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OPERATOR WEAK AMENABILITY OF THE FOURIER ALGEBRA

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ABSTRACT. We show that for any locally compact group G, the Fourier algebra $\mathrm{A}(G)$ is operator weakly amenable.

Let G be a locally compact group. It is shown by Johnson in [16] (and by Despić and Ghahramani in [6]) that the group algebra $\mathrm{L}^1(G)$ is always weakly amenable. It is natural to ask whether the same holds for the Fourier algebra $\mathrm{A}(G)$. In [17] it is shown for $G = \mathrm{SO}(3)$ that $\mathrm{A}(G)$ is not weakly amenable. We note that $\mathrm{A}(G)$ is weakly amenable (and, in fact amenable) whenever G is Abelian, since $\mathrm{A}(G) \cong \mathrm{L}^1(\widehat{G})$ where $\widehat{G}$ is the dual group. Also, $\mathrm{A}(G)$ is weakly amenable whenever the connected component of the identity in G is Abelian [11]. It is conjectured that this characterizes the weak amenability of $\mathrm{A}(G)$.

Since $\mathrm{A}(G)$ is the predual of the von Neumann algebra $\mathrm{VN}(G)$, it admits a natural structure as an operator space. Using this structure, Ruan [23] developed a completely bounded cohomology theory and proved that $\mathrm{A}(G)$ is operator amenable exactly when G is an amenable group. This is analogous to Johnson's result [15] that $\mathrm{L}^1(G)$ is amenable exactly when G is amenable. We note that the natural operator space structure on $\mathrm{L}^1(G)$ as the predual of $\mathrm{L}^\infty(G)$ is such that all bounded maps from $\mathrm{L}^1(G)$ into any operator space are automatically completely bounded. Thus the notions of amenability and operator amenability coincide on $\mathrm{L}^1(G)$, making Ruan's result truly a dual result of Johnson's, in the sense that $\mathrm{A}(G)$ is the dual of $\mathrm{L}^1(G)$ in a way which generalizes Pontryagin's Duality Theorem (see [9]).

The purpose of this note is to show that the natural operator space structure on $\mathrm{A}(G)$ allows us to obtain another analogous result to one for $\mathrm{L}^1(G)$: we show that $\mathrm{A}(G)$ is always operator weakly amenable. We note that this result was obtained in [12], for the case that the connected component of the identity in G is amenable.

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1. Preliminaries

If $\mathcal{X}$ is a Banach space we always let $\mathcal{X}^*$ denote its dual and $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded operators on $\mathcal{X}$. The symbol $\mathcal{H}$ (possibly with a

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subscript) will always denote a Hilbert space and $\mathcal{U}(\mathcal{H})$ will denote the group of unitary operators on $\mathcal{H}$ with the relativized weak operator topology.

Let G be a locally compact group. The *Fourier* and *Fourier-Stieltjes algebras*, $\mathrm{A}(G)$ and $\mathrm{B}(G)$, are defined in [10]. We recall that $\mathrm{B}(G)$ is the space of matrix coefficients of all continuous unitary representations of G; i.e. the space of functions of the form $s \mapsto \langle \pi(s)\xi|\eta\rangle$ where $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a continuous homomorphism and $\xi, \eta \in \mathcal{H}_\pi$. $\mathrm{B}(G)$ is the dual of the enveloping group C*-algebra $\mathrm{C}^*(G)$ via $\langle a, \langle \pi(\cdot)\xi|\eta\rangle\rangle = \langle \pi_*(a)\xi|\eta\rangle$, where $\pi_* : \mathrm{C}^*(G) \to \mathcal{B}(\mathcal{H}_\pi)$ is the representation induced by π. It can be shown that $\mathrm{B}(G)$ is a commutative Banach algebra (under pointwise operations) and $\mathrm{A}(G)$ is the closed ideal in $\mathrm{B}(G)$ generated by compactly supported matrix coefficients.

