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EVERY CLOSED CONVEX SET IS THE SET OF MINIMIZERS OF SOME C^∞-SMOOTH CONVEX FUNCTION

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ABSTRACT. We show that for every closed convex set C in a separable Banach space X there is a C^∞-smooth *convex* function $f : X \longrightarrow [0,\infty)$ so that $f^{-1}(0) = C$. We also deduce some interesting consequences concerning smooth approximation of closed convex sets and continuous convex functions.

It is well known that if a separable Banach space has a C^1-smooth equivalent norm, then every closed convex C set can be regarded as the set of minimizers of a C^1-smooth convex function f. One can obtain such a function f by considering the inf-convolution of the smooth norm with the indicator function of C (valued 0 on C and $+\infty$ elsewhere). Indeed, separable Asplund spaces have equivalent norms with dual LUR norms and, as shown by Asplund and Rockafellar, inf-convolutions preserve Fréchet smoothness when one of the convex functions is a norm whose dual is LUR (see [6], Proposition 2.3 for instance). However, the inf-convolution operation does not preserve C^2 smoothness of the norm, so this procedure does not provide C^2-smooth convex functions with a prescribed set of minimizers, and it seems to be an open question whether for every closed convex set C in a separable Banach space X there always exist C^2-smooth *convex* functions whose set of minimizers is C.

In this note we provide a simple proof that for every closed convex set C in a separable Banach space X there is a C^∞-smooth *convex* function $f : X \longrightarrow [0,\infty)$ so that $f^{-1}(0) = C$. Surprisingly enough, the result is not true for some nonseparable Banach spaces (see the Remark below). We also deduce some interesting consequences of our main theorem.

For instance, this result allows us to recover some classical theorems on smooth approximation of convex sets in $\mathbb{R}^n$. Namely, every bounded closed convex set in $\mathbb{R}^n$ can be approximated in the Hausdorff distance by C^∞-smooth convex bodies. Also, every continuous convex function on $\mathbb{R}^n$ can be uniformly approximated on bounded sets by C^∞-smooth convex functions.

The analogue consequences of our main theorem in the infinite-dimensional setting are that every closed convex set in a separable Banach space X is the Mosco limit of C^∞-smooth convex bodies (see Definition 4 below for Mosco's notion of convergence), and that every continuous convex function can be uniformly approximated on compact sets by C^∞-smooth convex functions; on the other hand, every

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weakly continuous convex function on X can be approximated by C^∞-smooth convex functions, uniformly on weakly compact sets.

It is worth noting that there are infinite-dimensional separable Banach spaces (such as ℓ_1) for which it is impossible to approximate bounded convex bodies by C^1-smooth convex bodies uniformly on bounded sets (in general, every Banach space with no Fréchet smooth equivalent norm lacks this kind of approximation). So, the best approximation results that one can expect in the general case are those provided by our corollaries. Of course, if the space considered has a smooth equivalent norm, then one should expect stronger results, such as uniform smooth approximation on bounded sets, but this is a much subtler question. In general it is unknown whether every Banach space X with a C^p-smooth equivalent norm has the property that every bounded convex body can be uniformly approximated on bounded sets by C^p-smooth convex bodies. In the case where X is separable and $p = 1$ this is indeed true and can be done by using inf-convolution. For $p \geq 2$, the results of Deville, Fonf and Hájek's [3, 4] on smooth and analytic approximation of bounded convex bodies on some separable Banach spaces show that this is also true for the separable polyhedral spaces and the separable L^p spaces (in fact, they obtain uniform analytic approximation on bounded sets when an equivalent analytic norm is available in the space).

We now state and prove our main result.

Theorem 1. *For every closed convex set C in a separable Banach space X there exists a C^∞ smooth convex function $f : X \longrightarrow [0, \infty)$ so that $f^{-1}(0) = C$ (and, in particular, also $f'(x) \neq 0$ for all $x \in X \setminus C$).*

