# THE INFLUENCE OF MINIMAL SUBGROUPS
# ON THE STRUCTURE OF A FINITE GROUP

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ABSTRACT. We study the detailed structure of a finite group under the assumption that all minimal subgroups of the generalized Fitting subgroup of some normal subgroup of $G$ are well-suited in $G$.

## 1. INTRODUCTION

All groups considered in this paper will be finite. We use conventional notions and notations, as in Huppert [3].

Recall that a minimal subgroup of a finite group is a subgroup of prime order. For groups of even order, it is helpful to also consider the cyclic subgroups of order 4. Two subgroups $H$ and $K$ of a group $G$ are said to permute if $HK = KH$. It is easily seen that $H$ and $K$ permute iff the set of $HK$ is a subgroup of $G$. We say, following Kegel [6], that a subgroup of $G$ is $\pi$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$. A number of authors have considered how minimal subgroups can be embedded in a group. Buckley [2] proved that if $G$ is a group of odd order and all minimal subgroups of $G$ are normal in $G$, then $G$ is supersolvable. Later Shaalan [7] proved that if $G$ is a finite group and every cyclic subgroup of prime order or of 4 is $\pi$-quasinormal in $G$, then $G$ is supersolvable. Some authors extend these results using formation theory ([10], [11]). It is natural to limit the hypotheses of minimal subgroups to a smaller subgroup of $G$. Since every non-abelian simple group has a trivial Fitting subgroup, one cannot expect a detailed structure if one gives conditions on minimal subgroups of $F(G)$. So some authors have assumed the solvability of some normal subgroup of $G$ (i.e., the solvability of $G$) to get the structure of $G$, such as Asaad proved in [1]: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and suppose that $G$ is a group with a normal solvable subgroup $H$ such that $G/H \in \mathcal{F}$. If every subgroup of $F(H)$ of prime order or of 4 is $\pi$-quasinormal in $G$, then $G \in \mathcal{F}$. It is meaningful to remove the solvability of $H$ in the hypotheses of this result, but in this case, the Fitting subgroup $F(H)$ will sometimes be a trivial group in some instances as we mentioned above. However, as we show in this paper, we can obtain our results

about the structure of $G$ if we assume that the minimal subgroups of the generalized Fitting subgroup of some normal subgroups of $G$ are well suited in $G$.

**Definition.** Let $p$ be a prime and $G$ be a group. We define:

$\mathcal{P}_p(G) = \{x \mid x \in G, |x| = p\}$,
$\mathcal{P}_4(G) = \{x \mid x \in G, |x| = 4\}$,
$\mathcal{P}(G) = \bigcup_{p \in \pi(G)} \mathcal{P}_p(G)$,
$\mathcal{P}^*(G) = \mathcal{P}_4(G) \cup \mathcal{P}(G)$.

An element $x$ of a group $G$ is said to be $\pi$-quasinormal in $G$ if $\langle x \rangle$ is $\pi$-quasinormal in $G$.

## 2. Preliminaries

**Lemma 2.1** ([6])**.** (1) *A $\pi$-quasinormal subgroup of $G$ is subnormal in $G$.*

(2) *If $H \leq K \leq G$ and $H$ is $\pi$-quasinormal in $G$, then $H$ is $\pi$-quasinormal in $K$.*

(3) *If $H$ is a $\pi$-quasinormal Hall subgroup of $G$, then $H \lhd G$.*

(4) *Let $K \lhd G$ and $K \leq H$. Then $H$ is $\pi$-quasinormal in $G$ if and only if $H/K$ is $\pi$-quasinormal in $G/K$.*

**Lemma 2.2.** *If $P$ is a $\pi$-quasinormal $p$-subgroup of $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$.*

*Proof.* Now take $q$ any prime number distinct from $p$, and $Q$ any Sylow $q$-subgroup of $G$. Then $PQ$ is a subgroup of $G$. Now $P$ is a $\pi$-quasinormal, Sylow subgroup of $PQ$, so by Lemma 2.1(3), $P \lhd QP$ and thus, $Q \subseteq N_G(P)$. Since $O^p(G) = \langle Q \mid Q \in Syl_q(G), q \neq p \rangle$, thus $N_G(P) \geq O^p(G)$.

Let $G$ be a group. The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. Now $F^*(G)$ is an important subgroup of $G$ and is a natural generalization of $F(G)$. Its definition and important properties can be found in [5, X 13]. We would like to give the following basic facts which we will use in our proof.

**Lemma 2.3.** *Let $G$ be a group and $M$ a subgroup of $G$. Then we have:*

(1) *If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = soc(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *Suppose $K$ is a subgroup of $G$ contained in $Z(G)$. Then $F^*(G/K) = F^*(G)/K$.*

*Proof.* (1)–(3) can be found in [5, X 13].

