PSEUDONORMALITY AND STARCOMPACTNESS OF σ-PRODUCTS

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Abstract. In this paper we shall prove the following: For every non-trivial σ-product σ, of uncountable number of spaces, having at least two points, σ \( \setminus \sigma_n \) is not pseudonormal. And every non-trivial σ-product is not strongly starcompact.

1. Introduction

Throughout this paper we assume that each space is a \( T_1 \)-space having at least two points. We recall the definition of σ-products which were introduced by H. H. Corson [4].

Definition 1. Let \( S = \{ X_\alpha | \alpha \in \Omega \} \) be a family of spaces. “σ = σ(\( S \)) is called a σ-product of \( S \)” if there is a point \( x^* = (x^*_\alpha)_{\alpha \in \Omega} \in X = \Pi \{X_\alpha | \alpha \in \Omega\} \) (called the base point of σ) such that σ is the subspace of \( X \) consisting of \( x \in X \) such that \( Q(x) \) is finite. Here \( Q(x) = \{ \alpha | \alpha \in \Omega, x_\alpha \neq x^*_\alpha \} \). Let \( |\Omega| = \{a \subset \Omega : |a| = n\} \) for each \( n \in \omega \) and put \( |\Omega|^{\omega} = \bigcup\{|\Omega|^n : n \in \omega\} \). Here \(|a|\) denotes the cardinal number of \( a \).

Σ = \( \{x \in X : |Q(x)| \leq \omega\} \) is called a Σ-product of \( S \).

A σ-product σ (resp. Σ-product Σ) is called non-trivial if \( \sigma \neq X \) (resp. \( \Sigma \neq X\)).

Let \( X \) be a space and \( \tau \) be an infinite cardinal number such that \( |\Omega| = \tau \). In case \( X_\alpha = X \) for each \( \alpha \in \Omega \), let us denote \( \sigma(\( S \)) \) by \( \sigma(X^\tau) \). For \( a \in X \), we denote by \( a^* = (a_\alpha)_{\alpha \in \Omega}, a_\alpha = a \) for each \( \alpha \in \Omega \).

For a finite subset \( F \) of \( \Omega \), \( \Pi \{X_\alpha | \alpha \in F\} \) is said to be a finite subproduct of \( \sigma \).

The following fact concerning σ-products is known.

Fact. Let \( \sigma = \sigma(\( S \)) \) and \( \sigma_n = \{x \in \sigma : |Q(x)| \leq n\} \) for each \( n \in \omega \). Then \( \sigma_n \) is closed in \( \sigma \).

In this paper we investigate normality-type properties of special subspaces of σ-products and compactness-type properties of σ-products.

In 1959, Corson [4] proved that for every non-trivial Σ-product Σ, a subspace \( \Sigma \setminus \{x\} \) is not normal for every point \( x \in \Sigma \). In 1978, A. P. Kombarov [8] proved that if a set \( Z \) is closed in the τ-envelope \( Y = Y(x^*, \tau) = \{y \in X = \Pi \{X_\alpha | \alpha \in \Omega\} : |Q(y)| < \tau\} \) and \( |\bigcup\{Q(z) : z \in Z\}| < \tau \), then \( Y \setminus Z \) is a non-normal subset.
of $Y$. As corollaries of this theorem we have non-normality of $\Sigma \setminus \{x\}$, where $\Sigma$ is a non-trivial $\Sigma$-product and $x \in \Sigma$, and non-normality of $\sigma \setminus \{x\}$, where $\sigma$ is a non-trivial $\sigma$-product of uncountable number of spaces and $x \in \sigma$.

A space $X$ is called pseudonormal if any two disjoint closed sets, one of which is countable, are separated by open sets in $X$. Obviously, any normal space is pseudonormal.

In 1996, Kombarov [9] proved that if $Y$ is a $\tau$-envelope of spaces $X_\alpha, \alpha \in \Omega$, $|\Omega| \geq \max\{\omega_1, \tau\}$, then a subspace $Y \setminus \{x\}$ is not pseudonormal for every $x \in Y$. In particular he obtained the following.

**Theorem A (Kombarov [9])**. Let $S = \{X_\alpha| \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(S)$. Then $\sigma \setminus \sigma_0$ is not pseudonormal.

In this paper we shall prove a generalization of Theorem A.

## 2. Normality and Pseudonormality

**Theorem 1**. Let $S = \{X_\alpha| \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(S)$. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.

**Lemma 1**. Let $S = \{X_\alpha| \alpha \in \Omega\}$ such that $|\Omega| \geq \omega_1$. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Here $2 = \{0, 1\}$ is the discrete space of two points. Then there is a homeomorphism $f$ from $\sigma$ onto $f(\sigma) \subseteq \sigma'$ such that $f(0^*) = x^*$ and $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$. Here $\sigma' = \sigma(S)$ with the base point $x^*$.