Proof. We may obviously assume that $\emptyset \neq C \neq X$. It is well known that, as a consequence of the Hahn-Banach theorem, every such closed convex set C is the intersection of the half-spaces of X which contain C; that is,

$$C = \bigcap_{i \in I} H_i,$$

where the H_i can be assumed to be of the form $H_i = \{x \in X : x_i^*(x) \leq \alpha_i\}$ for some $x_i^* \in X^*$ with $\|x_i^*\| = 1$, and $\alpha_i \in \mathbb{R}$. Then we have that $X \setminus C = \bigcup_{i \in I}(X \setminus H_i)$, and since the complements $X \setminus H_i$ are open and $X \setminus C$ is a Lindelöf space, there exists a countable subcovering

$$X \setminus C = \bigcup_{n=1}^{\infty}(X \setminus H_n),$$

where the $H_n = \{x \in X : x_n^*(x) \leq \alpha_n\}$ form a subsequence of the family $(H_i)_{i \in I}$. Therefore C can be written as a countable intersection of closed half-spaces,

$$C = \bigcap_{n=1}^{\infty}\{x \in X : x_n^*(x) \leq \alpha_n\}. \tag{1}$$

Now, let $\theta : \mathbb{R} \longrightarrow [0, \infty)$ be a C^∞ smooth *convex* function so that $\theta(t) = 0$ for $t \leq 0$, and $\theta(t) > 0$ whenever $t > 0$; furthermore, ensure that $\theta(t)$ be an affine function of slope 1 for $t \geq 1$, say $\theta(t) = t + b$ for $t \geq 1$, where $-1 < b < 0$. It is easy to construct such a function θ by integrating twice a suitable C^∞ smooth nonnegative function whose support is precisely the interval $[0, 1]$. Then define $\theta_n : \mathbb{R} \longrightarrow [0, \infty)$ by

$$\theta_n(t) = \theta(t - \alpha_n);$$

clearly θ_n is a C^∞ smooth convex function so that θ_n vanishes precisely on the interval $(-\infty, \alpha_n]$, and θ_n restricts to an affine function on $[\alpha_n + 1, \infty)$, namely $\theta_n(t) = t - \alpha_n + b$ for $t \geq \alpha_n + 1$.

Let us define our function $f : X \longrightarrow [0, \infty)$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{\theta_n(x_n^*(x))}{(1 + |\alpha_n|)2^n}$$

for all $x \in X$. It is clear that f is a convex function. Let us see that f is well defined and C^∞ smooth. We can write f as a function series, $f(x) = \sum_{n=1}^{\infty} f_n(x)$, where

$$f_n(x) = \frac{\theta_n(x_n^*(x))}{(1 + |\alpha_n|)2^n}.$$

In order to see that f is C^∞ smooth it is enough to check that the series of derivatives $\sum_{n=1}^{\infty} f_n^{(j)}(x)$ converges uniformly on each ball $B(0, R)$, with $R > 1$, for all $j = 0, 1, 2, ...$ Since the derivatives of the function θ are all bounded and θ_n is a mere translation of θ, there are constants $M_j > 0$ so that $\|\theta_n^{(j)}\|_\infty = \|\theta^{(j)}\|_\infty = M_j$ for all $j = 1, 2, ...$, while for $j = 0$ we have

$$0 \leq \theta_n(t) = \theta(t - \alpha_n) \leq \max\{\theta(1), t - \alpha_n + b\}$$

for all $t \in \mathbb{R}$. By using these bounds, and bearing in mind that $\|x_n^*\| = 1$, we can estimate, for $\|x\| \leq R$,

$$|f_n(x)| = \Big| \frac{\theta_n(x_n^*(x))}{(1 + |\alpha_n|)2^n} \Big| \leq \frac{\theta(1) + R + |\alpha_n| + |b|}{(1 + |\alpha_n|)2^n} := \delta_n^{(0)},$$

and since $\sum_{n=1}^{\infty} \delta_n^{(0)} < \infty$, it follows that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on the ball $B(0, R)$. For $j \geq 1$ it is easily seen that the j-linear map $f_n^{(j)}(x) \in \mathcal{L}_s^j(X)$ is given by

$$f_n^{(j)}(x) = \frac{\theta_n^{(j)}(x_n^*(x))}{(1 + |\alpha_n|)2^n} \overbrace{x_n^* \otimes ... \otimes x_n^*}^{j}$$

(there $\mathcal{L}_s^j(X)$ stands for the set of continuous symmetric j-linear forms on X). Then, by taking into account that $\|x_n^* \otimes ... \otimes x_n^*\| \leq 1 = \|x_n^*\|$, and $\|\theta_n^{(j)}\|_\infty = M_j$, we get that

$$\|f_n^{(j)}(x)\| \leq \frac{M_j R}{(1 + |\alpha_n|)2^n} := \delta_n^{(j)}$$

whenever $\|x\| \leq R$ and, since $\sum_{n=1}^{\infty} \delta_n^{(j)} < \infty$, this ensures that $\sum_{n=1}^{\infty} f_n^{(j)}$ converges uniformly on bounded sets, for all $j \in \mathbb{N}$. Therefore, f is of class C^∞.

