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REGULARITY CRITERIA INVOLVING THE PRESSURE FOR THE WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper we consider the Cauchy problem for the n-dimensional Navier-Stokes equations and we prove a regularity criterion for weak solutions involving the summability of the pressure. Related results for the initial-boundary value problem are also presented.

1. Introduction

We consider the Cauchy problem for the Navier-Stokes equations in $\mathbf{R}^n$, $n \geq 3$,

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla p = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ \mathbf{v}_{|t=0} = \mathbf{v}_0(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{1.1}$$

For any domain $\Omega \subseteq \mathbf{R}^n$, we let $H_2(\Omega)$ denote the closure, with respect to the L^2-norm, of the space $\mathcal{D}(\Omega) = \{\phi \in (C_0^\infty(\Omega))^n : \nabla\cdot\phi = 0\}$. Then, it is well-known that for a given $\mathbf{v}_0 \in H_2(\mathbf{R}^n)$ there exists at least a weak-solution (à la Leray-Hopf) $\mathbf{v}$ to (1.1), that is,

$$\mathbf{v} \in L^\infty(0,T;L^2(\mathbf{R}^n)) \cap L^2(0,T;H^1(\mathbf{R}^n)),$$

which satisfies the (weak) energy inequality (see (2.4) in Section 2; Leray [19] and Hopf [13]). However, the uniqueness and regularity of weak solutions is still an open problem.[1] It is also well-known, that uniqueness and full regularity is obtained under suitable extra-assumptions. In particular, if a weak solution satisfies at least

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[1]For the complete definition of weak solutions and for all the classical results cited and used in this paper see, for instance, the recent review paper of Galdi [10].

one of the following conditions

$$\text{(1.2)} \qquad \begin{cases} \text{i)} & \mathbf{v} \in L^r(0,T;L^s(\mathbf{R}^n)) \quad \text{with} \quad \dfrac{2}{r} + \dfrac{n}{s} \leq 1, \quad \text{for} \quad s > n, \\ \\ \text{ii)} & \mathbf{v} \in C(0,T;L^n(\mathbf{R}^n)), \\ \\ \text{iii)} & \mathbf{v} \in L^\infty(0,T;L^n(\mathbf{R}^n)) \quad \text{and} \quad \|\mathbf{v}\|_{L^\infty(0,T;L^n)} \quad \text{is "small,"} \end{cases}$$

then it is unique in the class of weak solutions[2]; see Prodi [21] and Serrin [23]. Furthermore, it is of class C^∞ in $(0,T] \times \mathbf{R}^n$; see Leray [19], Sohr [24], and Giga [11]. Regularity results, under slightly weaker assumptions, are also given in Beirão da Veiga [5], Kozono and Sohr [18], and Berselli [7]. Furthermore, *interior* regularity (in the space variables) of weak solutions that satisfy "locally" condition (1.2) can be proved; see for instance Serrin [23], Takahashi [27, 28], Struwe [26], and the references of Section 5 in [10]. However, the methods used to prove the interior regularity results are quite different from those used in this paper.

Full regularity of weak solutions can also be proved under alternative assumptions on the gradient of the velocity ∇u. Specifically, if

$$\text{(1.3)} \qquad \nabla u \in L^r(0,T;L^s(\mathbf{R}^n)) \quad \text{with} \quad \frac{2}{r} + \frac{n}{s} = 2, \quad \text{for} \quad s > \frac{n}{2},$$

then weak solutions are of class $C^\infty(\mathbf{R}^n \times (0,T])$; see Beirão da Veiga [2] and Galdi [10]; see also Berselli [8] for a simple proof when $n = 3$.