**Proof**. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega_1 \subset \Omega$. Let $f: \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega_1} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha \in \Omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega_1 \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise}. \end{cases}$$

Then $f$ has the desired properties. To prove that $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$, let $y \in (\sigma' \setminus \sigma'_n) \setminus f(\sigma \setminus \sigma_n)$. Then $Q(y) \cap (\Omega \setminus \omega_1) \neq \emptyset$. Let us choose an element $\alpha \in Q(y) \cap (\Omega \setminus \omega_1)$ and put $U = \{z \in \sigma' \setminus \sigma'_n| z_\alpha \neq x^*_\alpha\}$. Then $U$ is an open neighborhood of $y$ in $\sigma' \setminus \sigma'_n$ such that $U \cap f(\sigma \setminus \sigma_n) = \emptyset$.

Since pseudonormality is inherited by closed subspaces, Theorem 1 follows from Proposition 1 below by using Lemma 1.

**Proposition 1**. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.

**Proof**. We denote $\sigma = \{f: \omega_1 \to 2|Q(f) \text{ is finite}\}$. Here $Q(f) = \{\alpha \in \omega_1| f(\alpha) = 1\}$.

Put $G = \sigma \setminus \sigma_n$. Let us choose a subset $A \subset \omega_1$ such that $|A| = n$. For each $\alpha \in \omega_1$, let us define $f^\alpha: \omega_1 \to 2$ as follows:

$$f^\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \in A \cup \{\alpha\}, \\ 0 & \text{if } \beta \in \omega_1 \setminus (A \cup \{\alpha\}). \end{cases}$$

Then

1. $f^\alpha \in \sigma_{n+1} \setminus \sigma_n$ for each $\alpha \in \omega_1 \setminus A$.

Let us choose subsets $\Gamma_1$ and $\Gamma_2$ of $\omega_1$ such that $|\Gamma_1| = \omega_1, |\Gamma_2| = \omega, \Gamma_1 \cap \Gamma_2 = \emptyset, (\Gamma_1 \cup \Gamma_2) \cap A = \emptyset, \omega_1 = \Gamma_1 \cup \Gamma_2 \cup A$. Put $E = \{f^\alpha| \alpha \in \Gamma_1\}$ and $F = \{f^\alpha| \alpha \in \Gamma_2\}$. 


Then $E \cap F = \emptyset$, $|F| = \omega$ and

(i) $E$ and $F$ are closed subsets in $G$;
(ii) $E$ and $F$ are not separated by open sets in $G$.

Proof of (i). To prove that $E$ is closed in $G$, let $f \in G \setminus E$. If $|Q(f)| \geq n + 2$, then $f \notin \sigma_{n+1}$. Put $U = G \setminus \sigma_{n+1}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. If $|Q(f)| = n + 1$, then $Q(f) \setminus A \neq \emptyset$ because $|A| = n$. Let $\alpha \in Q(f) \setminus A$. Then $f(\alpha) = 1$. If $|A \cap Q(f)| = n$, then $\alpha \in \Gamma_2$ because $f \notin E$. Put $U = \{g \in G|g(\alpha) = 1\}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. If $|A \cap Q(f)| < n$, then there exists an element $\beta \in A$ such that $f(\beta) = 0$. Put $U = \{g \in G|g(\beta) = 0\}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. Therefore it is proved that $E$ is closed in $G$. Quite similarly it is proved that $F$ is closed in $G$.

Proof of (ii). Let $U$ be an arbitrary open set in $G$ such that $E \subset U$. Then, for each $\alpha \in \Gamma_1$, there is a finite set $r_\alpha$ of $\omega_1$ such that $A \cup \{\alpha\} \subset r_\alpha$ and $f^{o^*} \in U_\alpha = \{g \in G|g(\beta) = 1\}$ for each $\beta \in A \cup \{\alpha\}$, $g(\beta) = 0$ for each $\beta \in r_\alpha \setminus (A \cup \{\alpha\}) \subset U$. By Šanin’s lemma, there are an uncountable set $\Gamma^* \subset \Gamma_1$ and a finite set $r^{*} \subset \omega_1$ such that $\{r_\alpha \setminus r^{*} | \alpha \in \Gamma^*\}$ is disjoint. Since $\Gamma_2$ is infinite and $r^{*}$ is finite, $\Gamma_2 \setminus r^{*} \neq \emptyset$.

Let $\alpha^{*} \in \Gamma_2 \setminus r^{*}$. Then

(2) $f^{a^{*}} \in F \cap \text{cl}U$.

It is obvious that $f^{\alpha^{*}} \in F$. Therefore it is sufficient to prove the following.

Claim. $f^{\alpha^{*}} \in \text{cl}U$.

To prove this, let $V$ be an arbitrary open set in $G$ such that $f^{\alpha^{*}} \subset V$. Then there is a finite set $r$ of $\omega_1$ such that $A \cup \{\alpha^{*}\} \subset r$ and $f^{a^{*}} \in V^{f} \equiv \{g \in G|g(\beta) = 1\}$ for each $\beta \in A \cup \{\alpha^{*}\}$, $g(\beta) = 0$ for each $\beta \in r \setminus (A \cup \{\alpha^{*}\}) \subset V$.

Since $\Gamma^*$ is uncountable and $r$ and $r^{*}$ are finite, there is an element $\beta^{*} \in \Gamma^*$ such that $r \cap (r_{\beta^{*}} \setminus r^{*}) = \emptyset$ and $\beta^{*} \in r_{\beta^{*}} \setminus r^{*}$. Then it is easy to see that

(3) $\alpha^{*} \neq \beta^{*}, \alpha^{*}, \beta^{*} \notin A$.