If π is a continuous unitary representation of G, let A_π be the norm closure of $\mathrm{span}\{\langle \pi(\cdot)\xi|\eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}$ in $\mathrm{B}(G)$. Then, by [1, 2.2], $\mathrm{A}_\pi^* \cong \mathrm{VN}_\pi$, where VN_π is the von Neumann algebra generated by $\pi(G)$. If λ is the left regular representation of G on $\mathrm{L}^2(G)$, then $\mathrm{A}(G) = \mathrm{A}_\lambda$ and we write $\mathrm{VN}(G) = \mathrm{VN}_\lambda$. Given two unitary representations π and σ of G, we let $\pi \oplus \sigma : G \to \mathcal{U}(\mathcal{H}_\pi \oplus \mathcal{H}_\sigma)$ denote their direct sum. If π and σ are disjoint (i.e. there are no subrepresentations π' of π and σ' of σ such that π' is spatially equivalent to σ'), then

$$\mathrm{VN}_{\pi\oplus\sigma} = \mathrm{VN}_\pi \oplus_\infty \mathrm{VN}_\sigma \quad \text{and} \quad \mathrm{A}_{\pi\oplus\sigma} = \mathrm{A}_\pi \oplus_1 \mathrm{A}_\sigma \tag{1.1}$$

by [22, 3.8.10] and [1, 3.13], respectively, where $\oplus_p$ denotes the ℓ^p-direct sum for $p = 1, \infty$.

Our standard reference for operator spaces will be [8]. Given two operator spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{CB}(\mathcal{X}, \mathcal{Y})$ the space of completely bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$ and denote the norm on it by $\|\cdot\|_{cb}$. If $\mathcal{X} = \mathcal{Y}$, we will denote the Banach algebra $\mathcal{CB}(\mathcal{X}, \mathcal{X})$ by $\mathcal{CB}(\mathcal{X})$. Dual spaces will always be given the standard operator dual structure ([8, Sec. 3.2], [3]). Given two operator spaces $\mathcal{X}$ and $\mathcal{Y}$, their direct product with the canonical product operator space structure will be denoted $\mathcal{X} \oplus_\infty \mathcal{Y}$. The direct sum $\mathcal{X} \oplus_1 \mathcal{Y}$ will be given the operator space structure it obtains from being imbedded into $(\mathcal{X}^* \oplus_\infty \mathcal{Y}^*)^*$. The product and sum are denoted by $\mathcal{X} \oplus_{CM} \mathcal{Y}$ and $\mathcal{X} \oplus_{CL} \mathcal{Y}$, respectively, in [7]. If $\mathcal{M}$ is a von Neumann algebra, its predual $\mathcal{M}_*$ will always be given the operator space structure it inherits from being imbedded in the dual $\mathcal{M}^*$. Moreover, $\mathcal{M}$ is then the standard dual of $\mathcal{M}_*$ ([8, 4.2.2], [3]). In particular, each space $\mathrm{A}_\pi \cong (\mathrm{VN}_\pi)_*$ will be endowed with this predual operator space structure.

The projective tensor product of Banach space theory admits an operator space analogue. Given two operator spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote their *operator space projective tensor product* by $\mathcal{X}\widehat{\otimes}\mathcal{Y}$. We will not need the explicit formula for the norm of this tensor product, but note that it is a completion of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$. We will use two important properties of this operator tensor product. First,

$$\left(\mathcal{X}\widehat{\otimes}\mathcal{Y}\right)^* \cong \mathcal{CB}(\mathcal{X}, \mathcal{Y}^*) \quad \text{via} \quad \langle x \otimes y, T\rangle = \langle y, Tx\rangle. \tag{1.2}$$

See [8, 7.1.5] or [4]. This is analogous to the usual dual formula for the Banach space projective tensor product. Second, if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, then

$$\mathcal{M}_*\widehat{\otimes}\mathcal{N}_* \cong \left(\mathcal{M}\bar{\otimes}\mathcal{N}\right)_* \tag{1.3}$$

where $\mathcal{M}\bar{\otimes}\mathcal{N}$ is the von Neumann tensor product of $\mathcal{M}$ and $\mathcal{N}$. See [8, 7.2.4]. In particular, since $\mathrm{VN}(G\times G) \cong \mathrm{VN}(G)\bar{\otimes}\mathrm{VN}(G)$ spatially, via the unitary which implements $\mathrm{L}^2(G)\otimes\mathrm{L}^2(G) \cong \mathrm{L}^2(G\times G)$, we thus have that $\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G) \cong \mathrm{A}(G\times G)$ completely isometrically. This identity holds isometrically for the Banach space projective tensor product only when G is Abelian. See [20].

