Abstract. We prove a result on the transfer of essentiality of extensions of modules over subnormalizing extensions of rings, then apply it to look at the semiprimeness of Hopf-Galois extensions, in particular that of crossed products.

The following is an important open question in the theory of Hopf algebra actions (see [12, Question 7.4.9, p.121]):

If $H$ is a finite-dimensional and semisimple Hopf algebra over the field $k$, and $A$ is semiprime, is any crossed product $A \#_\sigma H$ semiprime?

This possible extension of the Maschke-type theorem for crossed products [2], was inspired by the fact that it holds in the following cases:

1. when $H = kG$ ($G$ is a finite group, and $|G|^{-1} \in k$) (this is the Fisher-Montgomery theorem, [9]), or

2. when $H = kG^*$ [6].

The first case was extended in [3] to the case where $H$ is semisimple, pointed and cocommutative. The aim of the present note is to try to extend the second case, exploiting the fact that the dual Hopf algebra $H^*$ is pointed in this case. The result of this effort is Corollary 5. The technique that we use involves the so-called essential form of Maschke’s theorem. This approach was first used to provide another proof for the Fisher-Montgomery theorem in [14]. Various partial answers to the above question were also given in [1], [17], [13].

Throughout, $H$ will denote a finite-dimensional Hopf algebra over the field $k$. For all unexplained notation or definitions, the reader is referred to [12].

Recall that if $R$ is a subring of $S$, the extension $R \subset S$ is called a subnormalizing (or triangular) extension if there exist elements $x_1, x_2, x_3, \ldots \in S$ such that $S = \sum_i \sum_j Rx_i$ and for any $j$ we have $\sum_i Rx_i = \sum_i x_i R$. A subnormalizing extension is called finite if the sequence $x_1, x_2, x_3, \ldots \in S$ is finite (see [18] or [11]). Our first result is

**Theorem 1.** Let $R \subset S$ be a subnormalizing extension of rings, such that the elements $x_1, x_2, x_3, \ldots \in S$ from the definition form a basis for $S$ as a left and a right $R$-module. If $M \subset N$ is an essential extension of left $R$-modules, and $N$ is nonsingular, then $S \otimes R M \subset S \otimes R N$ is an essential extension of $R$-modules.

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Proof. Suppose the extension $S \otimes_R M \subset S \otimes_R N$ is not essential. We say $w \in S \otimes_R N$, $w \neq 0$, is a \textit{nasty} element if any $R$-multiple of it is either 0 or not in $S \otimes_R M$. It is clear that a nonzero $R$-multiple of a nasty element is again nasty. Among all nasty elements choose $x_1 \otimes n_1 + \ldots + x_t \otimes n_t \in S \otimes_R N$ with $j$ minimal subject to $n_j \notin M, n_{j+1} \in \mathbb{M}, \ldots, n_t \in M$.

For any $i$ there exists an automorphism $\sigma_i : R \to R$ such that $ax_i = x_i \sigma_i(a) + \text{lower terms}$ for all $a \in R$. If $a \in R$, then

$$aw = \ldots + x_j \otimes (\sigma_j(a)n_j + n') + x_{j+1} \otimes n_{j+1}' + \ldots + x_t \otimes n_t'$$

where $n', n_{j+1}', \ldots n_t' \in M$, and by the minimality of $j$ we have that

$$\text{l.ann}_R(w) = \sigma_j^{-1}(\text{l.ann}_R(\widehat{n}_j)),$$

where $\widehat{n}_j = n_j + M \in M/N$. But $\text{l.ann}_R(\widehat{n}_j)$ is an essential ideal of $R$, because the extension $M \subset N$ is essential. Thus $\text{l.ann}_R(w)$ is also an essential ideal.

Now, for $a \in \text{l.ann}_R(w)$ we have

$$0 = aw = x_1 \otimes n_1' + \ldots + x_{t-1} \otimes n_{t-1}' + x_t \otimes \sigma_t(a)n_t,$$

so $\text{l.ann}_R(w) \subseteq \sigma_t^{-1}(\text{l.ann}_R(n_t))$, and hence $\text{l.ann}_R(n_t)$ is also an essential ideal, contradicting the fact that $N$ is nonsingular.