We recall that the class (1.2) is important from the point of view of the relation between *scaling invariance* and partial regularity of weak solutions; see for instance Giga [11]. In fact, if a pair $\{\mathbf{v}, p\}$ solves (1.1), then so does the family $\{\mathbf{v}_\lambda, p_\lambda\}_{\lambda > 0}$ defined by

$$\mathbf{v}_\lambda := \lambda\, \mathbf{v}(\lambda x, \lambda^2 t), \qquad p_\lambda := \lambda^2\, p(\lambda x, \lambda^2 t).$$

Scaling invariance means that $\|\mathbf{v}_\lambda\|_{L^r(0,T;L^s)} = \|\mathbf{v}\|_{L^r(0,T;L^s)}$ and this happens if and only if r and s satisfy (1.2). Likewise, one has $\|\nabla \mathbf{v}_\lambda\|_{L^r(0,T;L^s)} = \|\nabla \mathbf{v}\|_{L^r(0,T;L^s)}$ if and only if r and s are as in (1.3).

The objective of this paper is to investigate what regularity, for the *initial-value-problem* (1.1), can be inferred assuming some conditions on the pressure p. This problem has been treated by several authors. Specifically, in [14] Kaniel proved that (in a bounded three-dimensional domain Ω) the condition

$$p \in L^\infty(0,T;L^q(\Omega)), \quad \text{for } q > \frac{12}{5},$$

implies that $\mathbf{v}$ is in the class (1.2)(i), which is a regularity class also for the initial boundary value problem; see [11, 24].

More recently, this result has been improved by Berselli [6], who proved that if

$$\text{(1.4)} \qquad p \in L^\alpha(0,T;L^{\frac{\alpha n}{\alpha+n-2}}(\Omega)), \quad \alpha > n, \quad \Omega \subseteq \mathbf{R}^n, \ n \geq 3,$$

then the velocity is in the class (1.2)(i).

Furthermore, in the framework of weak (Marcinkiewicz) L^p_* spaces, Beirão da Veiga [3] proved (as a corollary of a main result) that if

$$p \in L^\gamma_*(\Omega \times (0,T)), \quad \text{for} \quad \gamma > \frac{n+2}{2},$$

[2]For uniqueness, the smallness assumption in iii) can be removed; see Kozono and Sohr [17].

again $\mathbf{v}$ is in the class (1.2)(i). Note that $r = s = (n+2)/2$ satisfy (1.5) below.

We wish to mention another result of Beirão da Veiga [4], where a sufficient condition involving $p/(1+|\mathbf{v}|)$ was given.

It is worth remarking that all the above results are partial, in the following sense. The scaling invariance for the pressure requires that $\|p_\lambda\|_{L^r(0,T;L^s)} = \|p\|_{L^r(0,T;L^s)}$, and this happens if and only if

$$p \in L^r(0,T;L^s(\mathbf{R}^n)) \quad \text{with} \quad \frac{2}{r} + \frac{n}{s} = 2, \quad \text{for} \quad s > \frac{n}{2}. \tag{1.5}$$

Therefore, we expect that condition (1.5) is "optimal" as, in fact, conjectured in [4]. To give more weight to this conjecture, we recall that, for the Cauchy problem, the following inequality holds (in the sequel $\|\,.\,\|_\gamma$, for $1 \leq \gamma \leq \infty$, will denote the $L^\gamma(\mathbf{R}^n)$-norm of both scalar and vector-valued functions)

$$\|p\|_\gamma \leq C\|\mathbf{v}\|_{2\gamma}^2, \qquad 1 < \gamma < +\infty. \tag{1.6}$$

Equation (1.6) easily follows by applying the "div" operator on both sides of (1.1), and using the Calderón-Zygmund theorem. Therefore, we obtain (roughly speaking) that the pressure behaves as velocity squared. Now, from (1.6) we find that

$$\|p\|_{L^{r/2}(0,T;L^{s/2})} \leq C\|\mathbf{v}\|^2_{L^r(0,T;L^s)}, \quad \text{with} \quad \frac{2}{r} + \frac{n}{s} = 1,$$

and consequently, condition (1.2) implies (1.5).