Define $g^{*}: \omega_1 \rightarrow 2$ by

$$g^{*}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \cup \{\alpha^{*}, \beta^{*}\}, \\ 0 & \text{if } \alpha \in \omega_1 \setminus (A \cup \{\alpha^{*}, \beta^{*}\}) \end{cases}$$

Then $g^{*} \in \sigma_{n+2} \setminus \sigma_{n+1}$ and therefore $g^{*} \in G$. Moreover we have

(4) $g^{*} \in V^{f} \cap U_{\beta^{*}}$.

To prove that $g^{*} \in V^{f}$, let $\beta \in r \setminus (A \cup \{\alpha^{*}\})$. Then $\beta \notin r_{\beta^{*}} \setminus r^{*}$. Thus $\beta \neq \beta^{*}$. Hence $g^{*}(\beta) = 0$. Therefore $g \in V^{f}$. To prove that $g^{*} \in U_{\beta^{*}}$, let $\beta \in r_{\beta^{*}} \setminus (A \cup \{\beta^{*}\})$. If $\beta \notin r^{*}$, then $\beta \neq \alpha^{*}$. Thus $g^{*}(\beta) = 0$. If $\beta \notin r^{*}$, then $\beta \in r_{\beta^{*}} \setminus r^{*}$. Therefore $\beta \neq r$. Then $\beta \neq \alpha^{*}$ and so $g(\beta) = 0$. Hence $g \in U_{\beta^{*}}$.

It is known that there exists a non-normal $\sigma$-product such that each finite sub-product is normal (cf. [3]).

**Theorem 2.** If each $X_{\alpha} \in S$ is normal (resp. pseudonormal), then $\sigma_1$ is normal (resp. pseudonormal).

**Proof.** We shall write only the proof of normality because the proof of pseudonormality is quite similar. Let $A$ and $B$ be disjoint closed subsets in $\sigma_1$. Let $x^{*} \in A$. There are a finite set $\{\alpha_{i} | i = 1, 2, ..., m\}$ of $\Omega$ and open sets $U_{\alpha_{i}}$ in $X_{\alpha_{i}}$ such that $x^{*} \in W \equiv \{x \in \sigma_{1}|x_{\alpha_{i}} \in U_{\alpha_{i}} \text{ for } i = 1, 2, ..., m\}$ and $\text{cl}W \cap B = \emptyset$. Since $X_{\alpha_{i}}$ is normal, there is an open set $U'_{\alpha_{i}}$ in $X_{\alpha_{i}}$ such that $x^{*}_{\alpha_{i}} \in U'_{\alpha_{i}}$ and $\text{cl}U'_{\alpha_{i}} \subset U_{\alpha_{i}}$ for
\(i = 1, 2, ..., m\). Put \(W' \equiv \{x \in \sigma | x_{\alpha_i} \in U'_{\alpha_i} \text{ for } i = 1, 2, ..., m\}\). Then \(W'\) is open in \(\sigma_1\), \(x^* \in W'\) and \(cW' \cap B = \emptyset\).

Let \(Y_\alpha = \{x \in \sigma_1 | x_\beta = x^*_\beta \text{ for each } \beta \neq \alpha\}\), i.e., \(Y_\alpha = X_\alpha \times \{z^*_\alpha\}\). Here \(z^*_\alpha = (x^*_\beta)_{\beta \in \Omega \setminus \{\alpha\}}\). Then it is easy to see that

(1) \(Y_\alpha \subset W'\) for each \(\alpha \in \Omega \setminus \{\alpha_i | i = 1, 2, ..., m\}\);

(2) \(Y_\alpha\) is closed in \(\sigma_1\).

Put \(A_i = A \cap Y_\alpha \setminus W'\) and \(B_i = B \cap Y_\alpha \setminus W'\). Then \(A_i\) and \(B_i\) are disjoint closed sets in \(Y_\alpha\) and \(x^* \notin A_i, x^* \notin B_i\). Since \(Y_\alpha\) is normal, there are open sets \(V_i\) and \(V'_i\) in \(X_\alpha\) such that \(V_i \cap V'_i = \emptyset, x^*_\alpha \notin V_i \cup V'_i, A_i \subset V_i \times \{z^*_\alpha\}\) and \(B_i \subset V'_i \times \{z^*_\alpha\}\). Put \(G_i = \{x \in \sigma_1 | x_{\alpha_i} \in V_i \} \setminus \bigcup \{Y_\alpha | j = 1, 2, ..., m; j \neq i\}\) and \(H_i = \{x \in \sigma_1 | x_{\alpha_i} \in V'_i \setminus cU'_{\alpha_i}\} \setminus \bigcup \{Y_\alpha | j = 1, 2, ..., m; j \neq i\}\). Then \(G_i\) and \(H_i\) are open sets in \(\sigma_1\) such that

(3) \(A_i \subset G_i, B_i \subset H_i\);

(4) \(G_i \cap H_i = \emptyset\).

