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THE KOWALEVSKI TOP AS A REDUCTION OF A HAMILTONIAN SYSTEM ON $\mathfrak{sp}(4,\mathbb{R})^*$

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Abstract. We show that the Kowalevski top and Kowalevski gyrostat are obtained as a reduction of a Hamiltonian system on $\mathfrak{sp}(4,\mathbb{R})^*$. Therefore the Lax-pair representations for the Kowalevski top and Kowalevski gyrostat are obtained via a direct method by transforming the canonical Lax-pair representation of a system on $\mathfrak{sp}(4,\mathbb{R})^*$. Also we show that the nontrivial integral of motion of the Kowalevski top comes from a Casimir function of the Lie-Poisson algebra $\mathfrak{sp}(4,\mathbb{R})^*$.

1. Introduction

The Kowalevski top is the third integrable case of a heavy rigid body with a fixed point rotating in a constant gravitational field. The other two cases were found by Euler and Lagrange in the 18th century. The Kowalevski top, found by S. Kowalevski [8], is a highly nontrivial integrable system because one of its integrals of motion is of a complicated form.

One of the techniques employed in the study of integrable systems is to express the corresponding differential equations in a Lax-pair form with spectral parameter:

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)].$$

The Lax-pair representation of a system exhibits the integrals of motion of the system as eigenvalues of the linear operator $L(\lambda)$ and leads to the linearization of the flow on the Jacobi variety of the algebraic curve $det(L(\lambda) - sI)$.

In 1987 three groups of authors, M. Adler and P. van Moerbeke in [1], L. Haine and E. Horozov in [6] and A. Reyman and M.A. Semenov-Tian-Shansky in [11], found three different Lax-pair representations with spectral parameter. All three Lax-pairs were used to define a spectral curve and proved valuable in integrating the system.

The simplest Lax-pair (because the entries of the Lax-pair matrices are linear functions of the phase variables) was discovered by A. Reyman and M.A. Semenov-Tian-Shansky, [11]. They used R-matrix construction and symplectic reduction to obtain the Lax-pair. Later I.D. Marshall, in [10], gave a direct description of the Lax-pair by means of the R-matrix construction.

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The aim of this paper is to present a direct and elementary method to construct the Lax-pair form. This method does not involve the R-matrix construction. We will show how starting from a Hamiltoninan system on $\mathfrak{sp}(4,\mathbb{R})^*$ we can obtain the Kowalevski top. We first find an integral of motion of the Hamiltonian system on $\mathfrak{sp}(4,\mathbb{R})^*$ and then we reduce the system using this integral of motion. Acting with an affine transformation on phase variables of the reduced system we end up with the Kowalevski top.

Because of the semi-simplicity of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$, any Hamiltonian system on $\mathfrak{sp}(4,\mathbb{R})^*$ admits a canonical Lax-pair representation with matrices L and M in the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. Performing the above procedure on the matrices L and M we end up with a Lax-pair representation without a spectral parameter for the Kowalevski top. The appearance of the spectral parameter in the matrices of the Lax-pair form can be explained by a symmetry of the top. This direct method is actually valid in a more general setting, leading to the Lax-pair form of the Kowalevski gyrostat.

The paper [10] of I.D. Marshall contains a concise description of A. Reyman and M.A. Semenov-Tian-Shansky's result as well as a bihamiltonian description of the Kowlevski top. Connections between the Kowalevski top and the elastic problem are explained by V. Jurdjevic in [7].

The paper is organized as follows: Section 2 describes general facts about the Kowalevski top and the Kowalevski gyrostat. Section 3 contains a treatment of Hamiltonian systems on the duals of Lie algebras. In Section 4 we explain a particular Lie algebra, namely $\mathfrak{sp}(4,\mathbb{R})$. In Section 5 we study the form and the properties of a class of Hamiltonian systems on $\mathfrak{sp}(4,\mathbb{R})^*$. In Section 6 we present the reduction procedure of the Hamiltonian systems on $\mathfrak{sp}(4,\mathbb{R})^*$ which leads to the Kowalevski top or the Kowalevski gyrostat. Also we include a simple explanation for the Lax-pair representations of the Kowalevski top and the Kowalevski gyrostat obtained in [11].

In this paper we will make use of:

- The tensor product of matrices:
  if A is a $p \times q$ matrix and B is an $m \times n$ matrix, $A \otimes B$ is the $pm \times qn$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ \vdots & \vdots & & \vdots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{pmatrix}. \tag{1.1}$$

- The Pauli spin matrices $\sigma_1,\ i\sigma_2,\ \sigma_3$ and the 2×2 identity matrix $\mathbf{1}$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

- The vector notation $\hat{v}$ for the three dimensional column vector $\hat{v} = (v_1, v_2, v_3)^T$; also $||\ ||^2$ is the standard norm of a vector $||\hat{v}||^2 = v_1^2 + v_2^2 + v_3^2$.
- $\times$ and . are the usual cross respective scalar product of vectors in $\mathbb{R}^3$.
- The standard basis of $\mathbb{R}^3$: $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$, $e_3 = (0,0,1)^T$.