Very recently Chae and Lee [9], in the case of the three-dimensional Cauchy problem, have proved that if p satisfies

$$p \in L^r(0,T;L^s(\mathbf{R}^n)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} < 2, \quad \text{for} \quad s > \frac{3}{2}, \tag{1.7}$$

then $\mathbf{v}$ is in the class (1.2)(i). Notice that both (1.7) and (1.4) are stronger than (1.5). The objective of this paper is to show that (1.5) is sufficient for the full regularity of weak solutions. Specifically, we have

Theorem 1.1. *Let $\mathbf{v}_0 \in H_2(\mathbf{R}^n) \cap L^n(\mathbf{R}^n)$, for $n \geq 3$. If the pressure p satisfies condition* (1.5)*, then $\mathbf{v}, p \in C^\infty(\mathbf{R}^n \times (0,T])$. In the cases $n = 3, 4$, the condition on the initial data can be relaxed to $\mathbf{v} \in H_2(\mathbf{R}^n)$.*

The proof of this result will be achieved by a suitable modification of the method used in [9].

Remark 1.2. The pressure associated to a weak solution satisfies the following regularity property:

$$p \in L^r(0,T;L^s(\mathbf{R}^n)) \quad \text{with} \quad \frac{2}{r} + \frac{n}{s} = n;$$

see Sohr and von Wahl [25].

Remark 1.3. In the limit case $r = \infty$, $s = n/2$ of condition (1.5) it is possible to prove full regularity, provided that the norm $\|p\|_{L^\infty(0,T;L^{n/2})}$ is small enough. This is the counterpart of condition (1.2)(iii). In this respect, we recall that the regularity of a weak solution with $\mathbf{v} \in L^\infty(0,T;L^n(\mathbf{R}^n))$ is still an outstanding open problem.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To this end, we collect some preliminary results, due to Kato [15] and Giga [11].

Proposition 2.1. *The following properties hold:*

i) *Suppose that $\mathbf{v}_0 \in L^q(\mathbf{R}^n)$, $q \geq n$. Then, there is $T_0 > 0$ and a unique solution of (1.1) on $[0, T_0)$ such that*

$$\begin{cases} \mathbf{v} \in BC([0,T_0); L^q(\mathbf{R}^n)) \cap L^r(0,T_0; L^s(\mathbf{R}^n)), \\ \\ t^{1/r}\mathbf{v} \in BC([0,T_0); L^s(\mathbf{R}^n)), \end{cases} \tag{2.1}$$

where $2/r + n/s = n/q$, $s > n$.

ii) *Suppose that $\mathbf{v}_0 \in L^n(\mathbf{R}^n) \cap L^{\overline{q}}(\mathbf{R}^n)$, $1 < \overline{q} < n$. Then, there is $T_0 > 0$ and a unique solution of (1.1) on $[0, T_0)$ such that*

$$\mathbf{v} \text{ and } t^{1/2}\nabla\mathbf{v} \in BC([0,T_0); L^n(\mathbf{R}^n) \cap L^{\overline{q}}(\mathbf{R}^n)). \tag{2.2}$$

iii) *Let $(0, T^*)$ be the maximal interval such that $\mathbf{v}$ solves problem (1.1) in $C((0,T^*); L^q(\mathbf{R}^n))$, $q > n$. Then*

$$\|\mathbf{v}(\tau)\|_q \geq \frac{C}{(T^* - \tau)^{\frac{q-n}{2q}}}, \tag{2.3}$$

for some constant C independent of T^* and q.

iv) *Let $\mathbf{v}$ be a solution of (1.1) on $(0, T_0)$ in the class (2.1). Suppose that $\mathbf{v}_0 \in L^2(\mathbf{R}^n)$; then $\mathbf{v}$ is also a weak solution, that is,*

$$\mathbf{v} \in L^\infty(0,T_0; L^2(\mathbf{R}^n)) \cap L^2(0,T_0; H^1(\mathbf{R}^n))$$

and $\mathbf{v}$ satisfies the energy inequality

$$\|\mathbf{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla\mathbf{v}(\tau)\|_2^2 \, d\tau \leq \|\mathbf{v}_0\|_2^2 \quad \text{for all} \quad t \in [0, T_0]. \tag{2.4}$$

v) *Let $\mathbf{v}$ be a weak solution satisfying (1.2), for some $s > n$. Then $\mathbf{v}$ belongs to $C^\infty(\mathbf{R}^n \times (0, T])$.*