If $\mathcal{A}$ is an operator space which is also an algebra, it is called a *completely contractive Banach algebra* if the multiplication map $m_0 : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ extends to a complete contraction $m : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$. The Fourier algebra $\mathrm{A}(G)$ is a completely contractive Banach algebra since the multiplication map $m : \mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G) \to \mathrm{A}(G)$ corresponds to restriction to the diagonal subgroup, that is, the map $R : \mathrm{A}(G\times G) \to \mathrm{A}(G)$ given by $Ru(s) = u(s,s)$ $(s \in G)$. Since the adjoint $R^* : \mathrm{VN}(G) \to \mathrm{VN}(G\times G)$ is a *-homomorphism, it is a complete contraction and hence $R \cong m$ is a complete contraction.

If $\mathcal{A}$ is a completely contractive Banach algebra and $\mathcal{X}$ is an operator space which is also an $\mathcal{A}$-module for which the module multiplication maps $m_{l,0} : \mathcal{A}\otimes\mathcal{X} \to \mathcal{X}$ and $m_{r,0} : \mathcal{X}\otimes\mathcal{A} \to \mathcal{X}$ extend to complete contractions $m_l : \mathcal{A}\widehat{\otimes}\mathcal{X} \to \mathcal{X}$ and $m_r : \mathcal{X}\widehat{\otimes}\mathcal{A} \to \mathcal{X}$, then $\mathcal{X}$ is called a *completely contractive* $\mathcal{A}$*-module.* A linear map $D : \mathcal{A} \to \mathcal{X}$ is called a derivation if $D(ab) = a{\cdot}D(b) + D(a){\cdot}b$ for a, b in $\mathcal{A}$. If $\mathcal{X}$ is a completely contractive $\mathcal{A}$-module, then $\mathcal{X}^*$ is, too. $\mathcal{A}$ is called *operator amenable* if every completely bounded derivation $D : \mathcal{A} \to \mathcal{X}^*$, where $\mathcal{X}$ is a completely contractive $\mathcal{A}$-module, is inner (i.e. $D(a) = a{\cdot}f - f{\cdot}a$ for some f in $\mathcal{X}^*$). $\mathcal{A}$ is called *operator weakly amenable* if every completely bounded derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner. If $\mathcal{A}$ is commutative, this is equivalent to saying that the only completely bounded derivation $D : \mathcal{A} \to \mathcal{X}$, where $\mathcal{X}$ is any symmetric completely contractive $\mathcal{A}$-module (i.e. $a{\cdot}x = x{\cdot}a$ for a in $\mathcal{A}$, x in $\mathcal{X}$), is 0. See [2] for this result in the Banach algebra case and [12] for the adaptation to the operator theoretic setting.

2. A theorem of Groenbaek

In this section we adapt a theorem of Groenbaek [13] to the completely contractive Banach algebra setting. Let $\mathcal{A}$ be a completely contractive Banach algebra and $\mathcal{X}$ an operator space which is a completely contractive $\mathcal{A}$-module. Let $\mathcal{A}_1 = \mathcal{A}\oplus_1\mathbb{C}$ be the unitization of $\mathcal{A}$. Then $\mathcal{X}$ is a completely contractive $\mathcal{A}_1$-module by setting $1{\cdot}x = x$ and $x{\cdot}1 = x$ for x in $\mathcal{X}$, where $1 = 0\oplus 1$ in $\mathcal{A}_1$. Indeed, $\mathcal{A}_1\widehat{\otimes}\mathcal{X} \cong (\mathcal{A}\widehat{\otimes}\mathcal{X})\oplus_1\mathcal{X}$ and the left multiplication map $m_1 : \mathcal{A}_1\widehat{\otimes}\mathcal{X} \to \mathcal{X}$ corresponds to the complete contraction $m \boxplus \mathrm{id}_{\mathcal{X}} : (\mathcal{A}\widehat{\otimes}\mathcal{X})\oplus_1\mathcal{X} \to \mathcal{X}$ given by $m \boxplus \mathrm{id}_{\mathcal{X}}((a\otimes x)\oplus y) = a{\cdot}x + y$. Similarly the right multiplication map can be extended. See [7].

The projection $\pi : \mathcal{A}_1 \to \mathcal{A}$ is a complete quotient map. Then if we let $i : \mathcal{A} \to \mathcal{A}_1$ be the natural injection, we get that the identity map $\mathrm{id}_{\mathcal{A}\widehat{\otimes}\mathcal{X}}$ factors as

$$\mathcal{A}\widehat{\otimes}\mathcal{X} \xrightarrow{i\otimes\mathrm{id}} \mathcal{A}_1\widehat{\otimes}\mathcal{X} \xrightarrow{\pi\otimes\mathrm{id}} \mathcal{A}\widehat{\otimes}\mathcal{X}.$$

Hence we have for u in $\mathrm{M}_n(\mathcal{A}\widehat{\otimes}\mathcal{X})$ ($n{\times}n$-matrices over $\mathcal{A}\widehat{\otimes}\mathcal{X}$) that

$$\|u\| \leq \|\pi\otimes\mathrm{id}\|_{cb}\,\|(i\otimes\mathrm{id})_n u\| \leq \|u\|,$$

from which it follows that $\|(i\otimes\mathrm{id})_n u\| = \|u\|$, so $\mathcal{A}\widehat{\otimes}\mathcal{X}$ is completely isometrically imbedded in $\mathcal{A}_1\widehat{\otimes}\mathcal{X}$.

