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ON ANTI-AUTOMORPHISMS OF THE FIRST KIND IN DIVISION RINGS

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ABSTRACT. We provide a short proof of the well-known fact that a division ring of finite degree over its center that admits an anti-automorphism of the first kind, i.e., an anti-automorphism that fixes the center elementwise, also admits an involutive anti-automorphism.

Albert has shown that a division ring of finite degree over its center that admits an anti-automorphism of the first kind also admits an involutive anti-automorphism; cf. Theorem X.19 of [1]. A proof of this theorem can now be found in a number of textbooks, e.g. [2], [3], [4], [5], [6], or [7]. The purpose of this note is to give another short proof.

Definition 1. A finite dimensional central simple algebra A is called a crossed product if A has a maximal subfield $\mathbb{K}$ which is Galois over the center of A.

Theorem 2. *Let A be a crossed product and let $\mathbb{F}$ be the center of A. Suppose that A allows an anti-automorphism that fixes $\mathbb{F}$ elementwise. Then A allows an involutive anti-automorphism.*

Proof. Let $\sigma : A \to A$ be an anti-automorphism that fixes $\mathbb{F}$ elementwise. Since A is a crossed product, there exists a maximal subfield $\mathbb{K}$ of A such that $\mathbb{K}/\mathbb{F}$ is Galois. Denote the corresponding Galois group by G. By Theorem 3.1.27 of [5] for every $g \in G$ there exists an element $j_g \in A$ with $g(k) = j_g k j_g^{-1}$ for all $k \in \mathbb{K}$, such that $A = \bigoplus_{g \in G} \mathbb{K} j_g$. Using the Skolem-Noether theorem, we can assume that σ fixes $\mathbb{K}$ elementwise and so for every $g \in G$ we have $j_g k j_g^{-1} = g(k) = \sigma(g(k)) = \sigma(j_g k j_g^{-1}) = \sigma(j_g)^{-1} k \sigma(j_g)$. Therefore $\sigma(j_g) j_g \in C_A(\mathbb{K}) = \mathbb{K}$. Hence $\sigma(j_g) = a_g j_g^{-1}$ for some $a_g \in \mathbb{K}^*$. This implies $\sigma^2(j_g) = \sigma(j_g^{-1}) a_g = j_g$, and thus σ^2 is the identity. □

Corollary 3. *Let $\mathbb{D}$ be a division ring of finite degree over its center $\mathbb{F}$. Suppose that $\mathbb{D}$ allows an anti-automorphism that fixes $\mathbb{F}$ elementwise. Then $\mathbb{D}$ allows an involutive anti-automorphism.*

Proof. There exists an $n \in \mathbb{N}$ such that $M_n(\mathbb{D})$ is a crossed product by Theorem V.1 of [1]. We can extend σ to an anti-automorphism $\overline{\sigma}$ of $M_n(\mathbb{D}) = \mathbb{D} \otimes_{\mathbb{F}} M_n(\mathbb{F})$ by letting $\overline{\sigma}$ act as the transposition map on $1 \otimes M_n(\mathbb{F})$. Theorem 2 implies that

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there exists an involutive anti-automorphism on $M_n(\mathbb{D})$. By Theorem 3.1.70 of [5], we conclude that $\mathbb{D}$ allows an involutive anti-automorphism. □

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