Remark 2.1. The results of Proposition 2.1 hold also for the initial-boundary value problem (with homogeneous Dirichlet boundary conditions), if the domain Ω satisfies at least one of the following properties:

1) Ω is the half space $\mathbf{R}^n_+$, $n \geq 3$;
2) Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 3$, with C^∞-boundary $\partial\Omega$;
3) Ω is an exterior domain in $\mathbf{R}^n$, $n \geq 3$, i.e., a domain having a compact complement $\mathbf{R}^n \backslash \Omega$ with C^∞-boundary $\partial\Omega$.

For the proof in case 1) see Kozono [16]; if Ω satisfies 2), see Giga [11]. Finally, if Ω satisfies 3), see Iwashita [12].

Proof of Theorem 1.1. By using the results of the previous proposition, the weak solution $\mathbf{v}$ is smooth in some time-interval $(0, T^*)$, $T^* \leq T$. We suppose that this interval is maximal and *per absurdum* we suppose that $T^* < T$. In particular, $\mathbf{v}, p \in C^\infty(\mathbf{R}^n \times (0, T^*))$, and $\mathbf{v}$ is in the class (2.2). Therefore, the calculations we are going to do are completely justified and, in particular, all the boundary integrals arising in the integrations by parts, needed to obtain (2.5), vanish. Following Rionero and Galdi [22] and Beirão da Veiga [1], we multiply (1.1) by $|\mathbf{v}|^{n-2}\mathbf{v}$ and perform suitable integrations by parts over $\mathbf{R}^n$ to obtain, for $t \in (0, T^*)$,

$$\frac{1}{n}\frac{d}{dt}\|\mathbf{v}\|_n^n + \nu \int_{\mathbf{R}^n} |\mathbf{v}|^{n-2}|\nabla \mathbf{v}|^2\,dx + 4\nu \frac{n-2}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2 \leq \frac{2(n-2)}{n}\,\mathcal{I}, \tag{2.5}$$

where

$$\mathcal{I} \stackrel{def}{=} \int_{\mathbf{R}^n} |p|\,|\mathbf{v}|^{\frac{n-2}{2}}|\nabla|\mathbf{v}|^{n/2}|\,dx.$$

In the sequel we will use many times two basic inequalities. The first one is the classical interpolation inequality for $f \in L^p(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$

$$\|f\|_r \leq \|f\|_p^{\lambda}\|f\|_q^{1-\lambda} \quad \text{for } p < r < q, \quad \text{with } \lambda = \frac{p(q-r)}{r(q-p)}. \tag{2.6}$$

The other one is the following Sobolev type inequality

$$\|f\|_{\frac{n^2}{n-2}} \leq C\|\nabla|f|^{n/2}\|_2^{2/n}, \tag{2.7}$$

obtained by applying the embedding $H^1(\mathbf{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbf{R}^n)$ to the function $|f|^{n/2}$. In the sequel we denote with C possibly different positive constants that do not depend on $\mathbf{v}$ but, at most, on n and ν.

We shall distinguish several cases where the integral $\mathcal{I}$ in the right-hand side of (2.5) is estimated in different ways:

Case 1): $n \leq s < \infty$.
In this case we apply, in the order, the Hölder inequality with exponents n, $\frac{2n}{n-2}$, 2, the interpolation inequality (2.6), the Calderón-Zygmund inequality, and the Young inequality to obtain

$$\begin{aligned}
\mathcal{I} &\leq \|p\|_n\,\|\mathbf{v}\|_n^{\frac{n-2}{2}}\,\|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq \|p\|_s^{\frac{s}{2s-n}}\,\|p\|_{n/2}^{\frac{s-n}{2s-n}}\,\|\mathbf{v}\|_n^{\frac{n-2}{2}}\,\|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq C\|p\|_s^{\frac{s}{2s-n}}\,\|\mathbf{v}\|_n^{\frac{2(s-n)}{2s-n}}\,\|\mathbf{v}\|_n^{\frac{n-2}{2}}\,\|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq C\|p\|_s^{\frac{2s}{2s-n}}\,\|\mathbf{v}\|_n^{\frac{2sn-n^2-2n}{2s-n}} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.
\end{aligned}$$

Observe that, since $n \leq s < \infty$, the quantity

$$\delta \equiv \frac{2sn-n^2-2n}{2s-n} = n\left(1 - \frac{2}{2s-n}\right)$$

belongs to $(0, n)$.