Consider the sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\;i\;} \mathcal{A}_1\widehat{\otimes}\mathcal{X} \xrightarrow{\;m\;} \mathcal{X} \longrightarrow 0$$

where m denotes the left module multiplication map (denoted m_1 above), $\mathcal{K} = \ker m$ and $i : \mathcal{K} \to \mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ is the injection. Note that $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ is a completely contractive $\mathcal{A}$-module via $a \cdot (b \otimes x) = (ab) \otimes x$ and $(b \otimes x) \cdot a = b \otimes (x \cdot a)$. Let

$$[\mathcal{K}; \mathcal{A}] = \overline{\text{span}}\{a \cdot u - u \cdot a : u \in \mathcal{K} \text{ and } a \in \mathcal{A}\} \tag{2.1}$$

and note that $[\mathcal{K}; \mathcal{A}] \subset \mathcal{K}$. The following proposition is [13, Prop. 3.1]. We rework it here to ensure that it holds in our context.

Proposition 2.1. *If an operator S in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*) \cong (\mathcal{A}_1 \widehat{\otimes} \mathcal{X})^*$ is a derivation, then it annihilates $[\mathcal{K}; \mathcal{A}]$. In particular, 0 is the only derivation in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*)$ if $[\mathcal{K}; \mathcal{A}] = \mathcal{K}$.*

Proof. First note that

$$\mathcal{K} = \{u - 1 \otimes m(u) : u \in \mathcal{A}_1 \widehat{\otimes} \mathcal{X}\} = \overline{\text{span}}\{b \otimes x - 1 \otimes b \cdot x : b \in \mathcal{A}_1 \text{ and } x \in \mathcal{X}\}.$$

Then for any a and b in $\mathcal{A}_1$ and x in $\mathcal{X}$, we have

$$\begin{aligned} \langle b \otimes x - 1 \otimes b \cdot x, S \cdot a - a \cdot S \rangle &= \langle x, S(ab) - a \cdot S(b) \rangle - \langle b \cdot x, S(a) - a \cdot S(1) \rangle \\ &= \langle x, S(ab) - a \cdot S(b) \rangle - \langle x, S(a) \cdot b \rangle \\ &= \langle x, S(ab) - (a \cdot S(b) + S(a) \cdot b) \rangle \end{aligned}$$

from which it follows that $S \cdot a - a \cdot S \in \mathcal{K}^\perp$ for all a. However, $S \cdot a - a \cdot S \in \mathcal{K}^\perp$ for all a, if and only if for all u in $\mathcal{K}$,

$$0 = \langle u, S \cdot a - a \cdot S \rangle = \langle a \cdot u - u \cdot a, S \rangle.$$

Hence we obtain the first statement of the proposition.

Suppose that S in $\mathcal{CB}(\mathcal{A}_1, \mathcal{X}^*)$ is a derivation. For any u in $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$, we can write $u = (u - 1 \otimes m(u)) + 1 \otimes m(u)$, so $\mathcal{A}_1 \widehat{\otimes} \mathcal{X} = \mathcal{K} \oplus (1 \otimes \mathcal{X})$. (Note that $u \mapsto u - 1 \otimes m(u)$ is a completely bounded projection from $\mathcal{A}_1 \widehat{\otimes} \mathcal{X}$ onto $\mathcal{K}$, so the direct sum is one of operator spaces.) Observe that $\langle 1 \otimes x, S \rangle = \langle x, S1 \rangle = 0$ for any x in $\mathcal{X}$, so if $S \neq 0$, then there must be u in $\mathcal{K}$ such that $\langle u, S \rangle \neq 0$. This is possible only if $[\mathcal{K}; \mathcal{A}] \neq \mathcal{K}$. □