Proof of (3). Let \(x \in A_i \cup B_i\). Then \(x_{\alpha} = x^*_\alpha\) for each \(\alpha \neq \alpha_i\). Since \(x \notin W', x_{\alpha_i} \neq x^*_{\alpha_i}\). Thus \(x \notin Y_\alpha\) if \(j \neq i\). Hence, if \(x \in A_i\), then \(x \in G_i\). Let \(x \in B_i\). Then \(x_{\alpha_i} \in V_i\). If \(x_{\alpha_i} \in cU'_{\alpha_i}\), then \(x_{\alpha_i} \in U_\alpha\). Since \(x_{\alpha_i} = x^*_{\alpha_i} \in U_\alpha\) for each \(j \neq i, x \in W\). Therefore \(x \notin B\). This is a contradiction. Thus \(x_{\alpha_i} \notin cU'_{\alpha_i}\). Hence \(x \in H_i\).

Put \(G = W' \cup \bigcup_{i=1}^m G_i\) and \(H = \bigcup_{i=1}^m H_i\). Then \(G\) and \(H\) are open sets in \(\sigma_1\) such that

(5) \(A \subset G, B \subset H\)

and

(6) \(G \cap H = \emptyset\).

(5) is obvious. (6) follows from (4) and (7) and (8) below.

(7) \(W' \cap H_i = \emptyset\) for each \(i\).

(8) \(i \neq j \Rightarrow G_i \cap H_j = \emptyset\).

(7) is obvious.

Proof of (8). If \(x \in G_i \cap H_j\), then \(x_{\alpha_i} \neq x^*_{\alpha_i}\) and \(x_{\alpha_j} \neq x^*_{\alpha_j}\). Thus \(|Q(x)| \geq 2\), which contradicts \(x \in \sigma_1\).

Example 1. There exists a \(\sigma\)-product such that each finite subproduct is normal and \(\sigma_2\) is not normal.

To prove Example 1, we shall use the following lemma.

Lemma 2 ([2]). Let \(X\) be a space and \(A\) be a closed set of \(X\) which is not a \(G_\delta\)-subset of \(X\). Let \(F\) be Bing’s Example \(G\) or \(H\) constructed by \(P = X \setminus A\). Then \(X \times F\) is not normal.

Definition 2 (Bing’s Example \(G\) ([]). Let \(P\) be an uncountable set and \(Q = \{q | q \subset P\}\). Put \(F = \{f : Q \rightarrow 2\}\). For each \(p \in P\), define \(f_p\) as follows:

\[
\begin{align*}
f_p(q) &= 1 & \text{if } p \in q, \\
&= 0 & \text{if } p \notin q.
\end{align*}
\]

Put \(F_P = \{f_p | p \in P\}\). Define the topology of \(F\) as follows: each \(f_p\) has a neighborhood base in Cartesian product topology and for each \(f \in F \setminus F_P\), \(\{f\}\) is open. For each \(r \in R = Q^{<\omega}\), put \(V(f_p; r) = \{f \in F | f(q) = f_p(q) \text{ for each } q \in r\}\). Then \(\mathcal{V}(f_p) = \{V(f_p; r) | r \in R\}\) is a neighborhood base of \(f_p\).
Proof of Lemma 2. In case $F$ is Bing’s Example H, the proof is in [2]. The proof is quite similar for the case of Bing’s Example G. But, since [2] is not widely known, we shall sketch the proof of the case of Bing’s Example G. Let $C = A \times F$ and $D = \{(p, f_p) | p \in P\}$. Then $C$ and $D$ are disjoint closed subsets in $X \times F$ and are not separated by open sets in $X \times F$. To show this, let $O$ be an arbitrary open set in $X \times F$ such that $D \subset O$. For each $p \in P$, there is a member $V(f_p; r_p)$ of $V(f_p)$ such that $\bigcup_{p \in P} \{(p) \times V(f_p; r_p)\} \subset O$. Let us put $P_i = \{p | p \in P, |r_p| = i\}$ for each $i < \omega$. Since $A$ is not a $G_\delta$-set of $X$, there is an $i$ such that $A \cap \operatorname{cl}(P_i) \neq \emptyset$. Let us fix this $i$. Let $x_0 \in A \cap \operatorname{cl}(P_i)$. Let us put $P_i(r) = \{p \in P_r | r_p \supset r\}$ for each $r \in R$. Then we can prove that there exists an element $r^* \in R$ satisfying the following conditions: (1) $x_0 \in \operatorname{cl}(P_i(r^*))$, (2) $x_0 \notin \operatorname{cl}(P_i(r^* \cup \{q\})$ for each $q \in Q \setminus r^*$. Let us put $R^* = \{s | s \subset r^*\}$. For each $s \in R^*$, we define an element $q_s$ of $Q$ by $q_s = \bigcap\{q \in s | s \in R^*\}$. Then $\{q_s | s \in R^*\}$ is a finite cover of $P$. Therefore, we can choose a member $s_0$ of $R^*$ such that $x_0 \in \operatorname{cl}(P_i(r^*)) \cap q_{s_0}$. For this $s_0$, we choose an element $p^* \in q_{s_0}$. Then $(x_0, f_{p^*}) \in C$. Next we shall prove that $(x_0, f_{p^*}) \in \operatorname{cl}(O)$. Let $U$ be an arbitrary open neighborhood of $x_0$ in $X \times (V(f_{p^*}; r))$ be an arbitrary member of $V(f_{p^*})$. Then we can choose an open neighborhood $U'$ of $x_0$ in $X$ such that $U' \cap (\bigcup \{P_i(r^* \cup \{q\}) | q \in r^* \setminus \{s_0\}\} = \emptyset$. Then $U \cap U' \cap P_i(r^*) \cap q_{s_0} = \emptyset$. Let $p \in U \cap U' \cap P_i(r^*) \cap q_{s_0}$. Then $r_p \cap (r^* \setminus r^*) = \emptyset$. Define $f : Q \to 2$ by