Case 2): $\dfrac{n^2}{2(n-1)} \leq s < n.$

We apply, in the order, the Hölder inequality with exponents s, $\frac{2s}{s-2}$, 2, the interpolation inequality (2.6), the Sobolev inequality (2.7), and the Young inequality to obtain

$$\begin{aligned}
\mathcal{I} &\leq \|p\|_s \|\mathbf{v}\|_{\frac{s(n-2)}{s-2}}^{\frac{n-2}{2}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq \|p\|_s \|\mathbf{v}\|_n^{\frac{2sn-2s-n^2}{2s}} \|\mathbf{v}\|_{\frac{n^2}{n-2}}^{\frac{n(n-s)}{2s}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq C\|p\|_s \|\mathbf{v}\|_n^{\frac{2sn-2s-n^2}{2s}} \|\nabla|\mathbf{v}|^{n/2}\|_2^{n/s} \\
&\leq C\|p\|_s^{\frac{2s}{2s-n}} \|\mathbf{v}\|_n^{\frac{2sn-n^2-2s}{2s-n}} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.
\end{aligned}$$

Observe that, since $\dfrac{n^2}{2(n-1)} \leq s < n$, the quantity

$$\delta \equiv \frac{2sn-n^2-2s}{2s-n} = n\left(1 - \frac{2s}{n(2s-n)}\right)$$

belongs to $[0, n)$.

Case 3): $n/2 < s < \dfrac{n^2}{2(n-1)}$.

We apply, in the order, the Hölder inequality with exponents $\frac{n^2}{2(n-1)}$, $\frac{n^2}{2(n-2)^2}$, 2, the interpolation inequality (2.6), the Calderón-Zygmund inequality, the Sobolev inequality (2.7), and the Young inequality to obtain

$$\begin{aligned}
\mathcal{I} &\leq \|p\|_{\frac{n^2}{2(n-1)}} \|\mathbf{v}\|_{\frac{n^2}{n-2}}^{\frac{n-2}{2}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq \|p\|_s^{\frac{2s}{n^2-2s(n-2)}} \|p\|_{\frac{n^2}{2(n-2)}}^{\frac{n^2-2sn+2s}{n^2-2sn+4s}} \|\mathbf{v}\|_{\frac{n^2}{n-2}}^{\frac{n-2}{2}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq C\|p\|_s^{\frac{2s}{n^2-2s(n-2)}} \|\mathbf{v}\|_{\frac{n^2}{n-2}}^{2\frac{n^2-2sn+2s}{n^2-2sn+4s} + \frac{n-2}{2}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\
&\leq C\|p\|_s^{\frac{2s}{n^2-2s(n-2)}} \|\nabla|\mathbf{v}|^{n/2}\|_2^{\frac{2n^2-4sn+2n+4s}{n^2-2sn+4s}} \\
&\leq C\|p\|_s^{\frac{2s}{2s-n}} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.
\end{aligned}$$

Case 4): $s = \infty$.
We apply, in the order, the Hölder inequality with exponents $n, \frac{2n}{(n-2)}, 2$, the Calderón-Zygmund inequality, and the Young inequality to obtain

$$\begin{aligned}\mathcal{I} &\leq \|p\|_\infty^{1/2}\|p\|_{n/2}^{1/2}\|\mathbf{v}\|_n^{\frac{n-2}{2}}\|\nabla|\mathbf{v}|^{n/2}\|_2\\ &\leq C\|p\|_\infty^{1/2}\|\mathbf{v}\|_n\|\mathbf{v}\|_n^{\frac{n-2}{2}}\|\nabla|\mathbf{v}|^{n/2}\|_2\\ &\leq C\|p\|_\infty\|\mathbf{v}\|_n^n + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.\end{aligned}$$

