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SOME RESULTS ON EXTREMAL VECTORS AND INVARIANT SUBSPACES

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Abstract. In 1996 P. Enflo introduced the concept of extremal vectors and their connection to the Invariant Subspace Problem. The study of backward minimal vectors gives a new method of finding invariant subspaces which is more constructive than the previously known methods. In this article we study the properties and behaviour of extremal vectors, give some new formulas related to backward minimal vectors and improve results from papers by Ansari and Enflo (1998) and Enflo (1998).

In 1996 P.Enflo introduced a new method to find invariant subspaces using backward minimal vectors. This method gives hyperinvariant subspaces for all compact and all normal operators in a unified way. Studying a general bounded linear operator T on Hilbert space and extremal vectors associated with the operator T, several parameters are coming up naturally. In order to further use extremal vectors to find invariant subspaces it seems important to know more about the connections between these parameters. In this article we study the properties and behaviour of extremal vectors, investigate the connections between many of the parameters that come up in this study, give some new formulas related to backward minimal vectors and improve results from [1] and [2].

We start by recalling the definition of backward minimal vectors from Ansari-Enflo [1]. H will denote a separable Hilbert space over the real or complex numbers. $R(T)$ will denote the range of T.

Definition 1. Let $T : H \to H$ be a bounded operator with dense range. Let $x_0 \in H$, $x_0 \notin R(T)$, and $\varepsilon > 0$ with $\varepsilon < \|x_0\|$. There is a unique vector y^{ε}_{n,x_0} such that $\|T^n y^{\varepsilon}_{n,x_0} - x_0\| \leq \varepsilon$ and

$$\|y^{\varepsilon}_{n,x_0}\| = \inf\{\|y\| : \|T^n y - x_0\| \leq \varepsilon\}.$$

The points y^{ε}_{n,x_0} are called backward minimal points.

When there is no ambiguity, we will drop x_0 and ε in y^{ε}_{n,x_0}. It is clear that $\|T^n y^{\varepsilon}_{n,x_0} - x_0\| = \varepsilon$.

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In [1] the following orthogonality equation is given:

Theorem 2 (S.Ansari, P.Enflo, [1]). *There exists a constant $\delta_\varepsilon < 0$ such that*

$$y_\varepsilon = \delta_\varepsilon T^*(Ty_\varepsilon - x_0).$$

From this it follows that $T^n y_n = -\delta_n(I - \delta_n T^n T^{*n})^{-1} T^n T^{*n} x_0$ for every n. The minimality of $y^\varepsilon_{n,x_0} = y_n$ and the previous theorem also give the following:

Orthogonality Relation.

$$\text{(1)} \qquad \textit{If } r_n \perp y_n, \ \textit{ then } \ T^n r_n \perp T^n y_n - x_0.$$

We will consider the function $\varepsilon \longmapsto \|y_\varepsilon\|$ and prove the following theorem.

Theorem 3. *The function $\varepsilon \longmapsto \|y_\varepsilon\|$ is convex on $(0, \|x_0\|)$.*

Proof. For any $\lambda \in [0,1]$ and for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with $\varepsilon_2 < \varepsilon_1$, let $\varepsilon_\lambda = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. Since the element $\lambda T y_{\varepsilon_1} + (1-\lambda)T y_{\varepsilon_2} = T(\lambda y_{\varepsilon_1} + (1-\lambda) y_{\varepsilon_2})$ is inside the ball around x_0 with radius ε_λ, by the minimality of $\|y_{\varepsilon_\lambda}\|$ we have $\|y_{\varepsilon_\lambda}\| \le \|\lambda y_{\varepsilon_1} + (1-\lambda) y_{\varepsilon_2}\| \le \lambda \|y_{\varepsilon_1}\| + (1-\lambda)\|y_{\varepsilon_2}\|$. Hence, $\varepsilon \longmapsto \|y_\varepsilon\|$ is convex. □

Next we present the following new formula for the derivative of the function $\varepsilon \longmapsto \|y_\varepsilon\|$.

Theorem 4. *For the function $f(\varepsilon) = \|y_\varepsilon\|$ on $(0, \|x_0\|)$ we have*

$$\frac{d}{d\varepsilon} f(\varepsilon) = \frac{\|y_\varepsilon\|}{\cos\theta_\varepsilon \|Ty_\varepsilon\|},$$

where θ_ε is the angle between the vectors $Ty_\varepsilon - x_0$ and Ty_ε.