2. The Kowalevski top and the Kowalevski gyrostat

In this section we present a few known facts about the Kowalevski top and the Kowalevski gyrostat.

The Kowalevski top is a Hamiltonian system on the dual of the semidirect product Lie algebra $\mathfrak{so}(3) \ltimes \mathbb{R}^3$:

$$\frac{d\hat{H}}{dt} = \hat{H} \times \hat{\Omega} + \hat{h} \times \hat{b}, \quad \frac{d\hat{h}}{dt} = \hat{h} \times \hat{\Omega}. \tag{2.1}$$

The vector $\hat{\Omega} = (H_1, H_2, 2H_3)^T$ corresponds to the angular velocity of the body measured relative to the moving frame. The vector $\hat{b} = (b_1, b_2, 0)^T$ indicates the center of mass of the body. A comprehensive description of the physical meaning of the variables and parameters of the system may be found in V. Jurdjevic's paper [7].

The Hamiltonian of the system is the inhomogeneous quadratic function

$$\mathrm{Ham}(\hat{H}, \hat{h}) = \frac{1}{2}(H_1^2 + H_2^2 + 2H_3^2) + b_1 h_1 + b_2 h_2. \tag{2.2}$$

The integrals of motion of this system are: two Casimir functions of $\mathfrak{so}(3) \ltimes \mathbb{R}^3$, $I_1 = ||\hat{h}||^2$, $I_2 = \hat{H}\hat{h}$ and another one with a complicated form

$$I_3 = |2z^2 - bw|^2 \tag{2.3}$$

where $z = (H_1 + iH_2)/2$, $w = h_1 + ih_2$ and $b = b_1 + ib_2$.

The Kowalevski gyrostat is a Hamiltonian system on a 9-dimensional phase space, the dual of the semidirect product $\mathfrak{so}(3) \ltimes (\mathbb{R}^3 \oplus \mathbb{R}^3)$. It was introduced by A. Reyman and M.A. Semenov-Tian-Shansky as a generalization of the top:

$$\frac{d\hat{H}}{dt} = \hat{H} \times (\hat{\Omega} + \gamma e_3) + \hat{h} \times \hat{b} + \hat{g} \times \hat{c}, \quad \frac{d\hat{h}}{dt} = \hat{h} \times (\hat{\Omega} + \gamma e_3), \quad \frac{d\hat{g}}{dt} = \hat{g} \times (\hat{\Omega} + \gamma e_3). \tag{2.4}$$

As before $\hat{\Omega} = (H_1, H_2, 2H_3)^T$. The vectors $\hat{b} = (b_1, b_2, 0)^T$, $\hat{c} = (-b_2, b_1, 0)^T$ are constant vectors. The particular form of the vectors $\hat{\Omega}$, $\hat{b}$ and $\hat{c}$ implies the existence of two nontrivial integrals of motion [2]. The other trivial integrals of motion are the Casimir functions of the phase space, $I_1 = ||\hat{h}||^2$, $I_2 = ||\hat{g}||^2$ and $I_3 = \hat{h}\hat{g}$.

The Hamiltonian of the system involves a new parameter γ,

$$\mathrm{Ham}_\gamma(\hat{H}, \hat{h}, \hat{g}) = \frac{1}{2}(H_1^2 + H_2^2 + 2H_3^2) + \gamma H_3 + \hat{b}\hat{h} + \hat{c}\hat{g}. \tag{2.5}$$

3. General facts

To represent a differential system $\dot{x}(t) = f(x(t))$ in Lax-pair form means to write the system as

$$\dot{L}(x(t)) = [M(x(t)), L(x(t))], \tag{3.1}$$

where L and M are two matrices whose entries depend on $x(t)$.

When the matrices L and M depend on an undetermined parameter λ, called spectral parameter, we say that the Lax-pair is a Lax-pair representation with spectral parameter. Usually it is desirable that the Lax-pair is a Lax-pair with spectral parameter.

There are at least two advantages of having a Lax-pair representation for a system.

- The spectral invariants of L (i.e. $Tr(L^k)$, $det(L)$) are integrals of motion of the system. As an example,

$$\begin{aligned}(3.2)\quad \frac{d}{dt}(Tr(L^k)) &= Tr(\sum_{i=1}^{k} L\ldots\dot{L}\ldots L) = kTr(\dot{L}L^{k-1})\\ &= kTr([M,L]L^{k-1}) = kTr(LML^{k-1} - ML^{k-1}) = 0.\end{aligned}$$

  When the integrals of motion of the system are not known, computing the spectral invariants of L is an easy way to find them. The spectral parameter allows us to find more integrals of motion. But usually the procedure of finding a Lax-pair is much harder than finding the integrals of motion.
- A Lax-pair with spectral parameter generates a spectral curve $det(L(\lambda)-\mu I)$. The Jacobian of this curve is a torus on which, under some conditions, the flow can be linearized. These conditions were derived by P. Griffiths in [5]. The spectral curve is also helpful in finding concise explicit solutions for the system.