We have thus shown that, for a suitable $\delta = \delta(s) \in [0,1]$, the following differential inequality holds:

$$\frac{1}{n}\frac{d}{dt}\|\mathbf{v}\|_n^n + \nu\int_{\mathbf{R}^n}|\mathbf{v}|^{n-2}|\nabla\mathbf{v}|^2\,dx + 2\nu\frac{n-2}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2 \leq C\|p\|_s^r\|\mathbf{v}\|_n^{n(1-\delta)}, \tag{2.8}$$

for r, s merely satisfying the assumption (1.5). Once (2.8) has been established, by Gronwall's lemma (see for instance Lemma 3 in reference [9]), if $\delta \in [0,1)$, we have that,

$$\|\mathbf{v}(t)\|_n^n \leq C\left[\|\mathbf{v}_0\|_n^n + \left(\int_0^t\|p(\tau)\|_s^r\,d\tau\right)^{\frac{1}{1-\delta}}\right]$$

and consequently

$$\mathbf{v} \in L^\infty(0,T^*;L^n(\mathbf{R}^n)). \tag{2.9}$$

If $\delta = 1$, we obtain (2.9), by integrating (2.8) (with $\delta = 1$). Integrating (2.8) with respect to t and using (2.9), we get

$$\int_0^{T^*}\int_{\mathbf{R}^n}|\mathbf{v}|^{n-2}|\nabla\mathbf{v}|^2\,dx\,d\tau = M_1 < \infty, \qquad \int_0^{T^*}\|\nabla|\mathbf{v}|^{n/2}\|_2^2\,d\tau = M_2 < \infty.$$

The latter estimate, along with (2.7), implies that

$$|\mathbf{v}|^{n/2} \in L^2(0,T^*;H^1(\mathbf{R}^n)) \subset L^2(0,T^*;L^{\frac{2n}{n-2}}(\mathbf{R}^n)),$$

that is,

$$\mathbf{v} \in L^n(0,T^*;L^{\frac{n^2}{n-2}}(\mathbf{R}^n)). \tag{2.10}$$

The interpolation inequality (2.6), together with (2.9) and (2.10), then furnishes

$$\mathbf{v} \in L^r(0,T^*;L^s(\mathbf{R}^n)), \quad \text{with} \quad \frac{2}{r} + \frac{n}{s} = 1, \quad n \leq s \leq \frac{n^2}{n-2}. \tag{2.11}$$

By i) of Proposition 2.1, for each $\epsilon > 0$, there exists $0 < \xi < \epsilon$ such that $\mathbf{v}(\xi) \in L^s(\mathbf{R}^n)$, for some $n < s \leq \frac{n^2}{n-2}$. We can now use the local existence of smooth solution starting from $u(\xi)$, since for them we can estimate the life-span of existence and we can give the blow-up estimate (2.3). The estimate (2.11) implies that this cannot happen for all L^s-norm with $s \in (n, n^2/(n-2)]$, since

$$\infty = \int_\xi^{T^*}\frac{C^r}{(T^*-\tau)^{\frac{r(s-n)}{2s}}}\,d\tau \leq \int_\xi^{T^*}\|\mathbf{v}(\tau)\|_s^r\,d\tau < \infty \quad \text{for} \quad \frac{2}{r} + \frac{n}{s} = 1.$$

Thus, we proved that $\mathbf{v} \in C^\infty(\mathbf{R}^n \times (\xi, T])$. Since $\xi > 0$ can be chosen arbitrarily small and $\mathbf{v}_0 \in L^2(\mathbf{R}^n)$, by v) of Proposition 2.1 we conclude the proof.