Now we will let $\mathcal{X} = \mathcal{A}$, and $m : \mathcal{A}_1 \widehat{\otimes} \mathcal{A}_1 \to \mathcal{A}_1$ be the multiplication map. Put

$$\mathcal{K}_1 = \ker m, \quad \mathcal{K} = \mathcal{K}_1 \cap (\mathcal{A}_1 \widehat{\otimes} \mathcal{A}) \quad \text{and} \quad \mathcal{K}_0 = \mathcal{K}_1 \cap (\mathcal{A} \widehat{\otimes} \mathcal{A}). \tag{2.2}$$

Since $\mathcal{A} \widehat{\otimes} \mathcal{A}$ imbeds into $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}$, it is easily seen that $\mathcal{K}$ is the same as in the notation above. If $\mathcal{A}$ is commutative, then $\mathcal{K}_1$ and hence $\mathcal{K}$ and $\mathcal{K}_0$ are closed ideals in $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_1$.

With only trivial modifications to his proof, we get the following theorem of Groenbaek [13].

Theorem 2.2. *If $\mathcal{A}$ is a commutative completely contractive Banach algebra, then the following are equivalent:*

- **(i)** *$\mathcal{A}$ is operator weakly amenable.*
- **(ii)** $[\mathcal{K}; \mathcal{A}] = \mathcal{K}$.
- **(iii)** $\overline{\mathcal{K}_1^2} = \mathcal{K}_1$.
- **(iv)** $\overline{\mathcal{K}^2} = \mathcal{K}$.
- **(v)** $\overline{\mathcal{A}^2} = \mathcal{A}$ *and* $\overline{\mathcal{K}_0^2} = \overline{(\mathcal{A} \widehat{\otimes} \mathcal{A}) \cdot \mathcal{K}_1}$.

Furthermore, if $\mathcal{A}$ has a bounded approximate identity, the above conditions are equivalent to

(iv) $\overline{\mathcal{K}_0^2} = \mathcal{K}_0$.

3. The Fourier algebra

We would now like to apply the above result to the Fourier algebra $\mathrm{A}(G)$ of a locally compact group G. Unless it is specified otherwise, let G be a *non-compact* locally compact group.

If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ and $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ are continuous unitary representations of G, let $\pi{\times}\sigma : G{\times}G \to \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\sigma)$ be the *Kronecker product* of π and σ, given by $\pi{\times}\sigma(s,t) = \pi(s) \otimes \sigma(t)$. We note, for future reference, that $\mathrm{VN}_\pi \bar{\otimes} \mathrm{VN}_\sigma = \mathrm{VN}_{\pi\times\sigma}$. Let $\lambda : G \to \mathcal{U}(\mathrm{L}^2(G))$ be the left regular representation of G and $1 : G \to \mathbb{T} \cong \mathcal{U}(\mathbb{C})$ be the trivial representation.

Proposition 3.1. *The representations $1{\times}1, 1{\times}\lambda, \lambda{\times}1$ and $\lambda{\times}\lambda$ of $G{\times}G$ are all disjoint (i.e. no two of these representations have subrepresentations which are unitary equivalent).*

Proof. Since G is non-compact, λ has no fixed points, for a fixed point would give a constant function in $\mathrm{A}(G)$. Hence none of $1{\times}\lambda, \lambda{\times}1$ or $\lambda{\times}\lambda$ have any fixed points for all of $G{\times}G$, so they are all disjoint from $1{\times}1$. $1{\times}\lambda$ fixes the entire subgroup $G{\times}\{e\}$ while neither $\lambda{\times}1$ nor $\lambda{\times}\lambda$ have any fixed points for that subgroup, so $1{\times}\lambda$ is disjoint from $\lambda{\times}1$ and $\lambda{\times}\lambda$. Similarly, $\lambda{\times}1$ and $\lambda{\times}\lambda$ are disjoint. □

Since λ and 1 are disjoint representations of G, we find that $\mathrm{VN}_{\lambda\oplus 1} \cong \mathrm{VN}_\lambda \oplus_\infty \mathrm{VN}_1 = \mathrm{VN}(G) \oplus_\infty \mathbb{C}$, by (1.1), and hence obtain the completely isometric identification $\mathrm{A}(G)_1 = \mathrm{A}(G) \oplus_1 \mathbb{C} \cong \mathrm{A}_{\lambda\oplus 1}$. The implied map is clearly an algebra isomorphism.

If u and v are complex functions on G, let $u{\times}v$ denote the complex valued function on $G{\times}G$ given by $u{\times}v(s,t) = u(s)v(t)$. Also, let 1 denote the constant function on G (as well as the trivial representation).