This implies that $S \otimes_R M$ has no nasty elements and the result follows. \hfill \Box

The next lemma is an application of the Taft-Wilson theorem, and was proved in [7] (see also [8, Exercise 7.7.9, p.338]).

**Lemma 2.** Let $H$ be a finite-dimensional pointed Hopf algebra acting on the algebra $A$. Then $A \# H$ is a finite subnormalizing extension of $A$, and the elements $x_1, \ldots, x_m \in A \# H$ may be chosen to form a basis of $H \subset A \# H$.

We are now in a position to state and prove our main result.

**Theorem 3.** Let $H$ be a finite-dimensional pointed Hopf algebra, and $A$ a left $H$-module algebra with subalgebra of invariants $A^H$, such that $A/A^H$ is a faithfully flat right $H^*$-Galois extension. Then the following hold:

a) If $M \subset A N$ is an essential extension of left $A$-modules, and $N$ is left nonsingular, then $N$ is an essential extension of $M$ as left $A^H$-modules.

b) If $A$ is $H$-semiprime and $A$ is a left nonsingular ring, then $A$ is semiprime.

c) If $A^H$ is semiprime and $A$ is a left nonsingular ring, then $A$ is semiprime.

**Proof.** a) The proof uses the duality approach of [6] as in [16] and [4], in the form for Hopf algebra actions which was also used in [7]: since the induced functor $A \# H \otimes_A - : A - \mathcal{M} \to (A \# H)^* \# H^* - \mathcal{M}$ is an equivalence, it follows that

$$A \# H \otimes_A M \subset A \# H \otimes_A N$$

is an essential extension of left $(A \# H)^* \# H^*$-modules.

By Lemma 2 we can apply Theorem 1 for the subnormalizing extension $A \subset A \# H$ to obtain that (11) is an essential extension of left $A$-modules, and so it is also an essential extension of left $A \# H$-modules.

Now we have the functorial isomorphisms of left $A \# H$-modules $A \# H \otimes_A M \simeq A \otimes_{A^H} M$, and $A \# H \otimes_A N \simeq A \otimes_{A^H} N$, so $A \otimes_{A^H} M \subset A \otimes_{A^H} N$ is an essential extension of left $A \# H$-modules. But the induced functor $A \otimes_{A^H} - : A^H - \mathcal{M} \to A \# H - \mathcal{M}$ is also an equivalence, since $A/A^H$ is a faithfully flat Galois extension, and therefore we have that $M \subset A^H N$ is an essential extension.
b) The proof is similar to the proof of the Fisher-Montgomery theorem, as given in [13], but we sketch it for the convenience of the reader. Let \( 0 \neq N \) be an ideal of \( A \) such that \( N^2 = 0 \). Then \( I = \text{r.ann}_A(N) \) is an essential left ideal of \( A \). By a), we get that \( I \) is an essential \( A^H \)-submodule of \( A \), so \( I^H = I \cap A^H \) is an essential left ideal of \( A^H \). Let

\[
J = (I : H) = \{a \in I \mid h \cdot a \in I \forall h \in H\}.
\]

Since \( I^H \subseteq J \subseteq I \), it follows that \( J^H = I^H \), and since the extension is Galois, we get that \( J = I^H A \), by [12, Corollary 8.3.10, p.138]. But now

\[
\text{l.ann}_A(J) = \text{l.ann}_A(I^H) \supseteq \text{l.ann}_A(I) \supseteq N \neq 0.
\]

Thus \( L = \text{l.ann}_A(J) \) is a nonzero \( H \)-stable ideal of \( A \), which is again generated (as a left ideal) by its intersection with \( A^H \). Hence we have that \( L^H \cap I^H \neq 0 \). Finally, \( J \cap L \) is a nonzero \( H \)-stable ideal of \( A \) with \( (J \cap L)^2 = 0 \), a contradiction to the fact that \( A \) is \( H \)-semiprime.

c) follows from b), because \( H \)-stable ideals of \( A \) are generated by their intersection with \( A^H \).