(4) Denote $F^*(G/K) = L/K$. Consider a chief series of $G$ of the form

$$G = G_0 > \cdots > G_{m-1} > G_m = K > G_{m+1} > \cdots > G_n = 1.$$

By the definition of the generalized Fitting subgroup, $\forall x \in K$, $\overline{x} = xK$ induces an inner automorphism on the chief factor $\overline{G_{i-1}/G_i} = (G_{i-1}/K)/(G_i/K)$ of $G/K$, for $i = 1, 2, \cdots, m$, thus $x$ induces an inner automorphism on the chief factor $G_{i-1}/G_i$ of $G$, for $i = 1, 2, \cdots, m$. Since $K \leq Z(G)$, the automorphism induced by $x$ on the chief factor $G_{i-1}/G_i$ of $G$ is identity, for $i = m + 1, \cdots, n$, so $x$ induces an inner automorphism on the chief factor $G_{i-1}/G_i$ of $G$, for $i = 1, 2, \cdots, n$. Hence by [5, X, Lemma 13.1], $x$ induces an inner automorphism on any chief factor of $G$. Thus $x \in F^*(G)$, i.e., $L \leq F^*(G)$. Obviously $F^*(G) \leq L$, hence $F^*(G/K) = F^*(G)/K$.

**Lemma 2.4.** *Suppose $G$ is a group with a normal subgroup $H$ such that $G/H$ is supersolvable. If every element of $\mathcal{P}^*(H)$ is $\pi$-quasinormal in $G$, then $G$ is supersolvable.*

*Proof.* See [7, Theorem 3.1]. $\blacksquare$

**Lemma 2.5.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. Then $G \in \mathcal{F}$ if and only if $G$ has a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$ and every element of $\mathcal{P}^*(F(H))$ is $\pi$-quasinormal in $G$. In particular, if $G$ is a group with a solvable normal subgroup $H$ such that $G/H$ is supersolvable and every element of $\mathcal{P}^*(F(H))$ is $\pi$-quasinormal in $G$, then $G$ is supersolvable.*

*Proof.* See [1, Theorem and Corollary 1]. $\blacksquare$

## 3. Results

**Theorem 3.1.** *Suppose $G$ is a group with a normal subgroup $N$ such that $G/N$ is supersolvable. If every element of $\mathcal{P}^*(F^*(N))$ is $\pi$-quasinormal in $G$, then $G$ is supersolvable.*

*Proof.* Suppose that the theorem is false and let $G$ be a counter-example of smallest order. Then we have:

(1) Every proper normal subgroup of $G$ is supersolvable.

If $H$ is a maximal normal subgroup of $G$, we have that $H/H \cap N$ is supersolvable, $F^*(H \cap N)$ is contained in $F^*(N)$ by Lemma 2.3. So $H, H \cap N$ satisfy the hypotheses of the theorem. The minimal choice of $G$ implies that $H$ is supersolvable.

(2) $N = G$, and $F^*(G) = F(G) < G$.

If $N < G$, then $N$ is supersolvable by (1), in particular, $N$ is solvable, so $F^*(N) = F(N)$, then $G$ is supersolvable by applying Lemma 2.5, a contradiction.

If $F^*(G) = G$, then $G$ is supersolvable by Lemma 2.4, a contradiction. Thus $F^*(G) < G$ is supersolvable. Since $F^*(G)$ is solvable, $F^*(G) = F(G)$ by Lemma 2.3(3).

(3) Contradiction.

(3.1) Now $G$ has a unique maximal normal subgroup, say $M$.

Suppose not; then there are maximal normal subgroups $M, N$. It follows that $G = NM$ and $G/M \cong N/(M \cap N)$ is supersolvable, as $N$ is. Now since $M$ is supersolvable, $G$ must be solvable. Hence, $G$ is supersolvable by Lemma 2.5, contrary to our assumption.

(3.2) $G/M$ is non-abelian simple and $G' = G$.

By (3.1) $G/M$ is simple. If $G/M$ is cyclic group of prime order, then $G$ is solvable. This implies that $G$ is supersolvable by Lemma 2.5, which implies a contradiction. Thus $G/M$ is non-abelian simple; it follows that $G' \nleq M$, so $G' = G$.

(3.3) $\forall q \in \pi(G)$, $O^q(G) = G$.

If not, $O^q(G) < G$, then $O^q(G) \leq M$ is supersolvable. On the other hand, $G/O^q(G)$ is a $q$-group, so $G$ is solvable, a contradiction as in the proof of (3.2).

