DISTINCT GAPS BETWEEN FRACTIONAL PARTS OF SEQUENCES

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ABSTRACT. Let $\alpha$ be a real number, $N$ a positive integer and $\mathcal{N}$ a subset of \{0, 1, 2, \ldots, N\}. We give an upper bound for the number of distinct lengths of gaps between the fractional parts \{n$\alpha$\}, $n \in \mathcal{N}$.

1. INTRODUCTION

Questions on the distribution of fractional parts of sequences have a long history, and among the most intensively studied are those related to polynomial sequences. After the classical work of Weyl [11] on uniform distribution mod 1, other aspects of the distribution of fractional parts of polynomials, especially questions concerned with small fractional parts, have been investigated (see Schmidt [8] and Baker [1]). Recently, the distribution of gaps between fractional parts of sequences has attracted attention. Following the work of Rudnick and Sarnak [5] on the pair correlation of fractional parts of polynomials, other related questions have been studied in [2], [6] and [7]. We mention that the distribution of the local spacings between the fractional parts \{n$^d\alpha$\}, $n \in \mathbb{N}$, in the case $d = 1$ is completely different than in the case $d > 1$. If $d > 1$ one expects that for almost all $\alpha$ the distribution is Poissonian, and one knows for instance that the pair correlation is Poissonian indeed (see [5]). If $d = 1$ one knows for a fact that the distribution is not Poissonian, and this is a consequence of the following Three Gap Theorem of Steinhaus (see [4], [9] and [10]):

Let $\alpha$ be a real number and $N$ a nonnegative integer. Then the fractional parts \{n$\alpha$\}, $0 \leq n \leq N$, partition the unit interval into $N + 1$ intervals which have at most 3 different lengths.

The correlation of fractional parts \{n$\alpha$\}, $n \in \mathbb{N}$, have been recently investigated by Marklof [3]. In this paper we take a real number $\alpha$, a positive integer $N$, a subset $\mathcal{N}$ of \{0, 1, 2, \ldots, N\} and look at the set of fractional parts

$$\mathcal{M} = \mathcal{M}(\alpha, \mathcal{N}) = \{n\alpha : n \in \mathcal{N}\},$$

with the intention of proving a result which is independent of $\mathcal{N}$. Clearly, as far as uniform distribution or small fractional parts are concerned, no such result is possible (for instance $\mathcal{N}$ might coincide with the set of those $1 \leq n \leq N$ for which \{n$\alpha$\} $\in \left(\frac{1}{3}, \frac{2}{3}\right)$). The same goes for the spacing distribution: if $\alpha$ is irrational, then the set \{n$\alpha$\}, $n \in \mathbb{N}$, is dense in [0, 1], and one can choose for large $N$ a sparse
set $\mathcal{N}$ for which the distribution of $\mathcal{M}$ in $[0, 1]$ approaches any given distribution. What we will do is to look at the gaps between the elements of $\mathcal{M}$ and see whether any kind of Steinhaus phenomenon still holds in this generality. Thus we arrange the elements of $\mathcal{M}$ in ascending order and consider the number $l(\alpha, \mathcal{N})$ of distinct lengths of gaps between consecutive elements of $\mathcal{M}(\alpha, \mathcal{N})$. Hence $l(\alpha, \mathcal{N}) \leq 3$ when $\mathcal{N} = \{0, 1, 2, \ldots, N\}$, by the Three Gap Theorem. For a general subset $\mathcal{N}$ of $\{0, 1, 2, \ldots, N\}$, $l(\alpha, \mathcal{N})$ can be much larger. For example, if $N$ is a positive integer, $0 < \alpha < \frac{1}{2}$ and $\mathcal{N}$ consists of the squares $\{0, 1, 4, 9, \ldots, \lceil \sqrt{N} \rceil^2\}$, then the numbers $na$, $n \in \mathcal{N}$, coincide with their fractional parts, and all the gaps between consecutive elements of $\mathcal{M}$ have distinct lengths. Thus $l(\alpha, \mathcal{N})$ can be as large as $\sqrt{N}$. The object of this paper is to prove the following theorem, which shows that $l(\alpha, \mathcal{N})$ cannot be much larger than $\sqrt{N}$.