$$f(q) = \begin{cases} 1 & \text{if } p^* \in q \in r \text{ or } p \in q \in r_p, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in V(f_{p^*}; r) \cap V(f_{p^*}; r_p)$. Therefore $(U \times V(f_{p^*}; r)) \cap (\{p\} \times V(f_{p^*}; r_p)) \neq \emptyset$.

Proof of Example 1. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^\ast$. Then

(i) $0^* \neq \sigma(2^{\omega_1})$ is not a $G_\delta$-set in $\sigma_1$.

Proof. Assume that there exist countable open sets $\{W_n | n = 1, 2, \ldots\}$ in $\sigma_1$ such that $\{0^*\} = \bigcap_{n, n \leq \omega} W_n$. Then there are finite sets $a_n \subset \omega_1$ and open sets $U_n, n \in X_\alpha$ for each $\alpha \in a_n$ such that $0^* \in U_n \equiv \{x \in \sigma_1 | x_n = 0 \text{ for each } \alpha \in a_n\} \subset W_n$ for each $n$. Then $\{0^*\} = \bigcap_{n, n \leq \omega} U_n$. Since $\omega_1 \setminus \bigcup_{n, n \leq \omega} U_n \neq \emptyset$, choose an element $\alpha \in \omega_1 \setminus \bigcup_{n, n \leq \omega} U_n$. Let us define $x = (x_\beta)_{\beta < \omega_1}$ by $x_\alpha = 1, x_\beta = 0$ if $\beta \neq \alpha$. Then $x \in \bigcap_{n \leq \omega} U_n \setminus \{0^*\}$, which contradicts $\{0^*\} = \bigcap_{n \leq \omega} U_n$.

(ii) Put $P = \sigma_1 \setminus \{0^*\}$ and let $F$ be Bing’s Example G constructed by $P$. Then $\sigma_1 \times F$ is not normal.

Let $S = \{2_\alpha | \alpha < \omega_1\}$ U \{F\}$ where $2_\alpha = 2$ for each $\alpha$ and let $\sigma' = \sigma(S)$ with the base point $(0^*, f^*), f^* \in F \times F$. Then

(iii) $\sigma'_2$ is not normal.

Since normality is inherited by closed subspaces, (iii) follows from (ii) and (iv) below.

(iv) $\sigma_1 \times F$ is a closed subset of $\sigma'_2$.

Proof. It is obvious that $\sigma_1 \times F \subset \sigma'_2$. Let $y \in \sigma'_2 \setminus \sigma_1 \times F$. Then we can denote $y = (x, f), x \in \sigma, f \in F$. Since $(x, f) \notin \sigma_1 \times F, (x, f) \notin \sigma_1$. Hence $f = f^*$. Since $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open set in $(\sigma_2 \setminus \sigma_1) \times F, (((\sigma_2 \setminus \sigma_1) \times \{f^*\}) \cap \sigma_2$ and $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open neighborhood of $y$ in $\sigma'_2$ such that $((\sigma_2 \setminus \sigma_1) \times \{f^*\}) \cap (\sigma_1 \times F) = \emptyset$. 


3. Starcompactness

It is well known that every non-trivial $\sigma$-product is not countably compact. A space $X$ is called countably compact if every countable open cover of $X$ has a finite subcover, or, which is equivalent, every infinite subset has a limit point. A space $X$ is called strongly starcompact if for every open cover $\mathcal{U}$ there exists a finite set $B$ of $X$ such that $st(B, \mathcal{U}) = X$. Here $st(B, \mathcal{U}) = \bigcup\{U \in \mathcal{U} | U \cap B \neq \emptyset\}$. A space $X$ is called starcompact if for every open cover $\mathcal{U}$ there exists a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{U}', \mathcal{U}) = X$.

It is known that countably compact $\Rightarrow$ strongly star compact $\Rightarrow$ starcompact, and for $T_2$-spaces, the converses hold.

