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A NORM ON THE HOLOMORPHIC BESOV SPACE

BJARTE BÖE

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Abstract. We obtain a description of the holomorphic Besov space that is valid for the indices $1 \leq p, q < \infty$, $0 < s < 1$. Applications to inner-outer factorisation, and to inner functions in particular, are provided.

1. Introduction

The holomorphic Besov space AB^s_{pq} consists of the holomorphic functions in the unit disk for which the norm

$$\|f\|_{B^s_{pq}} = \left(\int_0^1 \left(\int_T |f'(re^{i\theta})|^p d\theta\right)^{\frac{q}{p}} (1-r)^{(1-s)q-1} dr\right)^{\frac{1}{q}} + \|f\|_{H^p} \tag{1}$$

is finite. Here T denotes the unit circle and H^p is the Hardy space of the disk. The boundary values of functions in AB^s_{pq} belong to the familiar Besov space B^s_{pq}, for which a number of equivalent norms are available (see e.g. [14, V. 5]). However, it is desirable to have a description of AB^s_{pq} that only depends on the modulus of the function. This is provided in [7], but for a limited range of indices, specifically for $1 < p < \infty,\ 1 \leq q < \infty,\ \frac{1}{p} < s < 1$ and for $2 \leq p < \infty,\ 0 < q < \infty,\ 0 < s < \frac{1}{2}$. The main aim of this paper is to extend this description to $0 < s < 1,\ 1 \leq p, q < \infty$, thus answering a question in [7]. This result is contained in the following statement.

Theorem 1.1. *For $0 < s < 1,\ 1 \leq p, q < \infty$ we have $f \in AB^s_{pq}$ if and only if $|f| \in B^s_{pq}$ and*

$$I(f) = \int_0^1 \left(\int_T \left(\int_T |f| d\mu_z - |f(z)|\right)^p d\theta\right)^{\frac{q}{p}} (1-r)^{-sq-1} dr < \infty.$$

The two norms are comparable in the sense that $\|f\|_{B^s_{pq}} \sim \||f|\|_{B^s_{pq}} + I(f)^{\frac{1}{q}}$.

Here $z = re^{i\theta}$ and

$$d\mu_z(t) = \frac{1-|z|^2}{|e^{it}-z|^2} \frac{dt}{2\pi},$$

so that μ is the harmonic measure at the point z. The usefulness of this norm will be demonstrated by a number of examples.

Section 2 contains the proof of Theorem 1.1. In section 3 we consider applications to the inner-outer factorisation of functions in the holomorphic Besov space, and obtain, in particular, that under the restrictions in Theorem 1.1, any function

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in AB^s_{pq} may be written as a quotient of bounded functions in the same space. This was previously obtained in [10] for the case $p = q = 2, s = \frac{1}{2}$, in [2] for $p = q = 2, s < \frac{1}{2}$ and in [7] for $p \geq 2, s < \frac{1}{2}$.

In section 4, we point out some conditions for an inner function to be in AB^s_{pq}, along with the corresponding norm estimates.

2. Proof of Theorem 1.1

It will be useful for us to norm the (non-holomorphic) Besov space B^s_{pq} using oscillation. We formulate this in the following lemma:

Lemma 2.1. *When $1 \leq t \leq p$ and $t < \frac{1}{s}$, an equivalent norm in B^s_{pq} is provided by*

$$(\int_0^1 (\int_T [\int_T |f - \int_T f d\mu_{re^{i\theta}}|^t d\mu_{re^{i\theta}}]^{\frac{p}{t}} d\theta)^{\frac{q}{p}} (1-r)^{-sq-1} dr)^{\frac{1}{q}} + \|f\|_p.$$

For a proof of this, see [7].

Also, since AB^s_{pq} is contained in H^p, every function f in AB^s_{pq} admits the inner-outer factorisation $f = O_\phi \cdot I$. Here I is an inner function (i.e., $I \in H^\infty$ and $|I| = 1$ almost everywhere on T), and

$$O_\phi(z) = \exp(\int_T \frac{e^{it} + z}{e^{it} - z} \log \phi(t) \frac{dt}{2\pi})$$

is the outer function with modulus $|f| = \phi$ on the boundary. Curiously enough, we will deduce properties of this factorisation from Theorem 1.1, but we will also need it in the proof.