Proof. Let $\varepsilon > 0$ and let $h > 0$. Consider the balls around x_0 with radii ε and $\varepsilon - h$. Let y_ε and $y_{\varepsilon-h}$ be the backward minimal points for these balls, respectively. Consider the plane spanned by x_0 and Ty_ε. Since $\theta_\varepsilon > \frac{\pi}{2}$, there exist $t > 0$ and $s > 0$ such that $\|x_0 - (1+t)Ty_\varepsilon\| = \varepsilon \sin\theta_\varepsilon$ and $\|x_0 - (1+s)Ty_\varepsilon\| = \varepsilon - h$. From the Pythagorean Theorem,

$$\|(1+t)Ty_\varepsilon - (1+s)Ty_\varepsilon\| = \sqrt{(\varepsilon - h)^2 - \varepsilon^2 \sin^2\theta_\varepsilon},$$

and thus

$$\|(1+s)Ty_\varepsilon - Ty_\varepsilon\| = \varepsilon\cos(\pi - \theta_\varepsilon) - \sqrt{(\varepsilon - h)^2 - \varepsilon^2 \sin^2\theta_\varepsilon}.$$

Since $\|Ty_\varepsilon\| = \sqrt{1 - \varepsilon^2 \sin^2\theta_\varepsilon} - \varepsilon\cos(\pi - \theta_\varepsilon)$, we have by the minimality of $\|y_{\varepsilon-h}\|$ that

$$\|y_{\varepsilon-h}\| \le \left(1 + \frac{\varepsilon\cos(\pi-\theta_\varepsilon) - \sqrt{(\varepsilon-h)^2 - \varepsilon^2\sin^2\theta_\varepsilon}}{\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon} - \varepsilon\cos(\pi-\theta_\varepsilon)}\right)\|y_\varepsilon\|.$$

On the other hand, if $Ty_{\varepsilon-h} = aTy_\varepsilon + Tr$, where $Tr \perp x_0 - Ty_\varepsilon$ and $a > 1$, then

$$\|aTy_\varepsilon - Ty_\varepsilon\| = \frac{h}{\cos(\pi - \theta_\varepsilon)},$$

and thus

$$\left(1 + \frac{h}{\cos(\pi-\theta_\varepsilon)(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon} - \varepsilon\cos(\pi-\theta_\varepsilon))}\right)\|y_\varepsilon\| \le \|y_{\varepsilon-h}\|.$$

So, we get that

$$\frac{\|y_\varepsilon\|}{\cos(\pi-\theta_\varepsilon)(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon))} \le \frac{\|y_{\varepsilon-h}\|-\|y_\varepsilon\|}{h}$$
$$\le \frac{\varepsilon\cos(\pi-\theta_\varepsilon)-\sqrt{\varepsilon^2-2\varepsilon h+h^2-\varepsilon^2\sin^2\theta}}{h(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon))}\|y_\varepsilon\|.$$

Letting $h\to 0$ in the right-hand side of the inequality, we get

$$\lim_{h\to 0}\frac{\varepsilon\cos(\pi-\theta_\varepsilon)-\sqrt{\varepsilon^2(1-\sin^2\theta_\varepsilon)-2\varepsilon h+h^2}}{h(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon))}$$
$$=\frac{1}{\cos(\pi-\theta_\varepsilon)(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon))},$$

and thus

$$-f'(\varepsilon)=\lim_{h\to 0}\frac{\|y_{\varepsilon-h}\|-\|y_\varepsilon\|}{h}$$
$$=\frac{\|y_\varepsilon\|}{\cos(\pi-\theta_\varepsilon)(\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon))}=\frac{\|y_\varepsilon\|}{\cos(\pi-\theta_\varepsilon)\|Ty_\varepsilon\|}.$$

If θ_ε is close to $\frac{\pi}{2}$ and $\varepsilon>0$ is small, then $\|Ty_\varepsilon\|=\sqrt{1-\varepsilon^2\sin^2\theta_\varepsilon}-\varepsilon\cos(\pi-\theta_\varepsilon)$ is close to 1. Hence, as $\varepsilon\to 0$, we have that

$$\frac{f'(\varepsilon)}{\left(\frac{\|y_\varepsilon\|}{\cos\theta_\varepsilon}\right)}\longrightarrow 1.$$

□

Using Theorems 2 and 4, one can also find a formula for the δ_ε introduced in Theorem 2.