**Theorem 1.** For any real number $\alpha$, any positive integer $N$ and any subset $\mathcal{N}$ of $\{0, 1, 2, \ldots, N\}$ one has

$$l(\alpha, \mathcal{N}) < (2 + 2\sqrt{2})\sqrt{N}.$$ 

2. Proof of Theorem 1

Fix a positive integer $N$, then choose a real number $\alpha$ and a subset $\mathcal{N}$ of $\{0, 1, 2, \ldots, N\}$ such that $l(\alpha, \mathcal{N})$ is largest. Note first that for fixed $\mathcal{N}$, the function $\alpha \mapsto l(\alpha, \mathcal{N})$ is periodic mod 1; thus we may assume in what follows that $0 \leq \alpha < 1$. In case $\alpha = 0$ all the fractional parts $\{na\}$ are zero, so the maximum value of $l(\alpha, \mathcal{N})$ is attained for some $\alpha \in (0, 1)$. Next, notice that for $\mathcal{N}$ fixed, the lengths of the gaps between the elements of $\mathcal{M}(\alpha, \mathcal{N})$ are continuous functions of $\alpha$. Thus there is an $\varepsilon = \varepsilon(\alpha, \mathcal{N}) > 0$ such that

$$(1) \quad l(\beta, \mathcal{N}) \geq l(\alpha, \mathcal{N})$$

for any $\beta \in (\alpha - \varepsilon, \alpha + \varepsilon)$. If $\alpha$ and $\mathcal{N}$ are chosen as above such that $l(\alpha, \mathcal{N})$ is largest, then one has equality in (1). Replacing if necessary $\alpha$ by an irrational number $\beta \in (\alpha - \varepsilon, \alpha + \varepsilon)$ we may assume in the following that $0 < \alpha < 1$ is irrational. This last assumption is not essential in our proof, but it makes the presentation cleaner. For instance, in this case the fractional parts $\{na\}$, $n \in \mathcal{N}$, will be distinct, and we will discuss in detail the order of these fractional parts. To proceed, recall Dirichlet’s theorem which asserts that for any positive integer $M$ there are coprime integers $a$, $q$ with $1 \leq q \leq M$ such that

$$(2) \quad \left| \alpha - \frac{a}{q} \right| < \frac{1}{qM}.$$ 

We use (2) with $M = 2N$, so let $a \in \mathbb{Z}$ and $1 \leq q \leq 2N$ such that $(a, q) = 1$ and

$$(3) \quad \left| \alpha - \frac{a}{q} \right| < \frac{1}{2qN}.$$ 

Since $0 < \alpha < 1$ we see that $0 \leq a \leq q$. From (3) it follows that for any $n \in \mathcal{N}$ one has

$$(4) \quad \left| n\alpha - \frac{na}{q} \right| < \frac{1}{2q}.$$ 

Let us consider the open intervals $J_k = \left( \frac{k}{q} - \frac{1}{2q}, \frac{k}{q} + \frac{1}{2q} \right)$, $k = 0, 1, \ldots, q - 1$. For any $n \in \mathcal{N}$ let $k(n) \in \{0, 1, \ldots, q - 1\}$ be such that $an \equiv k(n)$ (mod $q$). Then
the fractional part \( \{ \frac{n\alpha}{q} \} \) coincides with the center \( \frac{k(n)}{q} \) of the interval \( J_{k(n)} \), and from (4) it follows that \( \{ n\alpha \} \) belongs to \( J_{k(n)} \). Therefore for any \( n, n' \in \mathcal{N} \) for which \( k(n) \neq k(n') \) the order of the elements \( \{ n\alpha \}, \{ n'\alpha \} \in \mathcal{M} \) will simply be given by the order of the numbers \( k(n) \) and \( k(n') \). On the other hand, if \( n, n' \in \mathcal{N} \) are such that \( k(n) = k(n') \), then the order of \( \{ n\alpha \}, \{ n'\alpha \} \) is determined by the sign of \( \alpha - \frac{n}{q} \) and the order of the numbers \( n \) and \( n' \). To be precise, let \( \alpha - \frac{n}{q} = \eta \) and assume in what follows that \( \eta > 0 \). The case \( \eta < 0 \) is similar and will be left to the reader. Since \( n\alpha = n\eta + \frac{na}{q} \), where as we know \( |n\eta| < \frac{1}{2q} \), the relative “coordinate” of \( \{ n\alpha \} \) with respect to the center \( \frac{k(n)}{q} \) of \( J_{k(n)} \) will equal \( n\eta \). With our assumption on \( \eta \), the order of \( \{ n\alpha \}, \{ n'\alpha \} \) in case \( k(n) = k(n') \) will be the same as the order of \( n, n' \). Here the condition \( k(n) = k(n') \) is equivalent to the condition \( n \equiv n' \pmod{r} \). We now have a more clear picture of the distribution of the elements of \( \mathcal{M} \). Write \( \mathcal{N} = \bigcup_{r=0}^{q-1} \mathcal{N}_r \), where \( \mathcal{N}_r = \{ n \in \mathcal{N} : n \equiv r \pmod{q} \} \). Each \( \mathcal{N}_r \) corresponds uniquely to a \( J_k \), given by \( k = k(r) \equiv ar \pmod{q} \), respectively \( r = r(k) \equiv \overline{a}k \pmod{q} \), where \( \overline{a} \) denotes the inverse of \( a \pmod{q} \). For any \( r \), the map \( n \mapsto \{ n\alpha \} \) sends \( \mathcal{N}_r \) monotonically to a subset of \( J_{k(r)} \). We now distinguish two kinds of gaps \( \{ \{ n\alpha \}, \{ n'\alpha \} \} \) between consecutive elements \( \{ n\alpha \}, \{ n'\alpha \} \) of \( \mathcal{M} \), according as to whether \( k(n) = k(n') \) or \( k(n) \neq k(n') \), and count them separately. Denote by \( l_1 \), respectively \( l_2 \), the number of distinct lengths of gaps of the first kind, respectively of the second kind, between consecutive elements of \( \mathcal{M} \). Some gaps of the first kind might have the same lengths as certain gaps of the second kind. Anyway one has

