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EXPOSED 2-HOMOGENEOUS POLYNOMIALS ON HILBERT SPACES

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ABSTRACT. We show that every extreme point of the unit ball of 2-homogeneous polynomials on a separable real Hilbert space is its exposed point and that the unit ball of 2-homogeneous polynomials on a non-separable real Hilbert space contains no exposed points. We also show that the unit ball of 2-homogeneous polynomials on any infinite dimensional real Hilbert space contains no strongly exposed points.

We recall that a unit vector x in a real Banach space E is *exposed* if there is a unit vector $f \in E^*$ so that $f(x) = 1$ and $f(y) < 1$ for each $y \in B_E$ with $y \neq x$, where B_E is the closed unit ball of E. It is easy to see that every exposed point of B_E is an extreme point. A unit vector x in a real Banach space E is *strongly exposed* if there is a unit vector $f \in E^*$ so that $f(x) = 1$ and given any sequence (x_k) in B_E with $f(x_k) \to 1$ we can conclude that $x_k \to x$ in norm. It is easy to see that every strongly exposed point of B_E is its exposed point. We denote by $extB_E, expB_E$ and $sexpB_E$ the sets of extreme points, exposed points and strongly exposed points of B_E, respectively.

Let $\mathcal{P}(^2H)$ be the Banach space of continuous 2-homogeneous polynomials on a real Hilbert space H. Recently many authors (see [1]-[7]) studied extremal problems for polynomials on a Banach space. The extreme points of the unit ball of this space has been characterized by Grecu [6]. The object of this note is to determine the exposed and strongly exposed points of the unit ball of $\mathcal{P}(^2H)$. Grecu showed that for a real Hilbert space H, $P \in extB_{\mathcal{P}(^2H)}$ if and only if there exists an orthogonal decomposition of $H = H_1 \oplus H_2$ such that $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$, where $\pi_j : H \to H_j$ are the orthogonal projections of H onto H_j $(j = 1, 2)$. Using this result we show the following results:

(1) If H is a separable real Hilbert space, then every extreme point of the unit ball of $\mathcal{P}(^2H)$ is exposed.

(2) If H is a non-separable real Hilbert space, then the unit ball of $\mathcal{P}(^2H)$ contains no exposed points.

(3) If H is an infinite dimensional real Hilbert space, then the unit ball of $\mathcal{P}(^2H)$ contains no strongly exposed points.

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Theorem 1. *Let H be a separable real Hilbert space. Then every extreme point of the unit ball of $\mathcal{P}(^2H)$ is exposed.*

Proof. Since $expB_{\mathcal{P}(^2H)} \subset extB_{\mathcal{P}(^2H)}$, it suffices to show that if $P \in extB_{\mathcal{P}(^2H)}$, then $P \in expB_{\mathcal{P}(^2H)}$.

Let $P \in extB_{\mathcal{P}(^2H)}$. By Theorem 1.6 of [6], $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ where $H = H_1 \oplus H_2$ and $\pi_j : H \to H_j$ are the orthogonal projections of H onto H_j $(j = 1, 2)$. Clearly $\|\pi_j\| = 1$. Let $\{e_\alpha\}_{\alpha \in A}$ and $\{t_\beta\}_{\beta \in B}$ be orthonormal bases of H_1 and H_2, respectively. It is clear that $\{e_\alpha, t_\beta\}_{\alpha \in A, \beta \in B}$ is an orthonormal basis of H. Then for each $x \in H$, we have

$$x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta$$

and

$$P(x) = \sum_{\alpha \in A} \langle x, e_\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t_\beta \rangle^2.$$

Note that $P(e_\alpha) = 1$ for all $\alpha \in A$ and $P(t_\beta) = -1$ for all $\beta \in B$. Let $\{a_\alpha\}_{\alpha \in A}$ and $\{b_\beta\}_{\beta \in B}$ be collections of reals such that $a_\alpha > 0, b_\beta < 0$ and

$$\sum_{\alpha \in A} a_\alpha - \sum_{\beta \in B} b_\beta = 1.$$

Define $f \in \mathcal{P}(^2H)^*$ such that for each $Q \in \mathcal{P}(^2H)$,

$$f(Q) = \sum_{\alpha \in A} Q(e_\alpha) a_\alpha + \sum_{\beta \in B} Q(t_\beta) b_\beta.$$

Then $\|f\| = 1$. Indeed, for each $Q \in \mathcal{P}(^2H)$ with $\|Q\| = 1$, we have $|Q(e_\alpha)| \leq 1$ for all $\alpha \in A, |Q(t_\beta)| \leq 1$ for all $\beta \in B$ and

$$|f(Q)| \leq \sum_{\alpha \in A} |Q(e_\alpha)| a_\alpha + \sum_{\beta \in B} |Q(t_\beta)|(-b_\beta) \leq \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{1 \cdot \beta \in B} b_\beta = 1$$

and

$$f(P) = \sum_{\alpha \in A} P(e_\alpha) a_\alpha + \sum_{\beta \in B} P(t_\beta) b_\beta = \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{\beta \in B} 1 \cdot b_\beta = 1.$$

We will show that this functional f exposes the polynomial P.