The fact that $f^{-1}(0) = C$ follows immediately from equality (1) above and from the definitions of the functions θ_n and f.

Finally, every convex differentiable nonnegative function which vanishes precisely on a set C cannot have a zero derivative outside C; therefore our function f satisfies $f'(x) \neq 0$ for all $x \in X \setminus C$. □

Remark. The above proof more generally shows that every closed convex set C in a (possibly nonseparable) Banach space that is a countable intersection of half-spaces can be written as $C = f^{-1}(0)$ for some C^∞-smooth convex function $f : X \longrightarrow [0, \infty)$ (in particular, all convex sets in separable Banach spaces are of this form).

However, there are very simple closed convex sets in quite reasonable nonseparable Banach spaces for which Theorem 1 fails. For instance, take $X = c_0(\Gamma)$, where Γ is an uncountable set, and $C = \{0\}$. As shown by Petr Hájek [7], there is no C^2-smooth function on $c_0(\Gamma)$ which attains its minimum at exactly one point. This shows that Theorem 1 is false for $X = c_0(\Gamma)$ even if we drop convexity. It seems to be an intriguing open question how to characterize the nonseparable Banach spaces for which every closed convex set (or even a mere singleton) can be written as the set of minimizers of a C^∞-smooth (convex) function.

Corollary 2. *Every bounded closed convex set in $\mathbb{R}^n$ can be approximated in the Hausdorff distance by C^∞ smooth convex bodies.*

Proof. Let C be a bounded closed convex set of $\mathbb{R}^n$. By Theorem 1, there exists a convex C^∞ smooth function $f : X \longrightarrow [0, \infty)$ so that $C = f^{-1}(0)$, and $f'(x) \neq 0$ for all $x \in X \setminus C$. Consider the sets $U_n = f^{-1}([0, 1/n])$. Since $f'(x) \neq 0$ when $f(x) = 1/n$, the implicit function theorem [2] tells us that the level set $\{x : f(x) = 1/n\}$, which is the boundary of the convex body U_n, is a one-codimensional C^∞-smooth manifold, and this means that the convex body U_n is C^∞-smooth as well. By using standard compactness arguments, it is easy to see that U_n converges to C in the Hausdorff distance. $\square$

By applying the above results to the epigraph of a continuous convex function, it is easy to deduce the following.

Corollary 3. *Every convex function on $\mathbb{R}^n$ can be approximated uniformly on bounded sets by C^∞ smooth convex functions.*

Finally, we see what kind of approximation results can be deduced from Theorem 1 in the infinite-dimensional case. As remarked above, in general one cannot expect to have uniform approximation on bounded sets, so we are made to deal with weaker notions of convergence. In the case of functions, we can consider uniform convergence on compact sets. For convex bodies, perhaps the strongest notion of convergence that remains useful in the general case is that of Mosco's.

Definition 4. Let C_n, C be closed convex subsets of a Banach space $(X, \|\cdot\|)$. The sequence C_n is said to converge to C in the Mosco sense provided that

(1) For every $x \in C$ there is a sequence (x_n) so that $\lim_{n\to\infty} \|x_n - x\| = 0$ and $x_n \in C_n$ for each n.
(2) For every sequence (x_{n_j}) weakly converging to some x, if $x_{n_j} \in C_{n_j}$ for all j, then $x \in C$.

Here are some useful remarks about Mosco convergence.

(1) This definition makes sense for nonconvex sets as well but, when the convexity condition is dropped, a constant sequence may not converge to itself.
(2) If, in the second condition of the above definition, we replace weakly convergence with norm convergence, we are left with the definition of Kuratowski convergence.
(3) Obviously, Mosco convergence implies Kuratowski convergence, and both notions of convergence are equivalent in Schur spaces.
(4) In reflexive Banach spaces, Mosco convergence implies pointwise convergence of the distance functions to the corresponding sets (the so-called Wijsman convergence).

For further information concerning Mosco and Wijsman convergence, see [1].