In the cases $n = 3, 4$, the local existence of weak solutions implies that, for each $\epsilon > 0$ there exists $0 < \eta < \epsilon$ such that a given weak solution $\mathbf{v}$ satisfies $\mathbf{v}(\eta) \in H^1(\mathbf{R}^n) \subset L^n(\mathbf{R}^n)$. By choosing an arbitrary positive η and by using the same argument used before, now on the time interval $[\eta, T^*)$, we can conclude the proof. $\square$

3. Remarks for the problem in a general domain

In this section we prove some results concerning the initial-boundary value problem in a domain Ω as in Remark 2.1 (in particular, we consider the Navier-Stokes equation in a domain $\Omega \neq \mathbf{R}^n$). We observe that the results in references [3, 4, 14, 6] hold also in a bounded domain, while that of Chae and Lee [9], due to the use of the Calderón-Zygmund inequality, applies only to the Cauchy problem (in $\mathbf{R}^3$).

When considering the problem in a domain with boundaries it is difficult to find appropriate estimates involving the pressure in terms of the velocity, and a simple inequality as (1.6) is not available.

We begin by observing that estimate (2.5) still holds; for the details see [4, 6]. We notice next that, in Case 2) of the previous theorem, we did not use the Calderón-Zygmund inequality. Therefore, the same calculations are valid also in a (smooth) domain $\Omega \subset \mathbf{R}^n$. In fact, we can prove the following result.

Theorem 3.1. *Let Ω be a domain as in Remark 2.1. Let $\mathbf{v}_0 \in H_2(\Omega) \cap L^n(\Omega)$, for $n \geq 3$. If the pressure p satisfies*

$$p \in L^r(0, T; L^s(\Omega)) \quad with \quad \frac{2}{r} + \frac{n}{s} = 2, \quad for \quad \frac{n^2}{2(n-1)} \leq s \leq n,$$

then $\mathbf{v} \in C^\infty(\overline{\Omega} \times (0, T])$. Again, in the cases $n = 3, 4$, the condition on the initial data can be relaxed to $\mathbf{v} \in H_2(\Omega)$.

Proof. We multiply the Navier-Stokes equations by $|\mathbf{v}|^{n-2}\mathbf{v}$ and we integrate by parts over Ω to obtain

$$\frac{1}{n}\frac{d}{dt}\|\mathbf{v}\|_n^n + \nu \int_\Omega |\mathbf{v}|^{n-2}|\nabla \mathbf{v}|^2\, dx + 4\nu\frac{n-2}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2 \leq \frac{2(n-2)}{n}\mathcal{I}',$$

where

$$\mathcal{I}' \stackrel{def}{=} \int_\Omega |p||\mathbf{v}|^{\frac{n-2}{2}}|\nabla|\mathbf{v}|^{n/2}|\, dx,$$

and now $\|\,.\,\|_p$ denotes the norm of the Lebesgue space $L^p(\Omega)$.

When $n^2/2(n-1) \leq s < n$, we use the same estimate as in Case 2. The only result to be proved is when $s = n$. In this case we can estimate the term $\mathcal{I}'$ in the following way, by applying the Hölder inequality with exponents n, $\frac{2n}{n-2}$, 2, and the Young inequality:

$$\begin{aligned}\mathcal{I}' &\leq \|p\|_n \|\mathbf{v}\|_n^{\frac{n-2}{n}} \|\nabla|\mathbf{v}|^{n/2}\|_2 \\ &\leq C\|p\|_n^2\|\mathbf{v}\|_n^{n-2} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.\end{aligned}$$

The proof follows now by using the same techniques used in the proof of Theorem 1.1. For the local existence of smooth solutions with initial data in $L^n(\Omega)$, and for the regularity of solutions satisfying (1.2), see Remark 2.1. $\square$

Remark 3.2. Due to the aforementioned lack of estimate (1.6) when $\Omega \neq \mathbf{R}^n$, the validity of Theorem 1.1 remains an interesting open question in such a case. In this situation, the best results known, for $s > n$, are that given in [3, 6].

Finally, we prove the following theorem which greatly improves the result shown by O'Leary [20].