Let $Q \in \mathcal{P}(^2H)$ be such that $f(Q) = 1 = \|Q\|$. We claim that $Q(e_\alpha) = 1$ for all $\alpha \in A$ and $Q(t_\beta) = -1$ for all $\beta \in B$. Indeed,

$$1 = f(Q) = \sum_{\alpha \in A} Q(e_\alpha) a_\alpha + \sum_{\beta \in B} Q(t_\beta) b_\beta = \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{\beta \in B} 1 \cdot b_\beta = 1,$$

so $Q(e_\alpha) = 1$ for all $\alpha \in A$ and $Q(t_\beta) = -1$ for all $\beta \in B$ because of $a_\alpha > 0, b_\beta < 0$.

Let $\hat{Q}$ be the corresponding continuous symmetric bilinear form to the polynomial Q.

We claim:

(1) $\hat{Q}(e_\alpha, e_{\alpha'}) = 0$ $(\alpha \neq \alpha' \in A)$;
(2) $\hat{Q}(t_\beta, t_{\beta'}) = 0$ $(\beta \neq \beta' \in B)$;
(3) $\hat{Q}(e_\alpha, t_\beta) = 0$ $(\alpha \in A, \beta \in B)$.

Proof of (1). It follows that

$$\begin{aligned}1 = \|Q\| &\geq \sup_{x_\alpha^2 + x_{\alpha'}^2 = 1} |Q(x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'})| \\ &= \sup_{x_\alpha^2 + x_{\alpha'}^2 = 1} |\hat{Q}(x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'}, x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'})| \\ &= \sup_{x_\alpha^2 + x_{\alpha'}^2 = 1} |Q(e_\alpha) x_\alpha^2 + Q(e_{\alpha'}) x_{\alpha'}^2 + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}| \\ &= \sup_{x_\alpha^2 + x_{\alpha'}^2 = 1} |x_\alpha^2 + x_{\alpha'}^2 + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}| \\ &= \sup_{x_\alpha^2 + x_{\alpha'}^2 = 1} |1 + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}|\end{aligned}$$

which implies $\hat{Q}(e_\alpha, e_{\alpha'}) = 0$.

Proof of (2). By the similar proof of (1), we have

$$1 = \|Q\| \geq \sup_{x_\beta^2 + x_{\beta'}^2 = 1} |Q(x_\beta t_\beta + x_{\beta'} t_{\beta'})| = \sup_{x_\beta^2 + x_{\beta'}^2 = 1} |1 + 2\hat{Q}(t_\beta, t_{\beta'}) x_\beta x_{\beta'}|$$

which implies $\hat{Q}(t_\beta, t_{\beta'}) = 0$.

Proof of (3). By the similar proof of (1), we have

$$\begin{aligned}1 = \|Q\| &\geq \sup_{x_\alpha^2 + x_\beta^2 = 1} |Q(x_\alpha e_\alpha + x_\beta t_\beta)| \\ &= \sup_{x_\alpha^2 + x_\beta^2 = 1} |Q(e_\alpha) x_\alpha^2 + Q(t_\beta) x_\beta^2 + 2\hat{Q}(e_\alpha, t_\beta) x_\alpha x_\beta| \\ &= \sup_{x_\alpha^2 + x_\beta^2 = 1} |x_\alpha^2 - x_\beta^2 + 2\hat{Q}(e_\alpha, t_\beta) x_\alpha x_\beta|.\end{aligned}$$

By Lemma 2.1 of [3], we have $\hat{Q}(e_\alpha, t_\beta) = 0$. For $x \in H$, it follows that

$$\begin{aligned}Q(x) &= \hat{Q}(x, x) \\ &= \hat{Q}\Big(\sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta, \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta\Big) \\ &= \sum_{\alpha, \alpha' \in A} \langle x, e_\alpha \rangle \langle x, e_{\alpha'} \rangle \hat{Q}(e_\alpha, e_{\alpha'}) + \sum_{\beta, \beta' \in B} \langle x, t_\beta \rangle \langle x, t_{\beta'} \rangle \hat{Q}(t_\beta, t_{\beta'}) \\ &\quad + 2 \sum_{\alpha \in A, \beta \in B} \langle x, e_\alpha \rangle \langle x, t_\beta \rangle \hat{Q}(e_\alpha, t_\beta) \\ &= \sum_{\alpha \in A} Q(e_\alpha) \langle x, e_\alpha \rangle^2 + \sum_{\beta \in B} Q(t_\beta) \langle x, t_\beta \rangle^2 \qquad \text{(by claims (1)-(3))} \\ &= \sum_{\alpha \in A} \langle x, e_\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t_\beta \rangle^2 = P(x),\end{aligned}$$

showing that f exposes P. Therefore $P \in expB_{\mathcal{P}(^2H)}$. ☐

Theorem 2. *Let H be an infinite dimensional real Hilbert space. Then the unit ball of $\mathcal{P}(^2H)$ contains no strongly exposed points.*