Proof of Theorem 1.1. We observe first that

$$\int |f - f(z)| d\mu_z \geq \int (|f| - |f(z)|) d\mu_z = \int |f| d\mu_z - |f(z)|.$$

Applying Lemma 2.1 with $t = 1$ and since clearly $\||f|\|_{B^s_{pq}} \leq C \cdot \|f\|_{B^s_{pq}}$, we get

$$\||f|\|_{B^s_{pq}} + I(f)^{\frac{1}{q}} \leq C \cdot \|f\|_{B^s_{pq}}.$$

We now prove the converse inequality. Recall that $\phi = |f|$. For convenience we put $u = \log \phi$ and

$$u(z) = \int u d\mu_z, \; \phi(z) = \int \phi d\mu_z.$$

We write $f = O_\phi \cdot I$, and consider first the outer factor. Differentiating, we get

$$|O'_\phi(z)| = 2 \cdot |\int \frac{e^{it} u(t)}{(e^{it} - z)^2} \frac{dt}{2\pi}| \cdot \exp(u(z)).$$

Since $\int_T \frac{e^{it}}{(e^{it}-z)^2} dt = 0$, we can estimate the first factor in terms of oscillation:

$$|O'_\phi(z)| \leq \frac{2}{1 - |z|} \int |u - u(z)| d\mu_z \cdot \exp(u(z)).$$

From the mean-value theorem it is clear that when $u(t) \geq u(z)$ we have

$$(u(t) - u(z)) \cdot \exp(u(z)) \leq \exp(u(t)) - \exp(u(z)).$$

Combining this with the identity

$$\int_{\{t:u(t)\geq u(z)\}}(u(t)-u(z))d\mu_z=\frac{1}{2}\int|u-u(z)|d\mu_z$$

we get

$$\begin{aligned}|O'_\phi(z)|&\leq\frac{4}{1-|z|}\int|\exp(u(t))-\exp(u(z))|d\mu_z\\&\leq\frac{4}{1-|z|}\cdot(\int|\phi-\phi(z)|d\mu_z+\phi(z)-|O_\phi(z)|).\end{aligned}$$

To estimate the inner factor we use the Schwarz-Pick theorem:

$$|I'(z)|(1-|z|)\leq 2(1-|I(z)|)$$

and get in total

$$\begin{aligned}|f'(z)|(1-|z|)&\leq|O'_\phi(z)|(1-|z|)+|O_\phi(z)|\cdot|I'(z)|(1-|z|)\\&\leq 4\cdot(\int|\phi-\phi(z)|d\mu_z+\phi(z)-|f(z)|).\end{aligned}$$

Now it is clear from (1) and Lemma 2.1 with $t=1$ that

$$\|f\|_{B^s_{pq}}\leq C\cdot(\||f|\|_{B^s_{pq}}+I(f)^{\frac{1}{q}}).$$

We comment on possible extensions of this proof. First, it is clear that our proof also works for the holomorphic Lipschitz space since norms corresponding to (1) and Lemma 2.1 are available in this situation. This will give us a fairly simple proof of Theorem 2 in [6]. In fact, Theorem 1.1 here can be considered a Besov space version of the mentioned theorem (see also [9] for a different simple proof). Second, a statement corresponding to the case $q=\infty$ can also be obtained; see [7, 5.1] for the necessary assertions.

3. Inner-outer factorisation

Throughout this section we let the indices p,q,s be restricted by the conditions in Theorem 1.1. To start with, we point out that our current theorem gives very simple proofs of some essentially known properties concerning multiplication by inner functions. First, we have the so-called F-property (see e.g. [11]).

Corollary 3.1. *Assume that $f=O_\phi\cdot I$ is in AB^s_{pq} and J is an inner function dividing I. Then f/J is also in AB^s_{pq} and in addition we have $\|f/J\|_{B^s_{pq}}\leq C\cdot\|f\|_{B^s_{pq}}$*

Proof. This follows from Theorem 1.1 since

$$\int|f|d\mu_z-|f(z)|\geq\int|f|d\mu_z-|(f/J)(z)|.$$

Second, we can determine when $f\cdot I$ belongs to AB^s_{pq}. In this respect, the idea is that the size of f must be sufficiently small close to the spectrum of the inner function. The condition is given below, and should be compared to a similar result in [4, 3.3].

Corollary 3.2. *Assume that $f \in AB^s_{pq}$ and I is an inner function. Then $f \cdot I \in AB^s_{pq}$ if and only if*

$$\int_0^1 (\int_T (|f(re^{i\theta})|(1-|I(re^{i\theta})|))^p d\theta)^{\frac{q}{p}} (1-r)^{-sq-1} dr < \infty.$$

Proof. This follows from Theorem 1.1 and the identity

$$\int |f \cdot I| d\mu_z - |f \cdot I(z)| = \int |f| d\mu_z - |f(z)| + |f(z)|(1-|I(z)|).$$

We are also interested in estimating the effect of truncating the modulus. This is done easily using Theorem 1.1, since it expresses the norm in terms of absolute values. Actually, this application serves as the motivation for obtaining the description in Theorem 1.1.