For a closed convex subset C of a separable Banach space, let f be the C^∞-smooth convex function constructed in the proof of Theorem 1. We have that $C = f^{-1}(0)$. Put $C_n = f^{-1}([0, 1/n])$ for each n. Since $f'(x) \neq 0$ whenever $f(x) = 1/n$, by the implicit function theorem [2], C_n is a C^∞-smooth convex body.

We are going to see that the sequence (C_n) converges to C in the Mosco sense. Since C_n is a decreasing sequence of convex sets whose intersection is C, we only have to check the second condition of Definition 4. To this end, suppose $x_{n_j} \in C_{n_j}$ and (x_{n_j}) weakly converges to x. As f is continuous and convex, it is weakly lower semicontinuous. Hence $f(x) \leq \liminf f(x_{n_j}) \leq \lim \frac{1}{n_j} = 0$, and therefore $x \in C$. Thus we have shown the following

Corollary 5. *Every closed convex subset (not necessarily bounded) of a separable Banach space can be approximated in the sense of Mosco by C^∞-smooth convex bodies.*

As in the finite-dimensional case, when one applies these techniques to the epigraphs of continuous convex functions, one can deduce some results on smooth approximation of convex functions.

Corollary 6. *Every continuous convex function on a separable Banach space can be approximated, uniformly on compact sets, by C^∞ smooth convex functions.*

If we demand that our convex function be weakly continuous, then we can get uniform approximation on weakly compact sets (in particular, for reflexive Banach spaces, the approximation is uniform on bounded sets).

Corollary 7. *Every weakly sequentially continuous convex function on a separable Banach space can be approximated by C^∞ smooth convex functions, uniformly on weakly compact sets*

Next we provide a proof for Corollary 7. The same argument, with obvious modifications (just considering the norm topology instead of the weak topology), yields the proofs of Corollaries 6 and 3.

Take a weakly sequentially continuous convex function $g : X \longrightarrow \mathbb{R}$. Let C be the epigraph of g, a subset of $X \times \mathbb{R}$. By Corollary 5 there is a sequence (C_n) of C^∞ smooth convex bodies in $X \times \mathbb{R}$ that converge to C in the sense of Mosco. Since the bodies C_n contain C, which is the epigraph of a convex function defined on the whole of X, every C_n is itself the epigraph of a C^∞ smooth convex function $g_n : X \longrightarrow \mathbb{R}$ which lies below g; moreover, $g_n \leq g_{n+1} \leq g$ for all n. The fact that the functions g_n are C^∞ smooth is a standard consequence of the implicit function theorem.

We see that g_n converges to g uniformly on weakly compact sets. If not, there would exist some weakly compact subset K of X, some sequence $(x_n) \subset K$, and some $\varepsilon > 0$ so that

$$g(x_n) - g_n(x_n) = |g(x_n) - g_n(x_n)| > \varepsilon$$

for every $n \in J$, where J is some cofinal subset of $\mathbb{N}$; we may assume $J = \mathbb{N}$ for the sake of simplicity. Since K is weakly compact, there exists a subsequence of (x_n) that weakly converges to some $x \in K$; again we may assume that (x_n) weakly converges to x. Then, since g is weakly sequentially continuous, we have that $\lim_{n\to\infty} g(x_n) = g(x)$, and therefore

$$g(x) - g_n(x_n) \geq \varepsilon/2 \tag{2}$$

if n is large enough; we may as well assume that this is so for all $n \in \mathbb{N}$.

On the other hand, note that the sequence $(g_n(x_n)) \subset \mathbb{R}$ is bounded. Indeed, it is bounded above because $g_n(x_n) \leq g(x_n)$ and g is bounded on K; and it is also bounded below since $g_1(x_n) \leq g_n(x_n)$ and g_1 is bounded below on K (bear in mind that every convex function is bounded below on a bounded set). Then we can choose a convergent subsequence of $(g_n(x_n))$. As always, we may assume that $g_n(x_n)$ converges to some $\alpha \in \mathbb{R}$.

To sum up, we have seen that $(x_n, g_n(x_n))$ weakly converges to (x, α) in $X \times \mathbb{R}$. Since $(x_n, g_n(x_n))$ belongs to C_n (the epigraph of g_n), and C_n converges to C in the Mosco sense, we must conclude that $(x, \alpha) \in C$, that is, $g(x) \leq \alpha$. But this contradicts (2). $\square$

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