**Theorem 3.** *Every non-trivial $\sigma$-product is not strongly starcompact.*

**Proof.** Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega$. Let $\sigma = \sigma(2^\omega)$ be the $\sigma$-product with the base point $0^*$ and let $\sigma' = \sigma(\mathcal{S})$. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega \subset \Omega$. Define $f : \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega}$, let $f(x) = (y_\alpha)_{\alpha \in \Omega}$ be

$$y_\alpha = \begin{cases} x^*_\alpha & \text{if } \alpha \in \omega \text{ and } x_\alpha = 1, \\ a_\alpha & \text{otherwise.} \end{cases}$$

Then $f$ is a homeomorphism from $\sigma$ onto $f(\sigma)$ such that $f(0^*) = x^*$ and $f(\sigma)$ is a closed subset of $\sigma'$. To prove that $f(\sigma)$ is a closed subset of $\sigma'$, let $y \in \sigma' \setminus f(\sigma)$. Then there exists $\alpha \in \Omega \setminus \omega$ such that $y_\alpha \neq x^*_\alpha$ and put $U = \{z \in \sigma'| z_\alpha \neq x^*_\alpha\}$. Then $U$ is an open neighborhood of $y$ in $\sigma'$ such that $U \cap f(\sigma) = \emptyset$.

**Claim.** $\sigma'$ is not strongly starcompact.

**Proof.** Let $U'_0 = \{x \in \sigma'| x_0 \neq a_0\}$ and let $U'_n = \{x \in \sigma'| x_0 \neq x^*_0, x_1 \neq x^*_1, \ldots, x_{n-1} \neq x^*_{n-1}, x_n \neq a_n\}$ for each $n \geq 1$. Put $\mathcal{U} = \{U_n | n \in \omega\} \cup \{\sigma' \setminus f(\sigma)\}$. Then $\mathcal{U}$ is an open cover of $\sigma'$. (i) $\mathcal{U}$ is an open cover of $\sigma'$; (ii) there is no finite $B \subset \sigma'$ such that $st(B, \mathcal{U}) = \sigma'$.

**Proof of (ii).** Let $B$ be a finite set of $\sigma'$. Then $B \subset \sigma'_n$ for some $n$. Since $U_i \cap \sigma'_n = \emptyset$ for each $i \geq n + 1$, $st(B, \mathcal{U}) \subset \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. However $\bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma)) \neq \sigma'$. To show this, let us define $z = (z_\alpha)_{\alpha \in \Omega}$ as follows:

$$z_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } \alpha \leq n, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $z \in \sigma'$ and $z \notin \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. Therefore $st(B, \mathcal{U}) \neq \sigma'$.

**Theorem 4.** If each $X_\alpha \in \mathcal{S}$ is strongly star compact (resp. star compact), then $\sigma_1$ is strongly star compact (resp. star compact).

**Proofs.** Proofs are easy and so we omit them.

Since for $T_2$-spaces, starcompactness is equivalent to countable compactness, every non-trivial $\sigma$-product of $T_2$-spaces is not starcompact. However, for $T_1$-spaces, non-trivial $\sigma$-product can be starcompact.

We denote $\sigma(X^\omega)$ with the base point $x^*$ by $\sigma(X^\omega; x^*)$.

**Example 2.** There exists a starcompact space $X$ such that $X$ is not a $T_2$-space and not countably compact and (1) $\sigma = \sigma(X^\omega; a^*)$ is starcompact for some $a \in X$. (2) $\sigma' = \sigma(X^\omega; b^*)$ is not starcompact for some $b \in X$. 
Proof. Let $X = \mathbb{R}$ with the topology as follows: let $\mathcal{U}(0) = \{U \mid 0 \in U, |X \setminus U| \leq \omega\}$ be the neighborhoods of 0 in $X$ and for each $x \neq 0, \mathcal{U}(x) = \{U \mid U$ is a neighborhood of $x$ in usual topology of $\mathbb{R}\}$ be the base of $x$ in $X$. Then $X$ is a $T_1$-space and not a $T_2$-space and $X$ is starcompact and not countably compact. By Theorem 5 below, (1) $\sigma = \sigma(X^\omega;0^*)$ is starcompact. By Theorem 6, (2) $\sigma' = \sigma(X^\omega;1^*)$ is not starcompact.

**Theorem 5.** Let $X$ satisfy the condition: “there exists $a \in X$ such that if $U$ and $V$ are open sets in $X$ and $a \in U$, then $U \cap V \neq \emptyset$”. Let $\tau$ be an infinite cardinal number, and $\sigma = \sigma(X^\tau;a^\tau)$. Then (i) $\sigma$ is starcompact, (ii) $\sigma_n$ is starcompact ($\forall n$), (iii) $X^\sigma$ is starcompact. Moreover let $\sigma' = \sigma(X^\omega;b^\omega), b \in X, b \neq a$. Then (iv) $\sigma'_n$ is starcompact ($\forall n$).

**Proof.** Proof of (i). Let $\mathcal{G}$ be an arbitrary open cover of $\sigma$. Let us choose $G_0 \in \mathcal{G}$ such that $a^\sigma \in G_0$. There are a finite set $\{a_i \mid i = 1, 2, ..., m\} \subset \tau$ and open sets $U_{a_i}$ in $X_{a_i}$ such that $a^\sigma \in W_0 \equiv \{x \in a^\sigma | x_{a_i} \in U_{a_i} \mid i = 1, 2, ..., m\} \subset G_0$. For each $x \in a \setminus W_0$, let us choose $G_x \in \mathcal{G}$ such that $x \in G_x$. Then there are a finite set $\{\beta_j \mid j = 1, 2, ..., k\}$ and open sets $V_{\beta_j}$ in $X_{\beta_j}$ such that $x \in W_x \equiv \{y \in \sigma | y_{a_i} \in V_{\beta_j} \mid i = 1, 2, ..., m\} \subset G_x$. Since $a \in U_{a_i}, U_a \cap V_{\beta_j} \neq \emptyset$ for each $i = 1, 2, ..., m$. Thus $W_0 \cap W_x \neq \emptyset$ and so $G_0 \cap G_x \neq \emptyset$. Therefore $st(G_0, \mathcal{G}) = \sigma$.