Next we describe a standard procedure for finding a Lax-pair for a system. The procedure, encapsulated in Proposition 3.1, amounts to recognizing that a given system is a Hamiltonian system on the dual of a Lie algebra.

The dual of any Lie algebra $\mathfrak{g}^*$ has a canonical Poisson structure. The bracket of any two smooth functions $f, g \in C^\infty(\mathfrak{g}^*)$ is

$$(3.3)\qquad \{f,g\}(p) = p([d_p f, d_p g]) \quad \forall p \in \mathfrak{g}^*.$$

Therefore any smooth function $H \in C^\infty(\mathfrak{g}^*)$ can be used to define a Hamiltonian vector field X_H on $\mathfrak{g}^*$:

$$(3.4)\qquad X_H(f) = \{f, H\}$$

or

$$(3.5)\qquad X_H(p) = -ad^*(dH_p)(p).$$

The Hamiltonian system associated to X_H, also known as the Euler-Poisson equations, is

$$(3.6)\qquad \dot{p}(t) = X_H(p(t)) = -ad^*(dH_p)(p).$$

A Casimir function of $\mathfrak{g}^*$ is a smooth function $f \in C^\infty(\mathfrak{g}^*)$ such that

$$(3.7)\qquad \{f,g\}(p) = 0 \quad \forall p \in \mathfrak{g}^* \quad \forall g \in C^\infty(\mathfrak{g}^*).$$

The Casimir functions have the property that they are integrals of motion for any Hamiltonian system on $\mathfrak{g}^*$.

When the Lie algebra $\mathfrak{g}$ admits a non-degenerate invariant bilinear form, $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to R$ (i.e. $\langle x, [y,z]\rangle = \langle [x,y], z\rangle$), we can construct a Lax-pair for the Hamiltonian system. This is always possible if $\mathfrak{g}$ is semi-simple. Thus the Killing form has the desired properties.

Proposition 3.1. *Assume the Lie algebra $\mathfrak{g}$ admits a non-degenerate invariant bilinear form: $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to R$. Let $p(t)$ be a curve in $\mathfrak{g}^*$ and $L(t)$ be the curve obtained from $p(t)$ by the identification $\langle L(t), L'\rangle = p(t)(L')\ \forall L' \in \mathfrak{g}$. If $p(t)$ is an integral curve of the Hamiltonian system, then $L(t)$ is an integral curve of the system written in the Lax-pair form*

$$\dot{L}(t) = [dH_p, L(t)]. \tag{3.8}$$

Proof.

$$\begin{aligned} \langle \dot{L}(t), L'\rangle &= \dot{p}(t)(L') = -ad^*(dH_p)(p(t))(L') = -p(t)([dH_p, L']) \\ &= -\langle L(t), [dH_p, L']\rangle = -\langle [L(t), dH_p], L'\rangle \\ &\Rightarrow \quad \dot{L}(t) = [dH_p, L(t)]. \end{aligned} \tag{3.9}$$

□

4. The structure of the Lie algebra $\mathfrak{sp}(4, \mathbb{R})$

We will be particularly interested in the Lie algebra $\mathfrak{sp}(4, \mathbb{R})$. The Lie algebra $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R})$ of the Lie group $Sp(4, \mathbb{R})$, the group of 4×4 real symplectic matrices, can be described as follows:

$$\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R}) = \{X \in \mathfrak{gl}(4, \mathbb{R}) | JXJ = X^T\} \tag{4.1}$$

where $J = 1 \otimes i\sigma_2$ has the property that $J^2 = -id$.

The Cartan decomposition of $\mathfrak{sp}(4, \mathbb{R})$ with respect to the involutive automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$, $\tau(X) = -X^T$ is

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where

$$\begin{aligned} \mathfrak{k} &= \{X | X = \tau(X)\} = \{X = A \otimes 1 + S_2 \otimes i\sigma_2 | A^T = -A, S_2^T = S_2\}, \\ \mathfrak{p} &= \{X | X = -\tau(X)\} = \{X = S_1 \otimes \sigma_1 + S_3 \otimes \sigma_3 | S_1^T = S_1, S_3^T = S_3\}. \end{aligned}$$

The matrices A, S_1, S_2 and S_3 involved in $\mathfrak{k}$ and $\mathfrak{p}$ belong to $\mathfrak{gl}(2, \mathbb{R})$. $\mathfrak{k}$ is a 4-dimensional Lie subalgebra of $\mathfrak{g}$ and is Lie isomorphic with $\mathfrak{u}(2)$. The center of $\mathfrak{k}$ is a 1-dimensional subspace spanned by

$$A_4 = \frac{1}{2} 1 \otimes i\sigma_2.$$

$\mathfrak{p}$ is a 6-dimensional vector subspace of $\mathfrak{g}$.