Proposition 3.2. *We have the following completely isometric identifications:*

(i) $\mathrm{A}(G)_1 \widehat{\otimes} \mathrm{A}(G) \cong \mathrm{A}_{(\lambda\times\lambda)\oplus(1\times\lambda)} = \mathrm{span}\{u, 1{\times}v : u \in \mathrm{A}(G{\times}G)$ *and* $v \in \mathrm{A}(G)\}$.

(ii) $\mathrm{A}(G)_1 \widehat{\otimes} \mathrm{A}(G)_1 \cong \mathrm{A}_{(\lambda\times\lambda)\oplus(1\times\lambda)\oplus(\lambda\times 1)\oplus(1\times 1)} = \mathrm{span}\{u, 1{\times}v, v{\times}1, 1{\times}1 : u \in \mathrm{A}(G{\times}G)$ *and* $v \in \mathrm{A}(G)\}$.

These are implemented by algebra isomorphisms.

Proof. (i) We have $\mathrm{A}(G)_1^* \cong \mathrm{VN}_{\lambda\oplus 1}$ and $\mathrm{A}(G)^* \cong \mathrm{VN}(G) = \mathrm{VN}_\lambda$. Thus, using (1.3), (1.1) and the lemma above, we obtain

$$\begin{aligned}(\mathrm{A}(G)_1 \widehat{\otimes} \mathrm{A}(G))^* &\cong \mathrm{VN}_{\lambda\oplus 1} \bar{\otimes} \mathrm{VN}_\lambda = (\mathrm{VN}_\lambda \oplus_\infty \mathrm{VN}_1) \bar{\otimes} \mathrm{VN}_\lambda \\ &= (\mathrm{VN}_\lambda \bar{\otimes} \mathrm{VN}_\lambda) \oplus_\infty (\mathrm{VN}_1 \bar{\otimes} \mathrm{VN}_\lambda) \\ &= \mathrm{VN}_{\lambda\times\lambda} \oplus_\infty \mathrm{VN}_{1\times\lambda} = \mathrm{VN}_{(\lambda\times\lambda)\oplus(1\times\lambda)},\end{aligned}$$

where the last space is the dual of $\mathrm{A}_{(\lambda\times\lambda)\oplus(1\times\lambda)}$. The latter equality in the statement (i) is a straightforward identification of $\mathrm{A}_{(\lambda\times\lambda)\oplus(1\times\lambda)}$ in $\mathrm{B}(G{\times}G)$. That the identifications are implemented by algebra isomorphisms is clear.

The proof of (ii) is similar. □

Theorem 3.3. *If G is a locally compact group, then* $\mathrm{A}(G)$ *is operator weakly amenable.*

Proof. If G is compact, then $\mathrm{A}(G)$ is operator amenable by [8, 16.2.3] (or Theorem 4.2, *infra*) and hence operator weakly amenable. Hence we are left to consider non-compact G. Let $\pi = (1\times 1) \oplus (1\times\lambda) \oplus (\lambda\times 1) \oplus (\lambda\times\lambda)$ so that A_π is the subalgebra of $\mathrm{B}(G\times G)$ indicated in Proposition 3.2(ii) above. In the identification $\mathrm{A}(G)_1\widehat{\otimes}\mathrm{A}(G)_1 \cong \mathrm{A}_\pi$, the multiplication map $m : \mathrm{A}(G)_1\widehat{\otimes}\mathrm{A}(G)_1 \to \mathrm{A}(G)_1$ corresponds to the map $R : \mathrm{A}_\pi \to \mathrm{A}_{\lambda\oplus 1}$, which restricts functions to the diagonal subgroup $G_D = \{(s,s) : s \in G\} \cong G$, i.e. $Ru(s) = u(s,s)$ for s in G. Letting $\mathcal{K}_1$ and $\mathcal{K}_0$ be as in (2.2), we obtain identifications

$$\mathcal{K}_1 \cong \ker R = \mathrm{span}\{u - 1\times R(u), u - R(u)\times 1 : u \in \mathrm{A}_\pi\}$$

and

$$\mathcal{K}_0 \cong \ker R \cap \mathrm{A}(G\times G) = \mathrm{I}(G_D).$$

Here, $\mathrm{I}(G_D)$ denotes *the* ideal in $\mathrm{A}(G\times G)$ with hull G_D: since G_D is a subgroup, it is a set of spectral synthesis by [14] (or see [17])†. From spectral synthesis we obtain that $\overline{\mathrm{I}(G_D)^2} = \mathrm{I}(G_D)$ so

$$\overline{\mathcal{K}_0^2} = \mathcal{K}_0. \tag{3.1}$$

Since $\mathrm{A}(G\times G)$ is an ideal in A_π, we get

$$(\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G))\cdot\mathcal{K}_1 \cong \mathrm{A}(G\times G)\cdot\ker R \subset \mathrm{A}(G\times G)\cap\ker R = \mathrm{I}(G_D) \cong \mathcal{K}_0. \tag{3.2}$$