Theorem 3.3. *Let Ω be either the whole space $\mathbf{R}^n$, or a domain as in Remark 2.1. Let be $\mathbf{v}_0 \in H_2(\Omega) \cap L^n(\Omega)$. If ∇p satisfies the condition*

$$\nabla p \in L^r(0,T;L^s(\Omega)) \quad \text{with} \quad \frac{2}{r}+\frac{n}{s}=3, \quad \text{for} \quad s \in \left[\frac{n^2}{3n-2}, n\right], \tag{3.1}$$

then $\mathbf{v} \in C^\infty(\overline{\Omega} \times (0,T])$.

Proof. We multiply the Navier-Stokes equations by $|\mathbf{v}|^{n-2}\mathbf{v}$ and we integrate by parts over Ω to obtain

$$\frac{1}{n}\frac{d}{dt}\|\mathbf{v}\|_n^n + \nu\int_\Omega |\mathbf{v}|^{n-2}|\nabla \mathbf{v}|^2\,dx + 4\nu\frac{n-2}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2 \le \int_\Omega |\nabla p|\,|\mathbf{v}|^{n-1}\,dx. \tag{3.2}$$

We estimate the right-hand side

$$\mathcal{I}'' = \int_\Omega |\nabla p|\,|\mathbf{v}|^{n-1}\,dx$$

in the following way, by using, in the order, the Hölder inequality with $\frac{1}{s}+\frac{1}{s'}=1$, the interpolation inequality (2.6), the Sobolev inequality (2.7), and the Young inequality:

$$\begin{aligned}
\mathcal{I}'' &\le \|\nabla p\|_s\, \|\,|\mathbf{v}|^{n/2}\|_{\frac{2(n-1)s'}{n}}^{\frac{2(n-1)}{n}} \\
&\le \|\nabla p\|_s\, \|\,|\mathbf{v}|^{n/2}\|_2^{\frac{n^2-(n-1)(n-2)s'}{ns'}}\, \|\,|\mathbf{v}|^{n/2}\|_{\frac{2n}{n-2}}^{\frac{(n-1)s'-n}{s'}} \\
&\le C\|\nabla p\|_s \|\mathbf{v}\|_n^{\frac{n^2-(n-1)(n-2)s'}{2s'}}\, \|\nabla|\mathbf{v}|^{n/2}\|_2^{\frac{(n-1)s'-n}{s'}} \\
&\le C\|\nabla p\|_s^{\frac{2s'}{n+(3-n)s'}} \|\mathbf{v}\|_n^{\frac{n^2-(n-1)(n-2)s'}{n+(3-n)s'}} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2 \\
&= C\|\nabla p\|_s^{\frac{2s}{3s-n}} \|\mathbf{v}\|_n^{\frac{n^2-(n-1)(n-2)s'}{n+(3-n)s'}} + \frac{2\nu(n-2)}{n^2}\|\nabla|\mathbf{v}|^{n/2}\|_2^2.
\end{aligned}$$

Observe that

$$\frac{n^2-(n-1)(n-2)s'}{n+(3-n)s'} \le n, \qquad \forall\, s' \ge 1$$

and the proof follows as in the previous theorems. $\square$

Remark 3.4. The interest in the last result stems essentially on the limit case

$$\nabla p \in L^1(0, T; L^n(\Omega)).$$

In fact, for Ω a (Lipschitz) bounded subset of $\mathbf{R}^n$, setting

$$D^{1,q}(\Omega) \stackrel{def}{=} \left\{ \phi \in L^1_{loc}(\Omega) \text{ such that } \int_\Omega |\nabla \phi|^q \, dx < \infty \right\},$$

we have that $D^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ for $1/q^* = 1/q - 1/n$, when $q < n$. In the case where $q = n$ we have $D^{1,n}(\Omega) \not\subset L^\infty(\Omega)$. When $\Omega = \mathbf{R}^n$, the inclusion still holds if we impose that the functions "vanish at infinity." (This assumption is not too restrictive, since we can impose $p \to 0$ as $|x| \to \infty$, due to the fact that the pressure is known up to an additive constant). This observation implies, in the light of (3.1), that the condition in Theorem 1.1 can be improved to the following one:

$$p \in L^1(0, T; L^\infty(\mathbf{R}^n) \cup D^{1,n}(\mathbf{R}^n)).$$

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