\[
l(\alpha, \mathcal{N}) \leq l_1 + l_2.
\]

In order to get an upper bound for \( l_1 \), we allow \( r \) to run over the set \( \{ 0, 1, \ldots, q-1 \} \) and for each such value of \( r \) we look at the gaps formed by the image of \( \mathcal{N}_r \) in \( J_{n(r)} \). We already know that consecutive elements of \( \mathcal{N}_r \) correspond to consecutive elements of \( \mathcal{M} \). Moreover, if \( n < n' \) are consecutive elements of \( \mathcal{N}_r \), then the length of the gap between \( \{ n\alpha \} \) and \( \{ n'\alpha \} \) equals the difference between their coordinates in \( J_{k(r)} \), which is \( (n' - n)\eta \). Thus the lengths of these gaps in \( \mathcal{M} \) are proportional to the lengths of the gaps \( (n'-n) \) in \( \mathcal{N}_r \), by a factor \( \eta \) which is independent of \( r \). It follows that \( l_1 \) equals the cardinality of the set

\[
A = \bigcup_{r=0}^{q-1} \{ n' - n : n, n' \text{ consecutive in } \mathcal{N}_r \}.
\]

Now the point is that since each element of \( A \) is a positive multiple of \( q \), the sum of its \( l_1 \) (distinct) elements will be at least

\[
q + 2q + \cdots + l_1q = \frac{q l_1 (l_1 + 1)}{2}.
\]

On the other hand, if we add all the elements of \( A \) counted with multiplicities, the sum will equal

\[
\sum_{r=0}^{q-1} \sum_{n, n' \text{ consecutive in } \mathcal{N}_r} (n' - n) = \sum_{r=0}^{q-1} (\max \mathcal{N}_r - \min \mathcal{N}_r) < Nq.
\]

It follows that \( \frac{q l_1 (l_1 + 1)}{2} < Nq \), which implies

\[
l_1 < \sqrt{2N}.
\]
We now turn to \( l_2 \). Some of the above sets \( \mathcal{N} \) might be empty, resulting in some intervals \( J_k \) having no points from \( \mathcal{M} \). Let \( 0 \leq k_1 < k_2 < \cdots < k_s \leq q - 1 \) be those values of \( k \) for which \( J_k \cap \mathcal{M} \) is nonempty. Then for each pair \((k_j, k_{j+1})\) we have exactly one gap of the second kind. Its left and right endpoints are the largest \( \delta \) and, respectively, the smallest element of \( \mathcal{M} \cap J_{k_{j+1}} \). Thus the length of this gap, which we denote by \( \delta_j \), is given by

\[
\delta_j = \frac{k_{j+1} - k_j}{q} + \overline{\delta}_{j+1} + \underline{\delta}_j \eta.
\]

where for any \( j, \underline{\delta}_j \) and \( \overline{\delta}_j \), stand for the smallest, respectively the largest, element of \( \mathcal{N}_{r(k_j)} \). The distance between the centers of \( J_k \) and \( J_{k_{j+1}} \) equals \( \frac{k_{j+1} - k_j}{q} \) and the coordinates of \( \{\overline{\delta}_j, \underline{\delta}_j\} \) with respect to these centers are \( \overline{\delta}_j \eta \) and \( \underline{\delta}_j \eta \) respectively. Hence

\[
(7) \quad \delta_j = \frac{k_{j+1} - k_j}{q} + \underline{\delta}_{j+1} + \overline{\delta}_j \eta.
\]

A trivial upper bound for \( l_2 \) is

\[
(8) \quad l_2 \leq s \leq q.
\]

For each positive integer \( b \), let \( n(b) \) be the number of distinct lengths \( \delta_j \) of gaps of the second kind for which \( k_{j+1} - k_j = b \). Thus \( l_2 \) can be written as

\[
(9) \quad l_2 = \sum_{b \geq 1} n(b).
\]