On the other hand, again using that G_D is a set of spectral synthesis for $\mathrm{A}(G\times G)$,

$$\mathcal{K}_0 \cong \mathrm{I}(G_D) = \overline{\mathrm{A}(G\times G)\cdot\mathrm{I}(G_D)} \subset \overline{\mathrm{A}(G\times G)\cdot\ker R} \cong \overline{(\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G))\cdot\mathcal{K}_1}. \tag{3.3}$$

We thus have, assembling (3.1) and the inclusions (3.2) and (3.3),

$$\overline{(\mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G))\cdot\mathcal{K}_1} = \overline{\mathcal{K}_0^2}.$$

Hence condition (v) of Theorem 2.2 is satisfied, since $\overline{\mathrm{A}(G)^2} = \mathrm{A}(G)$ by the Tauberian Theorem for $\mathrm{A}(G)$. □

If $\mathcal{A}$ is a completely contractive Banach algebra and φ is a character of $\mathcal{A}$, then $\mathbb{C}$ can be made into an $\mathcal{A}$-module via $a\cdot z = \varphi(a)z = z\cdot a$ for a in $\mathcal{A}$ and z in $\mathbb{C}$. If φ is continuous, it is automatically completely bounded, and hence $\mathbb{C}$ is a completely contractive $\mathcal{A}$-module. A *point derivation* is a derivation $D : \mathcal{A} \to \mathbb{C}$. If $\mathcal{A}$ admits continuous non-zero point derivations, then it is not (operator) weakly amenable.

Corollary 3.4. $\mathrm{A}(G)$ *has no continuous point derivations.*

In contrast to the case for $\mathrm{A}(G)$, if G is Abelian and non-compact, then $\mathrm{B}(G) \cong \mathrm{M}(\widehat{G})$ (the measure algebra of the non-discrete Abelian group $\widehat{G}$) admits continuous point derivations by [5], and hence is not (operator) weakly amenable. If G is a locally compact group containing a closed non-compact Abelian subgroup H such that the restriction map $R_H : \mathrm{B}(G) \to \mathrm{B}(H)$ is surjective, then $\mathrm{B}(G)$ is not (operator) weakly amenable. Note that if R_H is surjective, then $R_H^* : \mathrm{W}^*(H) \to \mathrm{W}^*(G)$ (enveloping von Neumann algebras) is a *-homomorphism, and hence a complete contraction. See [11] for further results on the weak amenability of $\mathrm{B}(G)$.

†*Note added in proof*: The spectral synthesis result, in full generality, can be found in [24].

It has been recently announced by H. G. Dales, F. Ghahramani and A. Ya. Helemskii that the measure algebra $\mathrm{M}(G)$ has continuous point derivations whenever G is not discrete. Hence we deduce that $\mathrm{M}(G)$ is weakly amenable if and only if G is discrete. The reasonable dual conjecture to this is: $\mathrm{B}(G)$ *is operator weakly amenable if and only if* G *is compact.*

4. Operator amenability of the Fourier algebra

In this section we give a quick proof that $\mathrm{A}(G)$ is operator amenable when G is an amenable [SIN]-group. This proof uses elements of both [8, Sec. 16] (i.e. of [23, 3.5]) and [17, 5.3].

We say that G has the *small invariant neighbourhood* property, or that G is a [SIN]-group, if there is a neighbourhood base $\mathcal{V}$ of the identity in G such that $sVs^{-1} = V$ for V in $\mathcal{V}$ and s in G, i.e. each V in $\mathcal{V}$ is invariant under inner automorphisms. Any compact, Abelian or discrete group is a [SIN]-group. [SIN]-groups are all unimodular. See [21] for more information.

We let $\mathrm{I}(G_D)$ be as in the proof of Theorem 3.3. If we let $\mathrm{I}_0(G_D)$ denote the set of functions in $\mathrm{A}(G{\times}G)$ which are compactly supported with support disjoint from G_D, then $\overline{\mathrm{I}_0(G_D)} = \mathrm{I}(G_D)$, since G_D is a set of spectral synthesis for $\mathrm{A}(G{\times}G)$.