Since $F^*(G) = F(G)$ is not the identity group, we may choose a minimal prime divisor $q$ of $\mid F(G) \mid$, now that the Sylow $q$-subgroup $Q$ of $F(G)$ is a non-trivial normal subgroup of $G$. For every element $x$ of $\mathcal{P}^*(Q)$, $x$ is $\pi$-quasinormal in $G$, thus $N_G(\langle x \rangle) \geq O^q(G) = G$ by Lemma 2.2. Thus $\langle x \rangle$ is normal in $G$. Since $G/C_G(\langle x \rangle)$ is abelian, we have that $G = G' \leq C_G(\langle x \rangle)$, and therefore $G$ centralizes every element of $\mathcal{P}^*(Q)$. By Huppert's result ([4, Satz IV 5.12]), $G/C_G(Q)$ is a $q$-group. Since $G' = G$, we have that $C_G(Q) = G$, i.e., $Q \leq Z(G)$. Hence by Lemma 2.3(4)

$F^*(G/Q) = F^*(G)/Q$. Consider the group $\overline{G} = G/Q$. For any cyclic subgroup $\overline{L}$ of prime order $r$ of $F^*(\overline{G})$, since $Q$ is a Sylow $q$-subgroup of $F^*(G)$ with the minimal prime divisor, we have that $r > q$ (in particular, $r \neq 2$). Applying the Schur-Zassenhaus Theorem ([3]), we can denote $\overline{L} = LQ/Q$, where $L$ is a cyclic subgroup of order $r$ of $F^*(G)$. Since $L$ is $\pi$-quasinormal in $G$ by hypotheses, so $\overline{L}$ is $\pi$-quasinormal in $\overline{G}$. It follows that $\overline{G}$ satisfies the hypotheses of the theorem, and from the minimal choice of $G$ it follows that $G/Q$ is supersolvable and so $G$ is solvable, thus $G' < G$, contrary to (3.2). The final contradiction. Thus, $G$ must be supersolvable, as required.

Denote the supersolvable residual of $G$ by $G^{\mathcal{U}}$. Using terminology of formations, it is easy to see that the above theorem can be expressed as the following:

**Theorem 3.1′.** *Suppose $G$ is a group. Then $G$ is supersolvable if and only if every element of $\mathcal{P}^*(F^*(G^{\mathcal{U}}))$ is $\pi$-quasinormal in $G$.*

**Corollary 3.2.** *Let $G$ be a group. If every element of prime order or order $4$ of $F^*(G)$ is $\pi$-quasinormal in $G$, then $G$ is supersolvable.*

**Theorem 3.3.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. Then $G \in \mathcal{F}$ if and only if $G$ has a normal subgroup $H$ such that $G/H \in \mathcal{F}$ and every element of $\mathcal{P}^*(F^*(H))$ is $\pi$-quasinormal in $G$.*

*Proof.* We only need to prove the "if" part.

By hypotheses every element of $\mathcal{P}^*(F^*(H))$ is $\pi$-quasinormal in $G$, thus is $\pi$-quasinormal in $H$ by Lemma 2.1(2). Corollary 3.2 implies that $H$ is supersolvable, so $F^*(H) = F(H)$ and hence $G \in \mathcal{F}$ by applying Lemma 2.5.

Using a proof similar to [1, Corollary 4], we also have a generalization of it.

**Corollary 3.4.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ under either of the following assumptions:*

(a) *$G$ is $2$-nilpotent and every element of odd prime order of $F^*(H)$ is $\pi$-quasinormal in $G$.*

(b) *The $Sylow_2$-subgroups of $G$ are abelian and every element of $F^*(H)$ of prime order is $\pi$-quasinormal in $G$.*

*Remark.* Theorem 3.3 is not true for saturated formations which do not contain $\mathcal{U}$. For example, if $\mathcal{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

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## REFERENCES

1. M. Asaad and P. Csorgo. The influence of minimal subgroups on the structure of finite group. Arch. Math., 1999,72:401-404. MR **2000f:**20022
2. J. Buckley. Finite groups whose minimal subgroups are normal. Math.Z, 1970,116:15-17. MR **41:**6967
3. D. Gorenstein. Finite Groups. Chelsea (New York 1968). MR **38:**229; MR **81b:**20002
4. B. Huppert. Endliche Gruppen I. Berlin: Springer-Verlag, 1967. MR **37:**302

5. B. Huppert, N. Blackburn. Finite Groups III. New York, Berlin: Springer-Verlag, 1982. MR **84i:**20001b

6. O. Kegel. Sylow-Gruppen und aubnormalteiler endlicher Gruppen. Math. Z., 1962, 78:205-221. MR **26:**5042

7. A. Shaalan. The influence of $\pi$-quasinormality of some subgroups. Acta. Math. Hungar, 1990, 56:287-293. MR **92c:**20034

8. Y. Wang. The influence of minimal subgroups on the structure of finite groups. Acta Math. Sinica, English series, 2000, 16:63-70. MR **2001c:**20035

9. M. Weinstein (editor). Between nilpotent and solvable. Polygonal Publishing House, Passaic (1982). MR **84k:**20002

10. A. Yokoyama. Finite soluble groups whose $\mathcal{F}$-hypercenter contains all minimal subgroups. Arch. Math., 1975,26:123-130. MR **53:**3106

11. A. Yokoyama. Finite soluble groups whose $\mathcal{F}$-hypercenter contains all minimal subgroups II. Arch. Math., 1976,27:572-576. MR **56:**5712

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