An invariant symmetric nondegenerate pairing $\langle\ ,\ \rangle : \mathfrak{sp}(4, \mathbb{R}) \times \mathfrak{sp}(4, \mathbb{R}) \to \mathbb{R}$ is the Killing form

$$\langle X, Y\rangle = Tr(XY). \tag{4.2}$$

The next ten matrices form a basis of $\mathfrak{sp}(4, \mathbb{R})$. With respect to this basis the quadratic form $\langle X, X\rangle$ is completely diagonal.

$$\begin{aligned} &A_1 = \tfrac{1}{2}\sigma_1 \otimes i\sigma_2, \quad A_2 = \tfrac{1}{2} i\sigma_2 \otimes 1, \quad A_3 = -\tfrac{1}{2}\sigma_3 \otimes i\sigma_2, \quad A_4 = \tfrac{1}{2} 1 \otimes i\sigma_2, \\ &B_1 = \tfrac{1}{2}\sigma_3 \otimes \sigma_3, \quad B_2 = \tfrac{1}{2} 1 \otimes \sigma_1, \quad B_3 = \tfrac{1}{2}\sigma_1 \otimes \sigma_3, \\ &C_1 = \tfrac{1}{2}\sigma_3 \otimes \sigma_1, \quad C_2 = -\tfrac{1}{2} 1 \otimes \sigma_3, \quad C_3 = \tfrac{1}{2}\sigma_1 \otimes \sigma_1. \end{aligned} \tag{4.3}$$

Table 4.1 gives the brackets between different elements of the above basis. The bracket of two matrices is taken to be $[A, B] = BA - AB$.

TABLE 4.1. $[A,B] = BA - AB$

$[\cdot,\cdot]$	A_1	A_2	A_3	A_4	B_1	B_2	B_3	C_1	C_2	C_3
A_1	0	$-A_3$	A_2	0	0	$-B_3$	B_2	0	$-C_3$	C_2
A_2	A_3	0	$-A_1$	0	B_3	0	$-B_1$	C_3	0	$-C_1$
A_3	$-A_2$	A_1	0	0	$-B_2$	B_1	0	$-C_2$	C_1	0
A_4	0	0	0	0	C_1	C_2	C_3	$-B_1$	$-B_2$	$-B_3$
B_1	0	$-B_3$	B_2	$-C_1$	0	A_3	$-A_2$	$-A_4$	0	0
B_2	B_3	0	$-B_1$	$-C_2$	$-A_3$	0	$-A_1$	0	$-A_4$	0
B_3	$-B_2$	B_1	0	$-C_3$	A_2	A_1	0	0	0	$-A_4$
C_1	0	$-C_3$	C_2	B_1	A_4	0	0	0	A_3	$-A_2$
C_2	C_3	0	$-C_1$	B_2	0	A_4	0	$-A_3$	0	A_1
C_3	$-C_2$	C_1	0	B_3	0	0	A_4	A_2	$-A_1$	0

On the dual $\mathfrak{g}^*$ we have the dual coordinates defined by the basis elements

$$H_j(p) = p(A_j), \quad P_j(p) = p(B_j), \quad Q_j(p) = p(C_j),$$

for all $p \in \mathfrak{g}^*$ and all $j = 1, 2, 3$ or 4.

The dual coordinates are functions on $\mathfrak{g}^*$ and can be viewed as left-invariant functions on $T^*Sp(4,\mathbb{R})$.

The identification of $\mathfrak{sp}(4,\mathbb{R})^*$ with $\mathfrak{sp}(4,\mathbb{R})$ via the Killing form is

$$p \in \mathfrak{sp}(4,\mathbb{R})^* \to L \in \mathfrak{sp}(4,\mathbb{R})$$

where

$$L = \sum_{j=1}^{3}(-H_j(p)A_j + P_j(p)B_j + Q_j(p)C_j) - H_4(p)A_4. \tag{4.4}$$

The above follows in a straightforward way from the formula of the Killing form

(4.5)
$$\begin{aligned}\langle X,Y\rangle &= \langle\sum_{j=1}^{3}(S_jA_j+P_jB_j+Q_jC_j)+S_4A_4, \sum_{j=1}^{3}(S'_jA_j+P'_jB_j+Q'_jC_j)+S'_4A_4\rangle \\ &= \sum_{j=1}^{3}(-S_jS'_j + P_jP'_j + Q_jQ'_j) - S_4S'_4.\end{aligned}$$

There are two other Lie algebras isomorphic to the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$, namely $\mathfrak{so}(2,3)$ and $(\mathbb{R}^{10}, [\ ,\ ])$. These Lie algebras are defined below.

$$\mathfrak{so}(2,3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{array}{l} a = -a^T\ ,\ b = c^T\ ,\ d = -d^T, \\ a \in \mathfrak{gl}(2,\mathbb{R}),\ b \text{ is a real } 3\times 2 \text{ matrix},\ d \in \mathfrak{gl}(3,\mathbb{R}) \end{array} \right\}.$$