Theorem 3.3. *Given $f = O_\phi \cdot I \in AB^s_{pq}$, the functions $O_{\min(\phi,1)} \cdot I$ and $O_{\max(\phi,1)}$ are also in AB^s_{pq}. In addition, we have the norm estimates $\|O_{\min(\phi,1)} \cdot I\|_{B^s_{pq}} \leq C \cdot \|f\|_{B^s_{pq}}$ and $\|O_{\max(\phi,1)}\|_{B^s_{pq}} \leq C \cdot \|f\|_{B^s_{pq}}$.*

Proof. Again, we use Theorem 1.1 and the identity

$$\int |O_\phi \cdot I| d\mu_z - |O_\phi \cdot I(z)|$$
$$= \int |O_\phi| d\mu_z - |O_\phi(z)| + |O_\phi(z)|(1-|I(z)|).$$

That the first term decreases when truncated, that is to say,

$$\int |O_\phi| d\mu_z - |O_\phi(z)| \geq \int |O_{\min(\phi,1)}| d\mu_z - |O_{\min(\phi,1)}(z)|,$$

may be seen when we integrate

$$\phi - \min(\phi,1) = \exp(\log\phi - \log\ \min(\phi,1)) - 1$$
$$\geq \exp(\int \log\ \min(\phi,1)\ d\mu_z) \cdot (\exp(\log\phi - \log\ \min(\phi,1)) - 1)$$

w.r.t. $d\mu_z$ and apply Jensen's inequality on the right-hand side. It is, furthermore, clear that the second term decreases when O_ϕ is replaced by $O_{\min(\phi,1)}$, so we may conclude $\|O_{\min(\phi,1)} \cdot I\|_{B^s_{pq}} \leq C \cdot \|O_\phi \cdot I\|_{B^s_{pq}}$.

In a similar way, the inequality

$$\int |O_\phi| d\mu_z - |O_\phi(z)| \geq \int |O_{\max(\phi,1)}| d\mu_z - |O_{\max(\phi,1)}(z)|$$

follows from

$$\max(\phi,1) - \phi = 1 - \exp(\log\phi - \log\ \max(\phi,1))$$
$$\leq \exp(\int \log\ \max(\phi,1) d\mu_z) \cdot (1 - \exp(\log\phi - \log\ \max(\phi,1)))$$

and integration w.r.t. $d\mu_z$ followed by Jensen's inequality. This proves the statement for $O_{\max(\phi,1)}$.

The functions in AB^s_{pq} are not necessarily bounded when $s \leq \frac{1}{p}$, but using our truncation result, we obtain that a weaker property holds:

Theorem 3.4. *Any function in AB^s_{pq} is a quotient of two bounded functions in AB^s_{pq}; in addition, the denominator may be taken to be an outer function.*

Proof. We put $f = O_\phi \cdot I$. Since $O_\phi = O_{\min(\phi,1)} \cdot O_{\max(\phi,1)}$, we may write

$$f = \frac{O_{\min(\phi,1)} \cdot I}{O_{\max(\phi,1)^{-1}}}.$$

We only need to check that $\frac{1}{O_{\max(\phi,1)}} \in AB^s_{pq}$, and this follows from $|(\frac{1}{O_{\max(\phi,1)}})'| \leq |O'_{\max(\phi,1)}|$, Theorem 3.3 and (1).

Remark. Some of the results in this section can be proved slightly more easily in the case $p \geq 2, s < \frac{1}{2}$; see [7].

To finish this section, it seems natural to point out that a somewhat different discussion of inner-outer factorisation in connection with Besov spaces is contained in the articles [12], [13].

4. Inner functions in AB^s_{pq}

In the present section we consider inner functions in AB^s_{pq}. Our results will be based on the combined use of Theorem 1.1 and Lemma 2.1 and the proofs are all simple. By B^s_{pq}-norm we will from now on only mean semi-norm, i.e., we ignore the L^p-term, but we keep the notation.

It is easy to verify the elementary inequality

$$\|f\|^r_{B^{s/r}_{rp,rq}} \leq C \cdot \|f\|^{r-1}_\infty \cdot \|f\|_{B^s_{pq}}, r \geq 1. \tag{2}$$

As it turns out, this norm inequality can be reversed for inner functions. This will be useful in that we can choose p freely when calculating the norm of inner functions.