Theorem 5. *Using the notation in Theorem 2, we have*

$$\delta_\varepsilon=\frac{f'(\varepsilon)f(\varepsilon)}{\varepsilon},\ \text{ where } f(\varepsilon)=\|y_\varepsilon\|.$$

Proof. Since $y_\varepsilon=\delta_\varepsilon T^*(Ty_\varepsilon-x_0)$, we have $\langle y_\varepsilon,z\rangle=\delta_\varepsilon\langle Ty_\varepsilon-x_0,Tz\rangle$ for all $z\in H$. Let $z=y_\varepsilon$. Then $\langle y_\varepsilon,y_\varepsilon\rangle=\delta_\varepsilon\langle Ty_\varepsilon-x_0,Ty_\varepsilon\rangle$, hence

$$\delta_\varepsilon=\frac{\|y_\varepsilon\|^2}{\langle Ty_\varepsilon-x_0,Ty_\varepsilon\rangle}=\frac{\|y_\varepsilon\|^2}{\|Ty_\varepsilon\|\|Ty_\varepsilon-x_0\|\cos\theta_\varepsilon}$$
$$=\frac{\|y_\varepsilon\|^2}{\varepsilon\|Ty_\varepsilon\|\cos\theta_\varepsilon}=\frac{f'(\varepsilon)f(\varepsilon)}{\varepsilon},$$

where $f(\varepsilon)=\|y_\varepsilon\|$. □

The next result is an improvement of Theorem 3 from [1]. This theorem was proved for a subsequence of a sequence $(\varepsilon_n)=(\varepsilon_{n-1}\sin\theta_{n-1})$, where $\theta_{n-1}=\theta_{\varepsilon_{n-1}}$ is the angle between the vectors $Ty_{\varepsilon_{n-1}}-x_0$ and $Ty_{\varepsilon_{n-1}}$ and the ε_n's are radii with $\varepsilon_n\to 0$. We will show that the theorem remains true for any sequence of ε_n's with $\varepsilon_n\to 0$.

Theorem 6. *For any bounded operator T with dense range, if $x_0\notin R(T)$, then*

$$\frac{1}{\varepsilon}\langle Ty_\varepsilon-x_0,Ty_\varepsilon\rangle\longrightarrow 0\quad\text{as}\quad\varepsilon\to 0.$$

Proof. Let (ε_n) be a sequence with $\varepsilon_n \to 0$. In Theorem 3 of [1] it is proved that if $\varepsilon_n = \varepsilon_{n-1} \sin\theta_{n-1}$, where θ_{n-1} is the angle between the vectors $Ty_{\varepsilon_{n-1}} - x_0$ and $Ty_{\varepsilon_{n-1}}$, then there is a subsequence (n_k) of natural numbers such that $\theta_{n_k} \longrightarrow \frac{\pi}{2}$. Consequently, $\cos\theta_{n_k} \longrightarrow 0$ for the same subsequence. From the definition of y_ε we also know that $\|y_\varepsilon\| \to \infty$ as $\varepsilon \to 0$. By Theorem 4 we have that the size of the derivative $f'(\varepsilon_{n_k})$ is asymptotically the same as the size of $\frac{\|y_{\varepsilon_{n_k}}\|}{\cos\theta_{n_k}}$, so $|f'(\varepsilon_{n_k})|$ is increasing as a function of k as $k \to \infty$. Since, by Theorem 3, the function $f(\varepsilon) = \|y_\varepsilon\|$ is convex on $(0, \|x_0\|)$, we have that $|f'(\varepsilon_n)|$ is increasing as $n \to \infty$ for all sequences ε_n, where $\varepsilon_n \to 0$. We will prove that $\theta_n \longrightarrow \frac{\pi}{2}$ as $n \to \infty$. If not, then there exists a sequence (r_n) with $r_n \to 0$ and $\sum r_n < \infty$, but $\theta_{r_n} > \frac{\pi}{2} + \delta$ for some $\delta > 0$ and for all $n \geq n_0$. If θ_{r_0} is the angle between the vectors $Ty_{r_0} - x_0$ and Ty_{r_0}, then there exists $t > 0$ such that $\|x_0 - (1+t)Ty_{r_0}\| = \varepsilon_1$. One can show that $t < K \cdot r_0$ for some constant $K > 0$. To see this note that $(1+t)\|Ty_{r_0}\| \leq \|Ty_{r_0}\| + 2r_0$, so $t\|Ty_{r_0}\| \leq 2r_0$ and $t \leq \frac{2r_0}{\|x_0\| - r_0} < K \cdot r_0$ for some $K > 0$. Thus, by the minimality of $\|y_{\varepsilon_1}\|$, we have $\|y_{\varepsilon_1}\| \leq (1+t)\|y_{r_0}\| < (1+Kr_0)\|y_{r_0}\|$. Let $r_1, r_2, \ldots, r_{k_1}$ be k_1 terms of the sequence (r_n) such that $r_0 > r_1 > r_2 > \cdots > r_{k_1} \geq \varepsilon_1$. By Definition 1 we also have that $\|y_{r_0}\| \leq \|y_{r_1}\| \leq \|y_{r_2}\| \leq \ldots \leq \|y_{r_{k_1}}\| \leq \|y_{\varepsilon_1}\| < (1+Kr_0)\|y_{r_0}\|$. For $n \geq 2$ let $\varepsilon_n = \varepsilon_{n-1} \cdot \sin\theta_{n-1}$. Let $r_{k_{n-1}+1}, \ldots, r_{k_n}$ be the terms from the sequence (r_n) such that $r_{k_{n-1}+1} > r_{k_{n-1}+2} > \cdots > r_{k_n} \geq \varepsilon_n$. Applying the above argument repeatedly, we get for $n \geq 2$ and $k_{n-1} < m \leq k_n$ that $\|y_{r_m}\| \leq \|y_{\varepsilon_n}\| \leq (1+K\varepsilon_0)(1+K\varepsilon_1)\ldots(1+K\varepsilon_n)\|y_{r_0}\|$. Since $\sum \varepsilon_n < \infty$, the infinite product $(1+K\varepsilon_0)(1+K\varepsilon_1)\ldots = M$ is finite, and so $\|y_{r_n}\| \leq M\|y_{r_0}\|$ for all n. Let (y_{n_k}) be a weakly convergent subsequence, converging weakly to y_0. Then, $Ty_{n_k} \xrightarrow{w} Ty_0$ and $\|Ty_{n_k} - x_0\| = r_{n_k} \longrightarrow 0$. So, $\|Ty_0 - x_0\| = 0$, that is, $Ty_0 = x_0$. This contradiction proves that $\theta_{\varepsilon_n} \longrightarrow \frac{\pi}{2}$ as $\varepsilon_n \to 0$ for any sequence of ε_n's. □