The vector space $\mathbb{R}^{10} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ with coordinates $(\hat{S}, \hat{P}, \hat{Q}, R)$ can be made into a Lie algebra by defining a Lie bracket $[\ ,\ ]$ by

$$\begin{aligned}&(\hat{S}, \hat{P}, \hat{Q}, R)\ ,\ (\hat{S}', \hat{P}', \hat{Q}', R')] \\ &= (-\hat{S}\times\hat{S}' + \hat{P}\times\hat{P}' + \hat{Q}\times\hat{Q}', \quad -\hat{P}\times\hat{S}' - \hat{S}\times\hat{P}' - R\hat{Q}' + R'\hat{Q}, \\ &\quad -\hat{Q}\times\hat{S}' - \hat{S}\times\hat{Q}' - R\hat{P}' + R'\hat{P}, \quad \hat{Q}\cdot\hat{P}' - \hat{P}\cdot\hat{Q}').\end{aligned} \tag{4.6}$$

The Lie algebra isomorphism between $(\mathbb{R}^{10}, [\ ,\ ])$ and $\mathfrak{sp}(4,\mathbb{R})$ is the map

$$(\hat{S}, \hat{P}, \hat{Q}, R) \to \sum_{j=1}^{3}(S_jA_j + P_jB_j + Q_jC_j) + RA_4. \tag{4.7}$$

The isomorphisms between the three Lie algebra mentioned above are just real forms of the low dimensional isomorphism $C_2 = B_2$, [3]. Because of the existence of these isomorphisms, the discussion in the next sections can be done using $\mathfrak{sp}(4,\mathbb{R})$, $\mathfrak{so}(2,3)$ or $(\mathbb{R}^{10}, [\ ,\ ])$. We prefer $\mathfrak{sp}(4,\mathbb{R})$ because we can then use the tensor product of matrices and hence the computations and the form of the results are greatly simplified.

5. A class of Hamiltonian systems on $\mathfrak{sp}(4,\mathbb{R})^*$

Using the dual coordinates, we can construct a family of Hamiltonian functions defined on $\mathfrak{sp}(4,\mathbb{R})^*$. This family depends on 7 parameters, $\gamma,\ b_j,\ c_j,\ j = 1,2,3.$

$$H_\gamma = \frac{1}{2}(H_1^2 + H_2^2 + 2H_3^2) + \gamma H_3 + b_1P_1 + b_2P_2 + b_3P_3 + c_1Q_1 + c_2Q_2 + c_3Q_3.$$

Theorem 5.1. *The Hamiltonian system generated by the Hamiltonian function* H_γ *is the system of differential equation* (5.1):

$$\begin{aligned}
\frac{d\hat{H}}{dt} &= \hat{H} \times (\hat{\Omega} + \gamma e_3) + \hat{P} \times \hat{b} + \hat{Q} \times \hat{c},\\
\frac{d\hat{P}}{dt} &= \hat{P} \times (\hat{\Omega} + \gamma e_3) - \hat{H} \times \hat{b} - H_4\hat{c},\\
\frac{d\hat{Q}}{dt} &= \hat{Q} \times (\hat{\Omega} + \gamma e_3) - \hat{H} \times \hat{c} + H_4\hat{b},\\
\frac{dH_4}{dt} &= \hat{b} \cdot \hat{Q} - \hat{c} \cdot \hat{P}.
\end{aligned} \tag{5.1}$$

$\hat{\Omega}$ *is the column vector* $\hat{\Omega} = (H_1, H_2, 2H_3)^T$ *and* $e_3 = (0,0,1)^T$.

Proof. The integral curve $p(t)$ of the Hamiltonian system $\dot{p}(t) = X_{H_\gamma}(p(t))$ can be identified via the Killing form with a curve $L(t)$ living inside of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. Moreover $L(t)$ is an integral curve of $\dot{L}(t) = [dH_p, L(t)]$.

We know that

$$L(t) = \sum_{i=1}^{3}(-H_i(p(t))A_i + P_i(p(t))B_i + Q_i(p(t))C_i) - H_4(p(t))A_4$$

and

$$dH_p = \sum_{i=1}^{3}((\hat{\Omega} + \gamma e_3)_iA_i + b_iB_i + c_iC_i),$$

or, viewed as elements of the Lie algebra $(\mathbb{R}^{10}, [\ ,\ ])$ mentioned in section 4, $L(t) = (-\hat{H}, \hat{P}, \hat{Q}, -H_4)$ and $dH_p = (\hat{\Omega} + \gamma e_3, \hat{b}, \hat{c}, 0)$.