Lemma 4.1. *If* G *is a [SIN]-group, then there is a bounded net* $\{u_V\}_{V\in\mathcal{V}}$ *in* $\mathrm{B}(G{\times}G)$ *such that* $u_V(s,s) = 1$ *for* s *in* G *and* $u_V v \to 0$ *for* v *in* $\mathrm{I}(G_D)$.

Proof. Let $\mathcal{V}$ be a neighbourhood base of the identity in G consisting of relatively compact neighbourhoods, each of which is invariant under inner automorphisms. Let $\pi : G{\times}G \to \mathcal{U}(\mathrm{L}^2(G))$ be given by $\pi(s,t)f = \lambda(s)\rho(t)f$ for f in $\mathrm{L}^2(G)$, where λ and ρ are the left and right regular representations of G, respectively. Then π is a continuous unitary representation of $G{\times}G$. For v in $\mathcal{V}$, let $u_V = \frac{1}{\mu(V)}\langle\pi(\cdot)\chi_V|\chi_V\rangle$, where χ_V is the indicator function of V and μ is the Haar measure. Then for (s,t) in $G{\times}G$,

$$u_V(s,t) = \frac{1}{\mu(V)}\int_G \chi_V(s^{-1}rt)\chi_V(r)d\mu(r) = \frac{\mu(sVt^{-1}\cap V)}{\mu(V)}.$$

Clearly $u_V(s,s) = 1$ for s in G. Hence, since each u_V is positive definite, we have $\|u_V\| = u_V(e,e) = 1$. If $v \in \mathrm{I}_0(G_D)$, then there is V in $\mathcal{V}$ such that $u_V v = 0$. Hence $u_V v \to 0$ for v in $\mathrm{I}(G_D)$, where V runs through decreasing elements of $\mathcal{V}$. □

We remark that if G is discrete, we can use the positive definite function χ_{G_D} in place of the net above.

Theorem 4.2. *If* G *is an amenable [SIN]-group, then* $\mathrm{A}(G)$ *is operator amenable.*

By [8, 16.1.4] (i.e. [23]), it suffices to show that $\mathrm{A}(G)$ has an operator bounded approximate diagonal, i.e. a bounded net $\{w_\beta\}_{\beta\in B}$ in $\mathrm{A}(G{\times}G) \cong \mathrm{A}(G)\widehat{\otimes}\mathrm{A}(G)$ such that

- **(i)** $\{Rw_\beta\}_{\beta\in B}$ is a bounded approximate identity for $\mathrm{A}(G)$, and
- **(ii)** $\|u{\times}1\, w_\beta - w_\beta\, 1{\times}u\| \to 0$ for all u in $\mathrm{A}(G)$.

Recall that $R : \mathrm{A}(G{\times}G) \to \mathrm{A}(G)$ is the map $Ru(s) = u(s,s)$ for s in G. Let $\{u_\alpha\}_{\alpha\in A}$ be a bounded approximate identity for $\mathrm{A}(G)$ (which we can obtain since G is amenable; see [19]), and $\{u_V\}_{V\in\mathcal{V}}$ be as in the lemma above. Let $B = A{\times}\mathcal{V}^A$ be the product directed set. For each $\beta = (\alpha, (V_{\alpha'})_{\alpha'\in A})$ in B let

$$w_\beta = u_{V_\alpha}\, u_\alpha{\times}u_\alpha.$$

Then $Rw_\beta = u_\alpha^2$, so (i) is satisfied. To see (ii), let $v \in \mathrm{A}(G)$, so $u_\alpha \times u_\alpha (v \times 1 - 1 \times v) \in \mathrm{I}(G_D)$ for each α, and hence

$$\begin{aligned} \lim_\beta \|u\times 1\, w_\beta - w_\beta\, 1\times u\| &= \lim_{\beta=(\alpha,(V_{\alpha'})_{\alpha'\in A})} \|u_{V_\alpha}\, u_\alpha\times u_\alpha(v\times 1 - 1\times v)\| \\ (\dagger) \qquad &= \lim_\alpha \lim_V \|u_V\, u_\alpha\times u_\alpha(v\times 1 - 1\times v)\| \\ &= \lim_\alpha 0 = 0. \end{aligned}$$

The equality of limits at (†) is from [18, p. 69]. □

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