At this point we do not know whether Theorem 5 from [1] holds for all n. This theorem says that if a quasinilpotent operator T has no hyperinvariant subspaces, then there is a subsequence (n_k) of natural numbers such that for some $\delta > 0$ we have

$$\langle T^{n_k} y_{n_k}^\delta, T^{n_k} y_{n_k}^\delta - x_0 \rangle \longrightarrow 0. \tag{2}$$

We can, however, give the following improvement.

Proposition 7. *Let T be a quasinilpotent operator. Let $x_0 \in H$. Then there is a subsequence (n_k) of natural numbers such that for some $\delta > 0$ and for every $m \in \mathbb{N}$ we have*

$$\langle T^{n_k+m} y_{n_k+m}^\delta, T^{n_k+m} y_{n_k+m}^\delta - x_0 \rangle \longrightarrow 0.$$

Proof. The proposition says that there are infinitely many such subsequences (n_k) with the property (2). To see this, for any $m = 1, 2, 3, \ldots$, find $l_m(T)$ with $\|l_m(T)\| \leq C$ such that $\|T^m l_m(T)x_0 - x_0\| < \frac{\delta}{2}$. Here, $l_m(T)$ denotes a polynomial in T. Let $\|T^{n_k} y_{n_k} - x_0\| \leq \frac{\delta}{2C}$. Then, for any fixed $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\|T^{n_k+m} l_m(T) y_{n_k} - x_0\| &\leq \|T^{n_k+m} l_m(T) y_{n_k} - l_m(T) T^m x_0\| + \|l_m(T) T^m x_0 - x_0\| \\
&\leq \|T^m l_m(T)\| \|T^{n_k} y_{n_k} - x_0\| + \frac{\delta}{2} \leq \|T^m\| C \cdot \frac{\delta}{2C} + \frac{\delta}{2} = (\|T^m\| + 1)\frac{\delta}{2} = \delta,
\end{aligned}$$

assuming that $\|T\| = 1$.

The minimality of $y^{\delta}_{n_k+m}$ gives

$$\|y^{\delta}_{n_k+m}\| \leq \|l_m(T)y^{\frac{\delta}{2C}}_{n_k}\| \leq C\|y^{\frac{\delta}{2C}}_{n_k}\|.$$

Thus, the ratio $\|y^{\delta}_{n_k+m}\|/\|y^{\frac{\delta}{2C}}_{n_k}\|$ is bounded by a constant. Now take $\varepsilon < \frac{\delta}{2C}$. Since the function $f(\varepsilon) = \|y_\varepsilon\|$ is convex, and since the modulus of its derivative goes to infinity as $\varepsilon \to 0$, we have $\frac{\|y^{\varepsilon}_{n_k}\|}{\|y^{\frac{\delta}{2C}}_{n_k}\|} \longrightarrow \infty$ as $\varepsilon \to 0$. From this it follows that for every $\varepsilon > 0$, $\theta^{\varepsilon}_{n_k} \longrightarrow \frac{\pi}{2}$ as $k \to \infty$. As mentioned above, since $\|y^{\delta}_{n_k+m}\|$ differs from $\|y^{\frac{\delta}{2C}}_{n_k}\|$ only by a constant, we also have that for every $\varepsilon < \frac{\delta}{2C}$, $\frac{\|y^{\varepsilon}_{n_k}\|}{\|y^{\delta}_{n_k+m}\|} \longrightarrow \infty$ as $k \to \infty$. Since $\|y^{\varepsilon}_{n_k+m}\| \geq \|y^{\varepsilon}_{n_k}\|$, we get $\frac{\|y^{\varepsilon}_{n_k+m}\|}{\|y^{\delta}_{n_k+m}\|} \longrightarrow \infty$ as $k \to \infty$. Hence, on the sphere with radius ε, if we have a subsequence (n_k) for which $\theta^{\varepsilon}_{n_k} \longrightarrow \frac{\pi}{2}$ as $k \to \infty$, then, in fact, $\theta^{\varepsilon}_{n_k+m} \longrightarrow \frac{\pi}{2}$ for any $m = 1, 2, 3, \ldots$ as $k \to \infty$. Hence for every $m \in \mathbb{N}$ we have