Proof. Since $sexpB_{\mathcal{P}(^2H)} \subset extB_{\mathcal{P}(^2H)}$, it suffices to show that

$$sexpB_{\mathcal{P}(^2H)} \cap extB_{\mathcal{P}(^2H)} = \emptyset.$$

Let $P \in extB_{\mathcal{P}(^2H)}$. By Theorem 1.6 of [6] $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ where $H = H_1 \oplus H_2$ and $\pi_j : H \to H_j$ are the orthogonal projections of H onto H_j $(j = 1, 2)$. Clearly $\|\pi_j\| = 1$. Without loss of generality, assume that $dim(H_1) = \infty$. Let $\{e_\alpha\}_{\alpha \in A}$ be an orthonormal basis of H_1. Let $\{\alpha_j\}_{j=1}^{\infty} \subset A$. It is clear that for each $x \in H$, we have $\pi_1(x) = \sum_{\alpha \in A} \langle \pi_1(x), e_\alpha \rangle e_\alpha$. Then

$$P(x) = \sum_{\alpha \in A} \langle \pi_1(x), e_\alpha \rangle^2 - \|\pi_2(x)\|^2.$$

Suppose that $P \in sexpB_{\mathcal{P}(^2H)}$. Then there is an $f \in \mathcal{P}(^2H)^*$ such that $\|f\| = 1 = f(P)$ and given any sequence $\{P_j\}$ in $B_{\mathcal{P}(^2H)}$ with $f(P_j) \to 1$, we have $\|P_j - P\| \to 0$. For each α_j, we have

$$\begin{aligned} & f(\sum_{\alpha \neq \alpha_j} \langle \pi_1(x), e_\alpha \rangle^2 - \langle \pi_1(x), e_{\alpha_j} \rangle^2 - \|\pi_2(x)\|^2) < 1 \\ = \quad & f(\sum_{\alpha \in A} \langle \pi_1(x), e_\alpha \rangle^2 - \|\pi_2(x)\|^2), \end{aligned}$$

so $f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) > 0$. We will show $f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) \to 0$ as $j \to \infty$. For each n,

$$\begin{aligned} & \sum_{1 \le j \le n} f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) = f(\sum_{1 \le j \le n} \langle \pi_1(x), e_{\alpha_j} \rangle^2) \\ \le \quad & \|f\| \| \sum_{1 \le j \le n} \langle \pi_1(x), e_{\alpha_j} \rangle^2 \| = \| \sum_{1 \le j \le n} \langle \pi_1(x), e_{\alpha_j} \rangle^2 \| \\ = \quad & \sup_{\|x\|=1} \sum_{1 \le j \le n} \langle \pi_1(x), e_{\alpha_j} \rangle^2 = \sup_{\|x\|=1} \|\pi_1(x)\|^2 = \|\pi_1\|^2 = 1. \end{aligned}$$

Thus $\sum_{1 \le j < \infty} f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) \le 1$, so we have $f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) \to 0$.

Define $P_j(x) = P(x) - \langle \pi_1(x), e_{\alpha_j} \rangle^2 \in \mathcal{P}(^2H)$. Then $\|P_j\| = 1$ and $|f(P_j) - 1| = |f(P_j - P)| = f(\langle \pi_1(x), e_{\alpha_j} \rangle^2) \to 0$. But

$$\|P_j - P\| = \|\langle \pi_1(x), e_{\alpha_j} \rangle^2\| = \sup_{\|x\|=1} \langle \pi_1(x), e_{\alpha_j} \rangle^2 = 1,$$

so we have a contradiction. Thus $sexpB_{\mathcal{P}(^2H)} = \emptyset$. □

Theorem 3. *Let H be non-separable real Hilbert space. Then the unit ball of $\mathcal{P}(^2H)$ contains no exposed points.*

Proof. Let P be an extreme point, so that

$$P(x) = \sum_{\alpha \in A} \langle x, t_\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t_\beta \rangle^2$$

relative to a suitably chosen orthonormal basis whose indexing set is the disjoint union $A \cup B$. Suppose that the functional f exposes P. As in the proof of Theorem 2 it follows that $f(\langle x, e_\alpha \rangle^2) > 0$ for every $\alpha \in A$ and similarly, $f(\langle x, e_\alpha \rangle^2) < 0$ for every $\alpha \in B$. But

$$\sum_{\alpha \in A} f(\langle x, t_\alpha \rangle^2) = f(\sum_{\alpha \in A} \langle x, t_\alpha \rangle^2) \le 1,$$

and hence A must be countable. Similarly, B is countable. Thus H is separable. □

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