Theorem 4.1. *For $0 < s < 1,\ p, q \geq 1,\ r \geq 1$ and I an inner function, we have $\|I\|_{B^s_{pq}} \leq C \cdot \|I\|^r_{B^{s/r}_{rp,rq}}$.*

Proof. Since

$$1 - |I(z)| \leq 1 - |I(z)|^2 = \int_T |I - I(z)|^2 d\mu_z,$$

we get $\|I\|_{B^s_{pq}} \leq C \cdot \|I\|^2_{B^{s/2}_{2p,2q}}$, where we have used Theorem 1.1 and Lemma 2.1 with $t = 2$. For general r we choose k such that $r \leq 2^k$. By repeated application of the case $r = 2$ we have

$$\|I\|_{B^s_{pq}} \leq C \cdot \|I\|^{2^k}_{B^{s/2^k}_{2^k p, 2^k q}} \leq C \cdot \|I\|^r_{B^{s/r}_{rp,rq}}$$

where the last inequality follows from (2).

Using Theorem 4.1 we can reduce the description of inner functions in AB^s_{pp} to the L^2-case AB^s_{22}, where we have the following simple norm in terms of Fourier coefficients (see e.g. [15, p. 165]):

$$(\sum_{n=1}^\infty |\hat{I}(n)|^2 \cdot n^{2s})^{\frac{1}{2}}.$$

In this way we obtain an extension of a result in [1] dealing with the case $1 \leq p \leq 2$:

Corollary 4.2. *For $p \leq 2, s < 1$ and $p > 2, s < \frac{2}{p}$ every inner function I satisfies*

$$\|I\|_{B^s_{pp}} \sim (\sum_{n=1}^{\infty} |\hat{I}(n)|^2 \cdot n^{sp})^{\frac{1}{p}}.$$

In general, Theorem 4.1 should be compared also to other results in [1].

We now consider Blaschke products in AB^s_{pq}. Recall that to a sequence $\{w_n\}$ in the unit disk satisfying the Blaschke condition $\sum(1-|w_n|) < \infty$ we have an associated Blaschke product given by

$$B(z) = \prod_{n=1}^{\infty} \frac{w_n - z}{1 - \bar{w_n} z} \frac{|w_n|}{w_n}.$$

In the Dirichlet space $AB^{\frac{1}{2}}_{22}$, the norm of a Blaschke product with n factors equals $n^{\frac{1}{2}}$ (see e.g. the Carleson formula [5]). Using Theorem 4.1 we then recover the following result from [3]:

Corollary 4.3. *A Blaschke product of n factors satisfies $\|B\|_{B^{1/p}_{pp}} \sim n^{1/p},\ p > 1$.*

In particular, this norm is essentially independent of the location of the zeros. From the last corollary, we see that only finite Blaschke products are contained in AB^s_{pp} when $s \geq \frac{1}{p}$. For $s < \frac{1}{p}$ this is no longer the case, and so it is natural to look for conditions for such inclusions. A sufficient condition is given in [8], and it is also proved in [8] that it is necessary for a finite product of interpolating Blaschke products. Our contribution here consists in showing that this condition is also necessary under weaker assumptions. To state this, we need the dyadic partition of the unit disk given by the boxes

$$Q_{jk} = \{re^{i\theta} : 1 - 2^{-j} < r \leq 1 - 2^{-j-1}, \\ 2\pi(k-1) \cdot 2^{-j} < \theta \leq 2\pi k \cdot 2^{-j}\}, 0 < k \leq 2^j.$$

The theorem is as follows.

Theorem 4.4. *A sufficient condition for the inclusion $B \in AB^s_{pq}$, $s < \frac{1}{p}, 1 \leq p, q < \infty$, is given by*

$$\sum_j (\sum_{2^{-j-1} < 1-|w_n| \leq 2^{-j}} (1-|w_n|)^{1-sp})^{\frac{q}{p}} < \infty. \tag{3}$$

If the number of zeros in a box Q_{jk} is bounded, then the condition is also necessary.

Proof. The sufficiency is proved in [8]. To get necessity, we remark that according to the identity

$$1 - \frac{|z-w|^2}{|1-\bar{w}z|^2} = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2}$$

we have $|\frac{z-w}{1-\bar{w}z}| < C < 1$ when $w, z \in Q_{jk}$, where C is independent of j and k. So if the Blaschke product has a zero in Q_{jk}, we will have $1 - |B(z)| \geq C$ for all $z \in Q_{jk}$. Theorem 1.1 tells us that the condition

$$\int_0^1 (\int_T (1 - |B(re^{i\theta})|)^p d\theta)^{\frac{q}{p}} (1-r)^{-sq-1} dr < \infty \tag{4}$$

is necessary. By restricting the integration in (4) to those boxes Q_{jk} containing a zero, we arrive at (3), under the assumption that there is an upper bound on the number of zeros in a box.

Remark. For a different necessary and sufficient condition, valid for finite products of interpolating Blaschke products, see [16].

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Institute of Mathematics, University of Bergen Godskes hus, Joh. Brunsgt. 12 5008 Bergen, Norway

E-mail address: bjarte.boee@mi.uib.no