A REMARK ON REAL COBOUNDARY COCYCLES IN \( L^\infty \)-SPACE

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Abstract. Let \( T \) be an ergodic automorphism of a probability measure space \((\Omega, \mathcal{A}, m)\) and let \( f \) be a real-valued measurable function on \( \Omega \). We deduce a necessary and sufficient condition for the existence of \( L^\infty \)-solutions of the cohomology equation \( f = h \circ T - h \), by using the recent result of Alonso, Hong and Obaya.

Let \((\Omega, \mathcal{A}, m)\) be a probability measure space and let \( T \) be an ergodic automorphism of \((\Omega, \mathcal{A}, m)\). Every real-valued measurable function \( f \) on \( \Omega \) defines an additive real \( T \)-cocycle

\[
S_n f = \sum_{j=0}^{n-1} f \circ T^j, \quad n > 0.
\]

Recently Alonso, Hong and Obaya proved in \([1]\) that a necessary and sufficient condition for the existence of \( L^p \)-solutions of the cohomology equation \( f = h \circ T - h \), with \( 0 < p < \infty \), is that \( f \in L^p(\Omega, m) \) and there exists a measurable subset \( A \subset \Omega \) with \( m(A) > 0 \) and an increasing sequence \( \{n_k\} \) of positive integers with

\[
\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f| \chi_A \ dm < \infty.
\]

For related results we refer the reader to \([2]\) and \([3]\). In this note we would like to consider the case \( p = \infty \) and prove the following theorem.

**Theorem.** Let \( f \in L^p(\Omega, m) \) for some \( p \) with \( 0 < p \leq \infty \). Then there exists a function \( h \in L^\infty(\Omega, m) \) with \( f = h \circ T - h \) if and only if there exists a measurable subset \( A \subset \Omega \) with \( m(A) > 0 \) and an increasing sequence \( \{n_k\} \) of positive integers with

\[
\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|S_j f \cdot \chi_A\|_\infty < \infty.
\]

**Proof.** Suppose \( f = h \circ T - h \) for some \( h \in L^\infty(\Omega, m) \). Then, since \( S_j f = h \circ T^j - h \) for \( j \geq 1 \), we have \( \|S_j f\|_\infty \leq 2\|h\|_\infty \), whence \([3]\) holds for \( A = \Omega \) and \( n_k = k \).

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Conversely, suppose the second condition of the Theorem holds. Since \(0 < r < p\) implies \(L^p(\Omega, m) \subset L^r(\Omega, m)\), we may assume from the beginning that \(0 < p \leq 1\). Then we have
\[
\int_A |S_j f|^p \, dm \leq 1 + \|S_j f \cdot \chi_A\|_\infty \quad \text{for } j \geq 1,
\]
which implies
\[
\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f|^p \, dm < \infty.
\]
Thus, by Theorem 3.2 of [1], there exists a function \(h\) in \(L^p(\Omega, m)\) with \(f = h \circ T - h\). It suffices to show that \(h\) is a function in \(L^\infty(\Omega, m)\).

To do so, we notice that there exists a positive number \(\delta\) with \(m(A \cap \{\omega : |h(\omega)| \leq \delta\}) > 0\).

Therefore, we may assume from the beginning that
\[
|h| \leq \delta \quad \text{on } A.
\]
Then, using the relations
\[
\|h \circ T^j \cdot \chi_A\|_\infty = \|S_j f + h \cdot \chi_A\|_\infty \leq \|S_j f \cdot \chi_A\|_\infty + \delta,
\]
we see from (3) that
\[
M := \sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|h \circ T^j \cdot \chi_A\|_\infty < \infty.
\]
By this inequality, we will prove below that \(h \in L^\infty(\Omega, m)\).

For this purpose, let \(\alpha\) be a positive number such that there exists a measurable subset \(B \subset \Omega\) with
\[
m(B) > 0 \quad \text{and} \quad |h| \geq \alpha \quad \text{on } B.
\]
Since \(T\) is an ergodic automorphism by hypothesis, the Birkhoff Ergodic Theorem implies that
\[
\lim_{k \to \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \chi_A(T^{-j}\omega) = m(A) > 0
\]
for almost all \(\omega \in \Omega\). Hence, we can choose a measurable subset \(B_1 \subset B\) with \(m(B_1) > 0\) and a positive integer \(K\) with
\[
\frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A(T^{-j}\omega) > 2^{-1} m(A) \quad \text{for all } \omega \in B_1.
\]
Thus, taking into account (5) and (6), we get
\[
\infty > M \geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \circ T^j \cdot \chi_A\|_\infty = \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \cdot \chi_A \circ T^{-j}\|_\infty
\]
\[
\geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|\alpha \chi_B \cdot (\chi_A \circ T^{-j})\|_\infty \geq \alpha \cdot \|\chi_{B_1} \cdot \left(\frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A \circ T^{-j}\right)\|_\infty
\]
\[
\geq \alpha \cdot 2^{-1} m(A).
\]
This shows that \(\alpha\) cannot be so large, and thus \(h \in L^\infty(\Omega, m)\) must follow. This completes the proof. \(\square\)
REFERENCES


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