Here we used the fact that if \( k_{j+1} - k_j = b \neq b' = k_{j'+1} - k_{j'} \), then \( \delta_j \neq \delta_{j'} \). This is a consequence of the inequalities \( k_{j+1} - k_j - \frac{1}{2} < q \delta_j < k_{j+1} - k_j + \frac{1}{2} \), which in turn follow from (7) and the inequality \( 0 \leq n \leq \frac{1}{q} \), valid for any \( n \in \mathcal{N} \). Note that

\[
(10) \quad \sum_{b \geq 1} n(b) b \leq \sum_j (k_{j+1} - k_j) \leq q.
\]

We claim that for any \( b \) one has

\[
(11) \quad n(b) \leq \left\lceil \frac{2N}{q} \right\rceil + 1,
\]

where \( \lceil \cdot \rceil \) denotes the greatest integer part function. In order to prove the claim, let \( j_1, \ldots, j_{n(b)} \) be such that \( \delta_{j_1}, \ldots, \delta_{j_{n(b)}} \) are distinct and

\[
k_{j_1} + 1 - k_j = \cdots = k_{j_{n(b)}}(b+1) - k_{j_{n(b)}} = b.
\]

By (7) we know that

\[
\delta_j = b + \eta(n_{j+1} - \overline{\delta}_j)
\]

for any \( j \in \{j_1, \ldots, j_{n(b)}\} \). The numbers \( \delta_{j_1}, \ldots, \delta_{j_{n(b)}} \) being distinct, it follows that as \( j \) runs over the set \( \{j_1, \ldots, j_{n(b)}\} \), the numbers \( n_{j_{j+1}} - \overline{\delta}_j \) are distinct. Recall that \( \overline{\delta}_j \in \mathcal{N}_{r(k_j)} \) and \( \underline{\delta}_{j+1} \in \mathcal{N}_{r(k_{j+1})} \), so they satisfy the congruences

\[
\overline{\delta}_j \equiv r(k_j) \equiv \overline{\delta}_k \pmod{q}
\]

and

\[
\underline{\delta}_{j+1} \equiv r(k_{j+1}) \equiv \underline{\delta}_{j+1} \pmod{q}.
\]
Hence
\[ \mu_{j+1} - \pi_j \equiv \pi (k_{j+1} - k_j) \equiv \pi b \mod q \]
for any \( j \in \{ j_1, \ldots, j_{\alpha(b)} \} \). Note also that for any \( j \) one has
\[ -N \leq \mu_{j+1} - \pi_j \leq N. \]

There are at most \( 1 + \left[ \frac{2N}{q} \right] \) integers in the interval \([-N, N]\) which are congruent to \( \pi b \mod q \), and this proves (11). Next, from (9) we know that the left-hand side of (10) is a sum of exactly \( l_2 \) terms, counting with multiplicities. By using (11) one sees that the left-hand side of (10) is at least as large as the sum
\[ \left( 1 + \left[ \frac{2N}{q} \right] \right) \cdot 1 + \left( 1 + \left[ \frac{2N}{q} \right] \right) \cdot 2 + \cdots + \left( 1 + \left[ \frac{2N}{q} \right] \right) u + v (u + 1), \]
where \( u \) and \( v \) are given by
\[ (12) \quad u = \left\lfloor \frac{l_2}{1 + \left[ \frac{2N}{q} \right]} \right\rfloor \]
and
\[ v = l_2 - \left( 1 + \left[ \frac{2N}{q} \right] \right) u. \]

We combine this with (10) to derive
\[ \left( 1 + \left[ \frac{2N}{q} \right] \right) u (u + 1) \leq q, \]
which implies
\[ (13) \quad u < \left( \frac{2q}{1 + \left[ \frac{2N}{q} \right]} \right)^{\frac{1}{2}}. \]

Relations (12) and (13) give
\[ \frac{l_2}{1 + \left[ \frac{2N}{q} \right]} - 1 < \left( \frac{2q}{1 + \left[ \frac{2N}{q} \right]} \right)^{\frac{1}{2}}, \]
from which we get the following upper bound for \( l_2 \):
\[ (14) \quad l_2 < 1 + \left[ \frac{2N}{q} \right] + \left( 2q \left( 1 + \left[ \frac{2N}{q} \right] \right) \right)^{\frac{1}{2}} < 1 + \frac{2N}{q} + (2q + 4N)^{\frac{1}{2}}. \]

Since \( q \leq 2N \), from (14) we obtain
\[ (15) \quad l_2 < 1 + \frac{2N}{q} + 2 \sqrt{2N}. \]

This inequality is sharp when \( q \) is at least of the size of \( \sqrt{N} \). If \( q \) is smaller, then we use (5). From (5) and (15) we find that
\[ (16) \quad l_2 < (2 + \sqrt{2}) \sqrt{N} \]
regardless of the size of \( q \). On combining (5), (6) and (16) we get
\[ l(\alpha, N) < (2 + 2\sqrt{2}) \sqrt{N}, \]
which completes the proof of Theorem 1.
References


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