$$\langle T^{n_k+m}y^{\delta}_{n_k+m}, T^{n_k+m}y^{\delta}_{n_k+m} - x_0\rangle \longrightarrow 0.$$

□

Suppose T is quasinilpotent and has no hyperinvariant subspaces. Assume $x_0 \notin R(T)$. Then, by the previous proposition, for some $\varepsilon > 0$ and for some subsequence (n_k) of integers,

$$\langle T^{n_k}y^{\varepsilon}_{n_k}, T^{n_k}y^{\varepsilon}_{n_k} - x_0\rangle \longrightarrow 0,$$

where $(y^{\varepsilon}_{n_k})$ are the backward minimal vectors with respect to x_0 and ε. As in [3], we can write (renaming $n = n_k$) that

$$T^n y^{\varepsilon}_{n,x_0} = (1-\varepsilon^2)x_0 + \gamma\varepsilon\sqrt{1-\varepsilon^2}s_0 + \sqrt{1-\varepsilon^2}\varepsilon\sqrt{1-\gamma^2}s_n, \tag{3}$$

where $\|s_0\| = \|s_n\| = 1$ and $s_n \xrightarrow{w} 0$.

S.Ansari and P.Enflo have shown that if T is quasinilpotent and has no hyperinvariant subspaces, then for every $x_0 \in H$ with $\|x_0\| = 1$, for every $\varepsilon > 0$, and for every $\gamma > \sqrt{2\varepsilon}$, there are x_1 with $\|x_1\| = 1$ and $\varepsilon_1 > 0$ with $0 < \varepsilon_1 < \varepsilon$ such that $\|x_0 - x_1\| \leq \varepsilon g(\gamma)$, where $g(\gamma)$ is a continuous function of γ, and such that for some subsequence (n_k) of integers we have

$$T^{n_k}y^{\varepsilon_1}_{n_k,x_1} = a_{n_k}x_1 + s_{n_k}, \tag{4}$$

where a_{n_k} is a constant, $s_{n_k} \perp x_1$, and each weak limit of s_{n_k} has norm less than $\gamma\varepsilon_1$ (see Theorem 6, [1]).

First we will show how to find x_1 and then we will determine the function $g(\gamma)$. Let

$$x_1 = \frac{\sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2(1-\gamma^2)}}x_0 + \frac{\gamma\varepsilon}{\sqrt{1-\varepsilon^2(1-\gamma^2)}}s_0, \tag{5}$$

where $x_0 \perp s_0$, and $\|x_0\| = \|s_0\| = 1$ as in (3). Then, clearly, $\|x_1\| = 1$, and

$$\|x_0 - x_1\|^2 = \|x_0 - \frac{1-\varepsilon^2}{\sqrt{1-\varepsilon^2(1-\gamma^2)}}x_0 - \frac{\gamma\varepsilon}{\sqrt{1-\varepsilon^2(1-\gamma^2)}}s_0\|^2$$

$$= \frac{1}{1-\varepsilon^2(1-\gamma^2)}(1-\varepsilon^2(1-\gamma^2) - 2\sqrt{1-\varepsilon^2}\sqrt{1-\varepsilon^2(1-\gamma^2)} + 1 - \varepsilon^2 + \gamma^2\varepsilon^2)$$

$$= 2\left(1 - \frac{\sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2(1-\gamma^2)}}\right) \approx 2\left(1 - \frac{1-\frac{\varepsilon^2}{2}}{1-\frac{\varepsilon^2}{2}(1-\gamma^2)}\right)$$

$$\approx 2\left(1 - (1-\frac{\varepsilon^2}{2})(1+\frac{\varepsilon^2}{2}(1-\gamma^2))\right) = \varepsilon^2\left(1-(1-\frac{\varepsilon^2}{2})(1-\gamma^2)\right).$$

The approximation $\approx$ comes from approximating the square root by the first two terms in its power series representation. Thus,

$$\|x_0 - x_1\| \approx \varepsilon\left(1 - \frac{1}{2}(1-\frac{\varepsilon^2}{2})(1-\gamma^2)\right) = \varepsilon g(\gamma) < \varepsilon,$$

where $g(\gamma) = 1 - \frac{1}{2}(1-\frac{\varepsilon^2}{2})(1-\gamma^2)$.