Now is easy to expand the system $\dot{L}(t) = [dH_p, L(t)]$ and to get the equations (5.1):

$$\left(-\frac{d\hat{H}}{dt}, \frac{d\hat{P}}{dt}, \frac{d\hat{Q}}{dt}, -\frac{d\hat{H}_4}{dt}\right) = [(\hat{\Omega} + \gamma e_3, \hat{b}, \hat{c}, 0)\ ,\ (-\hat{H}, \hat{P}, \hat{Q}, -H_4)]$$

$$= (-(\hat{\Omega} + \gamma e_3) \times (-\hat{H}) + \hat{b} \times \hat{P} + \hat{c} \times \hat{Q}, -(\hat{\Omega} + \gamma e_3) \times \hat{P} - \hat{b} \times (-\hat{H}) - H_4\hat{c},$$

$$-(\hat{\Omega} + \gamma e_3) \times \hat{Q} - \hat{c} \times (-\hat{H}) + H_4\hat{b}, \hat{c} \cdot \hat{P} - \hat{b} \cdot \hat{Q})$$

$$= (-\hat{H} \times (\hat{\Omega} + \gamma e_3) - \hat{P} \times \hat{b} - \hat{Q} \times \hat{c}, \hat{P} \times (\hat{\Omega} + \gamma e_3) - \hat{H} \times \hat{b} - H_4\hat{c},$$

$$\hat{Q} \times (\hat{\Omega} + \gamma e_3) - \hat{H} \times \hat{c} + H_4\hat{b}, -(\hat{b} \cdot \hat{Q} - \hat{c} \cdot \hat{P})).$$

□

Proposition 5.1. *The Hamiltonian system described in Theorem* 5.1 *has a Lax-pair representation without spectral parameter. In other words the system is equivalent to the Lax equation* $\dot{L} = [M, L]$ *where*

$$L = \frac{1}{2}\begin{pmatrix} (H_3 - H_4)i\sigma_2 & -H_2\mathbf{1} - H_1 i\sigma_2 \\ H_2\mathbf{1} - H_1 i\sigma_2 & -(H_3 + H_4)i\sigma_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (P_1 - Q_2)\sigma_3 + (P_2 + Q_1)\sigma_1 & P_3\sigma_3 + Q_3\sigma_1 \\ P_3\sigma_3 + Q_3\sigma_1 & -(P_1 + Q_2)\sigma_3 + (P_2 - Q_1)\sigma_1 \end{pmatrix} \tag{5.2}$$

and

$$M = \frac{1}{2}\begin{pmatrix} -(2H_3 + \gamma)i\sigma_2 & H_2\mathbf{1} + H_1 i\sigma_2 \\ -H_2\mathbf{1} + H_1 i\sigma_2 & (2H_3 + \gamma)i\sigma_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (b_1 - c_2)\sigma_3 + (b_2 + c_1)\sigma_1 & b_3\sigma_3 + c_3\sigma_1 \\ b_3\sigma_3 + c_3\sigma_1 & -(b_1 + c_2)\sigma_3 + (b_2 - c_1)\sigma_1 \end{pmatrix}. \tag{5.3}$$

Proof. The proof follows in a straightforward way from Proposition 3.1. L is exactly the identification of a dual element p with an element of the Lie algebra, using the trace form (4.2). $M = d_pH_\gamma$ and

$$d_pH_\gamma = H_1A_1 + H_2A_2 + (2H_3 + \gamma)A_3 + b_1B_1 + b_2B_2 + b_3B_3 + c_1C_1 + c_2C_2 + c_3C_3.$$

□

Theorem 5.2. *The Hamiltonian system* (5.1) *has* $H_3 - H_4$ *as integral of motion iff between the b's and c's there is the relation*

$$b_3 = c_3 = 0, \quad b_1 = c_2 \quad \text{and} \quad b_2 = -c_1. \tag{5.4}$$

Proof.

$$\begin{aligned} \frac{d}{dt}(H_3 - H_4) &= b_2P_1 - b_1P_2 + c_2Q_1 - c_1Q_2 - b_1Q_1 - b_2Q_2 - b_3Q_3 \\ &\quad + c_1P_1 + c_2P_2 + c_3P_3 \\ &= (b_2 + c_1)P_1 - (b_1 - c_2)P_2 + c_3P_3 + (c_2 - b_1)Q_1 \\ &\quad - (c_1 + b_2)Q_2 + b_3Q_3, \\ \frac{d}{dt}(H_3 - H_4) &= 0 \quad \text{iff} \quad b_3 = c_3 = 0, \quad b_1 = c_2 \quad \text{and} \quad b_2 = -c_1. \end{aligned}$$

□

We should note that the condition $b_3 = c_3 = 0, b_1 = c_2$ and $b_2 = -c_1$ implies that the vectors $\hat{b}$ and $\hat{c}$ are perpendicular and the Hamiltonian H_γ has the particular form

$$H_\gamma = \frac{1}{2}(H_1^2 + H_2^2 + 2H_3^2) + \gamma H_3 + b_1(P_1 + Q_2) + b_2(P_2 - Q_1). \tag{5.5}$$

6. Reduction of the Hamiltonian system on $\mathfrak{sp}(4)^*$ to the Kowalevski top

In this section we show how the class of Hamiltonian systems defined in section 5 can be reduced to the Kowalevski top and gyrostat. This reduction completely explains the Lax-pair representation of the top found by A. Reyman and M.A. Semenov-Tian-Shansky. Recall that our assumption (5.4) holds.