The following is a corollary of the proof of Theorem 3 and is presumably known.

**Corollary 4.** If \( G \) is a finite group acting as automorphisms on the \( k \)-algebra \( R \), \( R^G \) denotes the subalgebra of invariants, and \( R/R^G \) is a Galois extension having an element of trace 1, then the following hold:

a) If \( M \subset R \) is an essential extension of left \( R \)-modules, then \( M \subset N \) is also an essential extension as left \( R^G \)-modules.

b) If \( R^G \) is semiprime, then \( R \) is semiprime.

**Proof.** As in the proof of part a) of Theorem 3 we want to prove that

\[
R \ast kG \otimes_R M \subset R \ast kG \otimes_R N
\]

is an essential extension of \( R \ast kG \)-modules. But since this is an essential extension of left \( (R \ast kG) \# kG^G \)-modules, i.e. a gr-essential extension of graded \( R \ast kG \)-modules, the claim follows easily from [14, 1.2.8, p. 9]. The rest of the proof is the same.

A direct consequence of Theorem 3 is the following.

**Corollary 5.** Let \( H \) be a finite-dimensional Hopf algebra such that \( H^* \) is pointed, and \( A \#_{\sigma} H \) a crossed product with invertible cocycle \( \sigma \). If \( A \) is \( H \)-semiprime and \( A \#_{\sigma} H \) is a left nonsingular ring, then \( A \#_{\sigma} H \) is also semiprime.

**Proof.** Denote the left weak action of \( H \) on \( A \) by \( h \cdot a \) for \( a \in A \) and \( h \in H \).

We remark first that if \( A \) is \( H \)-semiprime, then \( A \#_{\sigma} H \) is \( H^* \)-semiprime. Indeed, if \( N \) is a nonzero \( H^* \)-stable ideal of \( A \#_{\sigma} H \) with \( N^2 = 0 \), then \( N \cap A \) generates \( N \) as a left ideal (again by [12, Corollary 8.3.10, p.138]), so \( N \cap A \neq 0 \), and clearly \( (N \cap A)^2 = 0 \). Moreover, \( N \cap A \) is an \( H \)-stable ideal of \( A \), since for \( n \in N \cap A \) and \( h \in H \), we have by [12, 7.2.3 and 7.2.7] that

\[
h \cdot n = \sum (1 \# h_1)(n \# 1)(\sigma^{-1}(S(h_3), h_4) \# S(h_2)).
\]

This provides the desired contradiction.

Now apply part b) of Theorem 3 to finish the proof.
Remark 6. The relationship between the above Corollary and the question mentioned in the beginning of this note is as follows: the condition “A#H is nonsingular” is needed as a replacement for the condition “H is semisimple” (the latter would imply that H = kG for some finite group G, and this would bring us back to the graded case). This can be seen by taking A to be trivial. Then H nonsingular is equivalent to H semisimple, or H semiprime, because H is Frobenius (see [10, 13.2, p.362]).

A simple example where A#H is nonsingular (in fact, even simple) without H being semisimple is as follows: let \( F_2 \) be the field with two elements, \( E = F_2(\langle X \rangle) \) the field of rational fractions in the indeterminate \( X \), and let \( k \) be the subfield \( F_2(\langle X^2 \rangle) \). Let \( \delta = \frac{d}{dx} \in Der(E) \), and denote by \( L \) the 1-dimensional Lie algebra generated by \( \delta \). Consider the action of \( L \) on \( E \) as derivations. Then the invariants of this action are just \( k \) itself, and the Hopf algebra is \( H = U(L) \cong H^* \). The extension is Galois. This follows immediately from a result in [5] which may also be found in [12, 8.3.7, p. 137], or it can be seen directly after a short computation. Now, since the extension is Galois, the smash product \( E#H \) is isomorphic to \( \text{End}_k(E) \), which is a simple ring.

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