Next we derive a formula for ε_1.

Proposition 8. *Let all the assumptions about T hold as above. Let x_0, ε and γ be given. Let x_1 be defined as in (5). Then ε_1 in (4) is given by $\varepsilon_1 = \varepsilon\sqrt{1-\gamma^2}$.*

Proof. Let $x_0 \in H$ with $\|x_0\| = 1$. Then, given $\varepsilon > 0$ and $\gamma > 0$ we can write $T^{n_k}y^{\varepsilon}_{n_k,x_0} = (1-\varepsilon^2)x_0 + \gamma\varepsilon\sqrt{1-\varepsilon^2}s_0 + \sqrt{1-\gamma^2}\varepsilon\sqrt{1-\varepsilon^2}s_{n_k}$. If x_1 is as in (5), then, by Theorem 6 in [1], there exists an ε_1 with $0 < \varepsilon_1 < \varepsilon$ such that

$$(1-\varepsilon_1^2)x_1 + \varepsilon_1\sqrt{1-\varepsilon_1^2}s_{n_k} = \frac{\sqrt{1-\varepsilon_1^2}}{\sqrt{1-\varepsilon^2}}\Big((1-\varepsilon^2)x_0 + \gamma\varepsilon\sqrt{1-\varepsilon^2}s_0$$
$$+\sqrt{1-\gamma^2}\varepsilon\sqrt{1-\varepsilon^2}s_{n_k}\Big).$$

Note that the right-hand side represents a vector that is tangent to the ε_1-ball around x_1. Comparing the ratios of sides of the similar triangles in the plane spanned by x_1 and $T^n y^{\varepsilon}_{n,x_0}$, we get

$$\frac{1-\varepsilon_1^2}{\varepsilon_1\sqrt{1-\varepsilon_1^2}} = \frac{\sqrt{(1-\varepsilon^2)^2 + \gamma^2\varepsilon^2(1-\gamma^2)}}{\sqrt{1-\gamma^2}\varepsilon\sqrt{1-\varepsilon^2}}. \tag{6}$$

Squaring both sides of (6) we get

$$\frac{1-\varepsilon_1^2}{\varepsilon_1^2} = \frac{(1-\varepsilon^2)(1-\varepsilon^2+\gamma^2\varepsilon^2)}{(1-\varepsilon^2)\varepsilon^2(1-\gamma^2)} = \frac{1-\varepsilon^2(1-\gamma^2)}{\varepsilon^2(1-\gamma^2)},$$

where it follows that $\varepsilon_1 = \varepsilon\sqrt{1-\gamma^2}$. □

Denote by A^{δ_n} the continuous linear operator

$$-\delta_n(I - \delta_n T^n T^{*n})^{-1}T^n T^{*n}. \tag{7}$$

From the orthogonality equations we obtain that $A^{\delta_n}x_0 = T^n y_{n,x_0}$, that is, given x_0, the δ_n determines a backward minimal vector y_{n,x_0}. Recall that δ_n depends on ε.

The following proposition gives an asymptotic relationship between the norm of y_{n,x_0} and the absolute value of δ_n.

Proposition 9. *Let $\varepsilon_2 < \varepsilon_1 < \varepsilon_0$. Then, for n sufficiently large, we have*

$$|\delta_n(\varepsilon_0)| \leq \frac{\|y_n^{\varepsilon_1}\|^2}{2\varepsilon_1(\varepsilon_0 - \varepsilon_1)} \leq |\delta_n(\varepsilon_2)|\,.$$

Proof. Integrating $\delta_n(\varepsilon) = \frac{f_n(\varepsilon) f_n'(\varepsilon)}{\varepsilon}$ by parts over a small interval $[\varepsilon_1, \varepsilon_0]$ (where $f_n(\varepsilon) = \|y_n^\varepsilon\|$ and $f_n'(\varepsilon) = \frac{\|y_n^\varepsilon\|}{\cos\theta_n^\varepsilon \|Ty_n^\varepsilon\|}$), we get

$$\left|\int_{\varepsilon_1}^{\varepsilon_0} \varepsilon\delta_n(\varepsilon)d\varepsilon\right| = \left|\int_{\varepsilon_1}^{\varepsilon_0} f_n(\varepsilon)f_n'(\varepsilon)d\varepsilon\right|$$

$$= \left|f_n^2(\varepsilon)\big|_{\varepsilon_1}^{\varepsilon_0} - \int_{\varepsilon_1}^{\varepsilon_0} f_n(\varepsilon)f_n'(\varepsilon)d\varepsilon\,\right| = \left|\frac{f_n^2(\varepsilon)}{2}\right|_{\varepsilon_1}^{\varepsilon_0} = \left|\frac{f_n^2(\varepsilon_0)}{2} - \frac{f_n^2(\varepsilon_1)}{2}\right|.$$

