m$  and call  $\tilde{g}_j^m$  the so-obtained vector of polynomials for  $j = 1, \dots, l_m$ ; set  $f_j^1 = \tilde{g}_j^1$  for  $j = 1, \dots, l_1$  and  $f_j^m = (\tilde{g}_j^m, 0, \dots, 0)$  (0 repeated  $q_{m-1} - l_{m-1}$  times) for  $j = 1, \dots, l_m$  and  $2 \leq m \leq p-1$ ;  $f_j^m$  for  $j = 1, \dots, l_m$  are vectors of linear polynomials for  $2 \leq m \leq p-1$  and they are quadratic if  $m = 1$ . Thus, by induction on  $m$  and by Remark 21, one can extend this set to a minimal set of generators  $\{f_j^m\}_{j=1, \dots, q_m}$ , of the  $m$ -module of a minimal resolution of  $G(\pi_X^* L \otimes \pi_Y^* M)$  for  $m \leq p-1$  (we recall that we supposed  $\pi_X^* L \otimes \pi_Y^* M$  satisfies Property  $N_{p-1}$ ); we can do it in such way that, when we set  $t_{i,j} = 0$  for  $j \neq 0$ , we have that  $f_j^1$  is zero for  $j = l_1 + 1, \dots, q_1$  and the  $r$ -th coordinate of  $f_j^m$  is zero for  $r \leq l_{m-1}$  and  $j = l_m + 1, \dots, q_m$  (we can prove this by induction on  $m$ , by using Remark 20 for the case  $m = 1$ : it is sufficient to subtract linear combination of  $f_j^m$  for  $j = 1, \dots, l_m$  to  $f_j^m$  for  $j = l_m + 1, \dots, q_m$ ).

Obviously  $\tilde{S}'$  is a syzygy of  $(f_1^{p-1}, \dots, f_{q_{p-1}}^{p-1})$ .

If  $\pi_X^* L \otimes \pi_Y^* M$  satisfies Property  $N_p$ , then  $\tilde{S}'$  would be generated by linear syzygies of  $(f_1^{p-1}, \dots, f_{q_{p-1}}^{p-1})$ .

We state that  $\tilde{S}'$  cannot be generated by linear syzygies of  $(f_1^{p-1}, \dots, f_{q_{p-1}}^{p-1})$ . In fact, if it were, say  $\tilde{S}' = \sum_{\alpha} S_{\alpha}$  ( $S_{\alpha}$  linear syzygies of  $(f_1^{p-1}, \dots, f_{q_{p-1}}^{p-1})$ ), we set  $t_{i,j} = 0$  for  $j \neq 0$  in each member of the equality  $\tilde{S}' = \sum_{\alpha} S_{\alpha}$  and, by taking off the last index (a 0) of every variable and considering only the first  $l_{p-1}$  coordinates of  $S$  and  $S_{\alpha}$ , one would obtain that  $S$  would be generated by linear syzygies of

$(g_1^{p-1}, \dots, g_{l_{p-1}}^{p-1})$  (observe that by setting  $t_{i,j} = 0$  for  $j \neq 0$  in  $S_\alpha$  and taking the first  $l_{p-1}$  coordinates, we get a syzygy of  $(f_1^{p-1}, \dots, f_{l_{p-1}}^{p-1})$ ).

But  $S$  cannot be generated by linear syzygies by assumption.  $\square$

The line bundle  $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}(1, 1, 1)$  does not satisfy Property  $N_4$ ; precisely the resolution, with the notation of Introduction, is

$$0 \rightarrow S(-6) \rightarrow S(-4)^9 \rightarrow S(-3)^{16} \rightarrow S(-2)^9 \rightarrow S \rightarrow G \rightarrow 0.$$

This has been proved by Barcanescu and Manolache in [B-M] and can be seen also by using the program Macaulay [B-S] (to see only that  $\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1}(1, 1, 1)$  does not satisfy Property  $N_4$ , it is sufficient to use the autoduality of the resolution; see [B-M]).

From this and from Proposition 7 we deduce that  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(1, \dots, 1)$  ( $d$  times) does not satisfy Property  $N_4$  for  $d \geq 3$ . By also using Gallego-Purnapranja's Theorem 4, we deduce that, if  $a_1, \dots, a_d$  are integer numbers with  $a_1 \leq a_2 \leq \dots \leq a_d$  and  $a_1 = \dots = a_k = 1$ , the line bundle  $\mathcal{O}_{\mathbf{P}^1 \times \dots \times \mathbf{P}^1}(a_1, \dots, a_d)$  does not satisfy Property  $N_4$  if  $k \geq 3$  and it does not satisfy Property  $N_{2a_{k+1}+2a_{k+2}-2}$  if  $d - k \geq 2$ .

With the same argument as in the Remark in part II, Section 2 of [Gr1-2] we deduce Corollary 8.

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