Proofs of (ii) and (iii) are similar.

**Proof of (iv).** First we define $\mathcal{B}, y^\sigma$ as follows: put $\mathcal{B} = \{W | W$ is a basic open set in $\sigma'\}$. Here $W \subset \sigma'$ is called a basic open set in $\sigma'$ if $W = \{x \in \sigma' \mid x_{i} \in U_i \mid i \leq n\}, n \in \omega, U_i$ is an open set in $X_{a_i}$ for each $i \leq n$. Define $l(W) = n$. For each $s \in [\omega]^\omega$, define $y^s = (y^s_i)_{i \in \omega}$ as follows:

$$y^s_i = \begin{cases} a & \text{if } i \in s, \\ b & \text{if } i \notin s. \end{cases}$$

To prove (iv), let $\mathcal{G}$ be an arbitrary open cover of $\sigma'$. Let us prove that there exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{G}$ such that $st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$, for each $n \in \omega$.

(I) Let us choose an element $G_0 \in \mathcal{G}$ such that $b^\sigma \in G_0$. Then there is a set $W_0 \in \mathcal{B}$ such that

(0-1) $b^\sigma \in W_0 \subset G_0$.

Put $l(W_0) = k_0$. Then

(0-2) For $x \in \sigma'$, if $l > k_0$ for each $l \in Q(x)$, then $x \in W_0$.

(II) For each $n = 1, 2, ..., $ inductively we can choose $k_n, S_n$ and $W_n$ satisfying the conditions:

(1) $k_n \in \omega, k_n < k_{n+1} (\forall n \geq 1), k_0 = k_1$.

(2) $S_n \subset S_{n+1} (\forall n \geq 1)$.

(3) $(n-1) S_n = \{s \mid s \subset \omega, 1 \leq |s| \leq n, l \leq k_n (\forall l \in s)\}$.

(4) $(n-2) W_n = \{W_s | s \in S_n\} \cup \{W_0\} \subset \mathcal{B}$, $W_n$ is a partial refinement of $\mathcal{G}$.

(5) $(n-3) y^s \in W_s (\forall s \in S_n), k_n < l(W_s) \leq k_{n+1} (\forall s \in S_n)$.

(6) $(n-4) st(\bigcup W_n, \mathcal{G}) \supset \sigma'_n$.

Assume that $k_n, S_n$ and $W_n$ have been chosen for each $n \leq m$. Define $k_{m+1} = \max\{|l(W_s)| s \in S_m\}$ and $S_{m+1} = \{s \mid s \subset \omega, 1 \leq |s| \leq m + 1, l \leq k_{m+1} (\forall l \in s)\}$.

For each $s \in S_{m+1} \setminus S_m$, choose $G_s \in \mathcal{G}$ and $W_s \in \mathcal{B}$ such that $y^s \in W_s \subset G_s$.

Put $W_{m+1} = \{W_s | s \in S_{m+1}\} \cup \{W_0\}$. Then $k_{m+1}, S_{m+1}$ and $W_{m+1}$ satisfy the conditions. We only prove $(m + 1 - 4)$ because others are obvious.
Proof of \((m + 1 - 4)\). Let \(x \in \sigma'_{m + 1} \setminus \sigma'_m\). Put \(s = Q(x)\). Then \(|s| = m + 1\). Choose \(G_x \in \mathcal{G}\) and \(W_x \in \mathcal{B}\) such that \(x \in W_x \subset G_x\). Put \(s = \{l_i | i = 1, 2, \ldots, m + 1\}\) such that \(l_i < l_{i+1}(\forall i)\).

(i) If \(l_{m+1} \leq k_{m+1}\), then \(s \in S_{m+1}\) and \(W_x \cap W_s \neq \emptyset\). To show this, let \(W_s = \{y \in \sigma'| y_i \in U^s_i \text{ for } \forall i \leq l(W_x)\}\) and \(W_x = \{y \in \sigma'| y_i \in U^x_i \text{ for } \forall i \leq l(W_x)\}\). Here we may assume that \(l(W_s), l(W_x) \geq k_{m+1}\). Since \(a \in U^x_i\) for each \(i \in s\), \(U^s_i \cap U^s_i \neq \emptyset\) for each \(i \in s\). For each \(i \not\in s\), \(i \leq \min\{l(W_s), l(W_x)\}, b \in U^s_i \cap U^x_i\). Hence \(U^s_i \cap U^x_i \neq \emptyset\). Therefore \(W_x \cap W_s \neq \emptyset\). Thus \(x \in st(\bigcup \mathcal{W}_{m+1}, \mathcal{G})\).

(ii) Case \(l_{m+1} > k_{m+1}\). If \(l_1 > k_0\), then \(x \in W_0\) by (0.2).