Since $\theta_n^\varepsilon \longrightarrow \frac{\pi}{2}$ as $n \to \infty$, by Thorem 4 we know that $|f_n'(\varepsilon)| \longrightarrow \infty$ very fast as $\varepsilon \to 0$ and as $n \to \infty$. Therefore, the size of the above integral is mostly determined by the f_n's value at ε_1. Considering ε as a constant over the interval $[\varepsilon_1, \varepsilon_0]$ (e.g. taking $\varepsilon = \varepsilon_1$), we get

$$\left|\int_{\varepsilon_1}^{\varepsilon_0} \delta_n(\varepsilon)d\varepsilon\right| \approx \frac{f_n^2(\varepsilon_1)}{2\varepsilon_1}.$$

Now, for some $\varepsilon_2 < \varepsilon_1 < \varepsilon_0$ $(\varepsilon_2 \leq \varepsilon_1(\varepsilon_0 - \varepsilon_1))$, we have

$$|\delta_n(\varepsilon_0)| \leq \left|\delta_{n,aver[\varepsilon_1,\varepsilon_0]}(\varepsilon)\right| = \frac{1}{\varepsilon_0 - \varepsilon_1}\left|\int_{\varepsilon_1}^{\varepsilon_0} \delta_n(\varepsilon)d\varepsilon\right| \leq \frac{1}{\varepsilon_0 - \varepsilon_1}\cdot\frac{f_n^2(\varepsilon_1)}{2\varepsilon_1}$$

$$\leq \frac{f_n^2(\varepsilon_2)}{\varepsilon_2} \leq \left|\int_{\varepsilon_2}^{\varepsilon_1} \delta_n(\varepsilon)d\varepsilon\right| \leq \frac{1}{\varepsilon_1 - \varepsilon_2}\left|\int_{\varepsilon_2}^{\varepsilon_1} \delta_n(\varepsilon)d\varepsilon\right| = \left|\delta_{n,aver[\varepsilon_2,\varepsilon_1]}(\varepsilon)\right| \leq |\delta_n(\varepsilon_2)|\,.$$

Hence, for n large enough, we have

$$|\delta_n(\varepsilon_0)| \leq \frac{f_n^2(\varepsilon_1)}{2\varepsilon_1(\varepsilon_0 - \varepsilon_1)} \leq |\delta_n(\varepsilon_2)|\,.$$

□

It follows from Theorem 5 and Proposition 7 that for some subsequence (n_k), if $\varepsilon_2 < \varepsilon_1$, then

$$\frac{|\delta_{n_k}(\varepsilon_2)|}{|\delta_{n_k}(\varepsilon_1)|} \longrightarrow \infty \text{ as } n_k \to \infty,$$

and if $\varepsilon_0 > \varepsilon_1$, then

$$\frac{|\delta_{n_k}(\varepsilon_0)|}{|\delta_{n_k}(\varepsilon_1)|} \longrightarrow 0 \text{ as } n_k \to \infty.$$

Combining this with Proposition 9, we get the following theorem:

Theorem 10. *Let $\varepsilon_1 > 0$. There is a subsequence (n_k) of natural numbers such that for every $\varepsilon_2 < \varepsilon_1$ we have*

$$\frac{|\delta_{n_k}(\varepsilon_2)|}{\|y_{n_k}^{\varepsilon_1}\|^2} \longrightarrow \infty \text{ as } k \to \infty,$$

and for every $\varepsilon_0 > \varepsilon_1$ we have

$$\frac{|\delta_{n_k}(\varepsilon_0)|}{\|y_{n_k}^{\varepsilon_1}\|^2} \longrightarrow 0 \text{ as } k \to \infty.$$