Theorem 6.1. *Assume that the value of the integral of motion $H_3 - H_4$ is $-\gamma$. Then the Hamiltonian system* (5.1) *can be transformed into the Kowalevski gyrostat by the affine transformation* (6.1)*:*

$$\hat{H} \to \hat{H} \quad \hat{P} + \hat{b} \to \hat{h} \quad \hat{Q} + \hat{c} \to \hat{g}. \tag{6.1}$$

Proof. To shorten notation, let $\hat{\Omega}_\gamma = \hat{\Omega} + \gamma e_3$. Then

$$\begin{aligned} \frac{d\hat{H}}{dt} &= \hat{H} \times \hat{\Omega}_\gamma + \hat{P} \times \hat{b} + \hat{Q} \times \hat{c} = \hat{H} \times \hat{\Omega}_\gamma + (\hat{h} - \hat{b}) \times \hat{b} + (\hat{g} - \hat{c}) \times \hat{c} \\ &= \hat{H} \times \hat{\Omega}_\gamma + \hat{h} \times \hat{b} + \hat{g} \times \hat{c}, \end{aligned}$$

$$\begin{aligned} \frac{d\hat{h}}{dt} &= \frac{d\hat{P}}{dt} = \hat{P} \times \hat{\Omega}_\gamma - \hat{H} \times \hat{b} - H_4\hat{c} = (\hat{h} - \hat{b}) \times \hat{\Omega}_\gamma - \hat{H} \times \hat{b} - (H_3 + \gamma)\hat{c} \\ &= \hat{h} \times \hat{\Omega}_\gamma + (\hat{\Omega}_\gamma - \hat{H}) \times \hat{b} - (H_3 + \gamma)\hat{c} \\ &= \hat{h} \times \hat{\Omega}_\gamma + (H_3 + \gamma)e_3 \times \hat{b} - (H_3 + \gamma)\hat{c} = \hat{h} \times \hat{\Omega}_\gamma, \end{aligned}$$

$$\begin{aligned} \frac{d\hat{g}}{dt} &= \frac{d\hat{Q}}{dt} = \hat{Q} \times \hat{\Omega}_\gamma - \hat{H} \times \hat{c} + H_4\hat{b} = (\hat{g} - \hat{c}) \times \hat{\Omega}_\gamma - \hat{H} \times \hat{c} + (H_3 + \gamma)\hat{b} \\ &= \hat{g} \times \hat{\Omega}_\gamma + (\hat{\Omega}_\gamma - \hat{H}) \times \hat{c} + (H_3 + \gamma)\hat{b} \\ &= \hat{g} \times \hat{\Omega}_\gamma + (H_3 + \gamma)e_3 \times \hat{c} + (H_3 + \gamma)\hat{b} = \hat{g} \times \hat{\Omega}_\gamma. \end{aligned}$$

We used that $e_3 \times \hat{b} = \hat{c}$ and $e_3 \times \hat{c} = -\hat{b}$. □

Theorem 6.2. *Assume that the value of the integral of motion $H_3 - H_4$ is $-\gamma$. Then the Hamiltonian system* (5.1) *reduces to the Kowalevski top using the affine transformation* (6.2)*:*

$$\hat{H} \to \hat{H} \quad \hat{P} + \hat{b} \to \hat{h} \quad \hat{Q} \to -\hat{c}. \tag{6.2}$$

Proof. The proof is based on the fact that the top is obtained from the gyrostat by imposing the invariant condition $\hat{g} = \hat{0}$ together with $\gamma = 0$. □

Proposition 6.1. *The Lax-pair representation of the system* (5.1) *exhibited in Proposition* 5.1 *can be modified into a Lax-pair representation without spectral parameter for both the Kowalevski gyrostat and the Kowalevski top.*

The transformation which produces the Lax-pair for the gyrostat is that of Theorem 6.1. *The transformation which produces the Lax-pair for the top is that of Theorem* 6.2.

The matrices L, M of the Lax-pair for the gyrostat are

$$L = \begin{pmatrix} 0 & 0 \\ 0 & b_1\sigma_3 - b_2\sigma_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\gamma i\sigma_2 & -H_2\mathbf{1} - H_1 i\sigma_2 \\ H_2\mathbf{1} - H_1 i\sigma_2 & -(2H_3 + \gamma i\sigma_2)i\sigma_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (h_1 - g_2)\sigma_3 + (h_2 + g_1)\sigma_1 & h_3\sigma_3 + g_3\sigma_1 \\ h_3\sigma_3 + g_3\sigma_1 & -(h_1 + g_2)\sigma_3 + (h_2 - g_1)\sigma_1 \end{pmatrix} \tag{6.3}$$

and

$$M = \begin{pmatrix} 0 & 0 \\ 0 & b_2\sigma_1 - b_1\sigma_3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -(2H_3 + \gamma)i\sigma_2 & H_2\mathbf{1} + H_1 i\sigma_2 \\ -H_2\mathbf{1} + H_1 i\sigma_2 & (2H_3 + \gamma)i\sigma_2 \end{pmatrix}. \tag{6.4}$$

The matrices L, M of the Lax-pair for the top are the matrices L, M of the Lax-pair for the gyrostat on which we impose the condition $\hat{g} = \hat{0}$, $\gamma = 0$.