Suppose \(l_1 \leq k_0\). Then there is a \(j\) such that \(1 \leq j \leq m + 1, Q(x) \cap \{l|k_j < l \leq k_{j+1}\} \neq \emptyset\). Let \(j\) be the greatest such number. Then there exists \(t \in \{i | i = 1, 2, \ldots, m + 1\}\) such that \(l_t \leq k_j\) and \(l_{t+1} > k_{j+1}\). Then \(t \leq j\). Therefore \(s = \{l_i | i \leq t\} \in S_j\). Since \(l(W_s) \leq k_{j+1}\), it is easy to see that \(W_x \cap W_s \neq \emptyset\).

Since \(W_n\) is a partial refinement of \(\mathcal{G}\), there exists \(\mathcal{G}_n\) such that \(st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n\).

**Theorem 6.** There are a countable closed subset \(A\) of \(X\) and a pairwise disjoint open family \(\mathcal{U} = \{U(a) | a \in A\}\) such that \(a \ni U(a)\) for each \(a \in A\) and \(X \setminus A \neq \emptyset\). Let \(\sigma = (X^w; a^*)\), \(a \in A\). Then \(\sigma\) is not starcompact.

**Proof.** Let \(A = \{a_n | n = 1, 2, \ldots\}\), \(U(a_n) = U_n\) for each \(n\) and put \(U_0 = X \setminus A\). Then \(\bigcup_{n \in \omega} U_n = X\). Without loss of generality we may assume that \(a = a_1\). For each \(k = 1, 2, \ldots\), let \(\Lambda_k = \{(l_0, l_1, \ldots, l_{k-1}, 1) \in [\omega]^{k+1} | l_0 \neq 1, l_{k-1} \neq 1\}\) and put \(\Lambda = \bigcup_{1 \leq k} \Lambda_k\).

Define \(G_1 = \{x \in \sigma| x_0 \in U_1\}\) and \(G_\lambda = \{x \in \sigma| x_i \in U_i\text{ for } i = 0, 1, \ldots, k-1; x_k \in U_1\}\) for each \(\lambda = (l_0, l_1, \ldots, l_{k-1}, 1) \in \Lambda\) and put \(\mathcal{G} = \{G_\lambda | \lambda \in \Lambda\} \cup \{G_1\}\). Then

1. \(\mathcal{G}\) is an open cover of \(\sigma\).
2. For any finite subfamily \(\mathcal{G}'\) of \(\mathcal{G}\), \(st(\bigcup \mathcal{G}'\), \(\mathcal{G}) \neq \sigma\).

**Proof of (1).** Let \(x \in \sigma\). If \(x_0 \not\in U_1\), then \(x_0 \in U_i\) for some \(i \neq 1\). Since \(|Q(x) = \{i | x_i \neq a_1\}| < \omega\), there is a \(k\) such that \(x_i = a_1\) for each \(i \geq k\) and \(x_{k-1} \neq a_1\). Then \(x \in G_\lambda\) for some \(\lambda \in \Lambda\).

**Proof of (2).** Let \(\mathcal{G}'\) be an arbitrary finite subfamily of \(\mathcal{G}\). Then there exists \(k > 1\) such that

1. \(G_\lambda \notin \mathcal{G}'\) for each \(\lambda \in \bigcup_{m > k} \Lambda_m\).

Define \(x = (x_i)_{i \in \omega}\) as follows:

\[
x_i = \begin{cases} a_k & \text{if } i \leq k, \\ a_1 & \text{if } i > k.
\end{cases}
\]

Then

(2-2) \(x \notin st(\bigcup \mathcal{G}'\), \(\mathcal{G})\).

**Proof of (2-2).** Let \(x \in G \in \mathcal{G}\). Then \(G \neq G_1\). Therefore \(G = G_\lambda\) with \(\lambda = (l_0, l_1, \ldots, l_{m-1}, 1) \in \Lambda_m\). Then \(m > k\). To show this, assume that \(m \leq k\). Then \(x_m = a_k\) by the definition of \(x\). Since \(a_k \notin U_1, x \notin G_\lambda\), which is a contradiction.

Therefore \(m > k\). It is easy to see that \(l_i = k\) for each \(i \leq k\) and \(l_i = 1\) for each \(i \geq k + 1\). Thus \(l_0 \geq 2\) and so \(U_{l_0} \cap U_1 = \emptyset\). Hence \(G_\lambda \cap G_1 = \emptyset\). If \(m \geq k + 2\), then \(l_{m-1} = 1\). This contradicts the definition of \(\lambda\). Thus \(m = k + 1\).

Let \(G_n \in \mathcal{G}'\) with \(\mu = (s_0, s_1, \ldots, s_{k-1}, 1) \in \Lambda_k\). Then \(\mu \leq k\). Therefore \(l_\mu = k\). Since \(U_k \cap U_{l_\mu} = \emptyset, G_\lambda \cap G_n = \emptyset\).

**Remark.** For \(\sigma\) and \(\sigma'\) in Example 2, \(\sigma_n\) and \(\sigma'_n\) are starcompact for each \(n\) by Theorems 5 and 6.
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