Corollary 11. *If* $|\delta_{n_k}(\mu_k)| = \|y^{\varepsilon_k}_{n_k,x_0}\|^2$ *for all* k *large enough, then*

$$\limsup_k \mu_k = \varepsilon = \limsup_k \varepsilon_k.$$

Proof. WLOG suppose $\lim_k \sup \mu_k = \varepsilon$, but $\lim_k \sup \varepsilon_k \neq \varepsilon$ with $\varepsilon_k < \varepsilon$. Then there exist a $\delta > 0$ and $k_0 \in \mathbb{N}$ such that $\varepsilon_k < \varepsilon - \delta$ for all $k > k_0$. By Theorem 10 we have

$$\frac{|\delta_{n_k}(\mu_k)|}{\|y^{\varepsilon_k}_{n_k}\|^2} \leq \frac{|\delta_{n_k}(\mu_k)|}{\|y^{\varepsilon-\delta}_{n_k}\|^2} \longrightarrow 0$$

as $k \to \infty$, contradicting the assumption that $|\delta_{n_k}(\mu_k)| = \|y^{\varepsilon_k}_{n_k,x_0}\|^2$ for all k large enough. □

Now, having x_0 and ε fixed, we get δ_{n_k} and apply the operator $A^{\delta_{n_k}}$ from (7) to the vector x_1. We know that $\|A^{\delta_{n_k}} x_0 - x_0\| = \|T^{n_k} y^{\varepsilon}_{n_k,x_0} - x_0\| = \varepsilon$. We will show that when x_0 is replaced by x_1, we get a better approximation.

Proposition 12. *With the notation as above,*

$$\limsup_k \|A^{\delta_{n_k}} x_1 - x_1\| \leq \varepsilon\sqrt{1-\gamma^2}.$$

Proof. Let x_0 and ε be given. For every $k \in \mathbb{N}$ find an $\varepsilon_k < \varepsilon$ such that $|\delta^{\varepsilon}_{n_k,x_0}| = \|y^{\varepsilon_k}_{n_k,x_0}\|^2$. As a special case of Corollary 11, we have that $\varepsilon_k \to \varepsilon$. Let x_1 be as in (5). Then, if $\|y^{\varepsilon_k}_{n_k,x_0}\| = \|y^{\upsilon_k}_{n_k,x_1}\|$ for some sequence (υ_k) and for all k, then, by Proposition 8, $\lim_k \sup \upsilon_k \leq \varepsilon\sqrt{1-\gamma^2}$. On the other hand, $\|y^{\upsilon_k}_{n_k,x_1}\|^2 = |\delta^{\mu_k}_{n_k,x_1}|$ for some sequence (μ_k) with $\lim_k \sup \mu_k = \lim_k \sup \upsilon_k \leq \varepsilon\sqrt{1-\gamma^2}$. Hence, $\lim_k \sup \|T^{n_k} y^{\upsilon_k}_{n_k,x_1} - x_1\| \leq \varepsilon\sqrt{1-\gamma^2}$. □

We now start over with x_1 instead of x_0 and ε_1 instead of ε. Similarly to (5), for every $\gamma_1 > \sqrt{2\varepsilon_1}$, we define

$$x_2 = \frac{\sqrt{1-\varepsilon_1^2}}{\sqrt{1-\varepsilon_1^2(1-\gamma_1^2)}} x_1 + \frac{\gamma_1\varepsilon_1}{\sqrt{1-\varepsilon_1^2(1-\gamma_1^2)}} s_1.$$

Then

$$\|x_1 - x_2\| \approx \varepsilon(1 - \frac{\gamma^2}{2})\left(1 - \frac{1}{2}(1 - \frac{\varepsilon^2}{2}(1-\gamma^2))(1-\gamma_1^2)\right).$$

Also,

$$\|x_0 - x_2\| \approx \varepsilon\left(1 - \frac{1}{2}(1-\gamma^2)\left((1-\gamma_1^2)(1 - \frac{\varepsilon^2}{2}(1-\gamma^2)) - \frac{\varepsilon^2}{2}\right)\right).$$

Similarly to Proposition 8, we get that $\varepsilon_2 = \varepsilon_1\sqrt{1-\gamma^2} = \varepsilon(1-\gamma^2)$, and we repeat previous steps with x_3, ε_2 and $\gamma_2 > \sqrt{2\varepsilon_2}$ instead.

In general, we define

$$x_{n+1} = \frac{\sqrt{1-\varepsilon_n^2}}{\sqrt{1-\varepsilon_n^2(1-\gamma_n^2)}} x_n + \frac{\gamma_n\varepsilon_n}{\sqrt{1-\varepsilon_n^2(1-\gamma_n^2)}} s_n, \tag{8}$$

from where it follows that

$$\|x_{n+1} - x_n\| \approx \varepsilon \prod_{k=0}^{n-1}(1 - \frac{\gamma_k^2}{2})\left(\gamma^2 + \frac{\varepsilon^2}{2}\prod_{k=0}^{n}(1-\gamma_k^2)\right).$$

Remark. We conjecture that if the sequence (x_n) converges and the ε_n's go to zero, then the limit vector is noncyclic. For the further use of extremal vectors it is of interest to know whether the sequence (x_n), that is defined inductively as in (8), always converges or whether it may go weakly to zero.

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