Next we show how we can modify the Lax-pairs from Proposition 6.1 in such a way that they contain a spectral parameter. We will only show the work for the gyrostat. The same reasoning can be carried out for the top. The explanation for the appearance of the spectral parameter is based on a symmetry of the gyrostat.

Theorem 6.3. *If $(\hat{H},\ \hat{h},\ \hat{g})$ is a solution of the gyrostat fixed by the parameters $\hat{b}$ and $\hat{c}$, then $(\hat{H},\ \frac{1}{\lambda}\hat{h},\ \frac{1}{\lambda}\hat{g})$ is a solution of the gyrostat fixed by the parameters $\lambda\hat{b}$ and $\lambda\hat{c}$.*

Proof. The proof follows from the form of the equations (2.4) of the gyrostat. The decisive property of these equations is the pair of relations

$$\hat{h} \times \hat{b} = \frac{1}{\lambda}\hat{h} \times \lambda\hat{b} \quad \text{and} \quad \hat{g} \times \hat{c} = \frac{1}{\lambda}\hat{g} \times \lambda\hat{c}. \tag{6.5}$$ □

Observe that the matrices L and M in Proposition 6.1 of the Lax-pair representation for the gyrostat, without spectral parameter, are of the form

$$\begin{aligned} L &= c_l + f_1(\hat{H}) + f_2(\hat{h}, \hat{g}), \\ M &= c_m + f_3(\hat{H}), \end{aligned}$$

where c_l, c_m are constants involving b's and $f_1,\ f_2,\ f_3$ are linear functions of H's, h's, g's and γ. Their particular form allows us to prove the next theorem.

Theorem 6.4. *There is a Lax-pair representation with spectral parameter $\dot{L(\lambda)} = [M(\lambda), L(\lambda)]$ for the Kowalevski gyrostat. The matrices $L(\lambda)$, $M(\lambda)$ have the form*

$$L(\lambda) = \lambda c_l + f_1(\hat{H}) + \frac{1}{\lambda} f_2(\hat{h}, \hat{g}), \tag{6.6}$$

$$M(\lambda) = \lambda c_m + f_3(\hat{H}). \tag{6.7}$$

Proof. The proof is based on Theorem 6.3. Since $\dot{L} = [M, L]$ is equivalent with the gyrostat then $\dot{L(\lambda)} = [M(\lambda), L(\lambda)]$ will still be equivalent with the same gyrostat. Hence $\dot{L(\lambda)} = [M(\lambda), L(\lambda)]$ is a Lax-pair with spectral parameter for the gyrostat and coincides with the Lax-pair found by A. Reyman and M.A. Semenov-Tian-Shansky [11]. □

As a byproduct of the reduction of the Hamiltonian system on $\mathfrak{sp}(4)^*$ to the Kowalevski top we can relate the Casimir functions of $\mathfrak{sp}(4)^*$ to the Hamiltonian of the top and the nontrivial integral of motion (2.3) of the top.

The Poisson manifold $\mathfrak{sp}(4)^*$ has two independent Casimir functions,

$$\mathrm{Tr}(L^2) \quad \text{and} \quad \mathrm{Tr}(L^4), \tag{6.8}$$

where L is the matrix given by equation (5.2). They are integrals of motion for the Hamiltonian system (5.1). Therefore if we impose the conditions

$$H_3 = H_4, \quad \hat{P} = \hat{h} - \hat{b}, \quad \hat{Q} = -\hat{c}, \quad \gamma = 0 \tag{6.9}$$

on $\mathrm{Tr}(L^2)$ and $\mathrm{Tr}(L^4)$ we obtain the following integrals of motion for the Kowalevski top. Indeed under these conditions

$$\mathrm{Tr}(L^2) = -2\mathrm{Ham} + ||\hat{h}||^2 + 2(b_1^2 + b_2^2), \tag{6.10}$$

(6.11)

$$\mathrm{Tr}(L^4) = I_3 + ||\hat{h}||^2(2b_1^2 + 2b_2^2 - \mathrm{Ham}) + 2(\mathrm{Ham} - b_1^2 - b_2^2)^2 - (\hat{h}\hat{H})^2 + \frac{1}{4}||\hat{h}||^4,$$

where by Ham we mean the Hamiltonian of the top (2.2) and I_3 is the integral of motion (2.3) of the top with the complicated form described in Section 2.

Hence we have shown that the nontrivial integral of motion of the Kowalevski's top comes from one of the Casimir functions of the Poisson-Lie algebra $\mathfrak{sp}(4, \mathbb{R})^*$.

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