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THE INFLUENCE OF MINIMAL SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

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Abstract. We study the detailed structure of a finite group under the assumption that all minimal subgroups of the generalized Fitting subgroup of some normal subgroup of G are well-suited in G.

1. Introduction

All groups considered in this paper will be finite. We use conventional notions and notations, as in Huppert [3].

Recall that a minimal subgroup of a finite group is a subgroup of prime order. For groups of even order, it is helpful to also consider the cyclic subgroups of order 4. Two subgroups H and K of a group G are said to permute if $HK = KH$. It is easily seen that H and K permute iff the set of HK is a subgroup of G. We say, following Kegel [6], that a subgroup of G is π-quasinormal in G if it permutes with every Sylow subgroup of G. A number of authors have considered how minimal subgroups can be embedded in a group. Buckley [2] proved that if G is a group of odd order and all minimal subgroups of G are normal in G, then G is supersolvable. Later Shaalan [7] proved that if G is a finite group and every cyclic subgroup of prime order or of 4 is π-quasinormal in G, then G is supersolvable. Some authors extend these results using formation theory ([10], [11]). It is natural to limit the hypotheses of minimal subgroups to a smaller subgroup of G. Since every non-abelian simple group has a trivial Fitting subgroup, one cannot expect a detailed structure if one gives conditions on minimal subgroups of $F(G)$. So some authors have assumed the solvability of some normal subgroup of G (i.e., the solvability of G) to get the structure of G, such as Asaad proved in [1]: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and suppose that G is a group with a normal solvable subgroup H such that $G/H \in \mathcal{F}$. If every subgroup of $F(H)$ of prime order or of 4 is π-quasinormal in G, then $G \in \mathcal{F}$. It is meaningful to remove the solvability of H in the hypotheses of this result, but in this case, the Fitting subgroup $F(H)$ will sometimes be a trivial group in some instances as we mentioned above. However, as we show in this paper, we can obtain our results

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about the structure of G if we assume that the minimal subgroups of the generalized Fitting subgroup of some normal subgroups of G are well suited in G.

Definition. Let p be a prime and G be a group. We define:
$\mathcal{P}_p(G) = \{x \mid x \in G, |x| = p\}$,
$\mathcal{P}_4(G) = \{x \mid x \in G, |x| = 4\}$,
$\mathcal{P}(G) = \bigcup_{p\in\pi(G)} \mathcal{P}_p(G)$,
$\mathcal{P}^*(G) = \mathcal{P}_4(G) \cup \mathcal{P}(G)$.

An element x of a group G is said to be π-quasinormal in G if $\langle x\rangle$ is π-quasinormal in G.

2. Preliminaries

Lemma 2.1 ([6])**.** (1) *A π-quasinormal subgroup of G is subnormal in G.*

(2) *If $H \leq K \leq G$ and H is π-quasinormal in G, then H is π-quasinormal in K.*

(3) *If H is a π-quasinormal Hall subgroup of G, then $H \lhd G$.*

(4) *Let $K \lhd G$ and $K \leq H$. Then H is π-quasinormal in G if and only if H/K is π-quasinormal in G/K.*

Lemma 2.2. *If P is a π-quasinormal p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Proof. Now take q any prime number distinct from p, and Q any Sylow q-subgroup of G. Then PQ is a subgroup of G. Now P is a π-quasinormal, Sylow subgroup of PQ, so by Lemma 2.1(3), $P \lhd QP$ and thus, $Q \subseteq N_G(P)$. Since $O^p(G) = \langle Q \mid Q \in Syl_q(G), q \neq p\rangle$, thus $N_G(P) \geq O^p(G)$.

Let G be a group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Now $F^*(G)$ is an important subgroup of G and is a natural generalization of $F(G)$. Its definition and important properties can be found in [5, X 13]. We would like to give the following basic facts which we will use in our proof.

Lemma 2.3. *Let G be a group and M a subgroup of G. Then we have:*

(1) *If M is normal in G, then $F^*(M) \leq F^*(G)$.*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = soc(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *Suppose K is a subgroup of G contained in $Z(G)$. Then $F^*(G/K) = F^*(G)/K$.*

Proof. (1)–(3) can be found in [5, X 13].

(4) Denote $F^*(G/K) = L/K$. Consider a chief series of G of the form

$$G = G_0 > \cdots > G_{m-1} > G_m = K > G_{m+1} > \cdots > G_n = 1.$$

By the definition of the generalized Fitting subgroup, $\forall x \in K$, $\overline{x} = xK$ induces an inner automorphism on the chief factor $\overline{G_{i-1}}/\overline{G_i} = (G_{i-1}/K)/(G_i/K)$ of G/K, for $i = 1, 2, \cdots, m$, thus x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \cdots, m$. Since $K \leq Z(G)$, the automorphism induced by x on the chief factor G_{i-1}/G_i of G is identity, for $i = m+1, \cdots, n$, so x induces an inner automorphism on the chief factor G_{i-1}/G_i of G, for $i = 1, 2, \cdots, n$. Hence by [5, X, Lemma 13.1], x induces an inner automorphism on any chief factor of G. Thus $x \in F^*(G)$, i.e., $L \leq F^*(G)$. Obviously $F^*(G) \leq L$, hence $F^*(G/K) = F^*(G)/K$.

Lemma 2.4. *Suppose G is a group with a normal subgroup H such that G/H is supersolvable. If every element of $\mathcal{P}^*(H)$ is π-quasinormal in G, then G is supersolvable.*

Proof. See [7, Theorem 3.1].

Lemma 2.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if G has a solvable normal subgroup H such that $G/H \in \mathcal{F}$ and every element of $\mathcal{P}^*(F(H))$ is π-quasinormal in G. In particular, if G is a group with a solvable normal subgroup H such that G/H is supersolvable and every element of $\mathcal{P}^*(F(H))$ is π-quasinormal in G, then G is supersolvable.*

Proof. See [1, Theorem and Corollary 1].

3. Results

Theorem 3.1. *Suppose G is a group with a normal subgroup N such that G/N is supersolvable. If every element of $\mathcal{P}^*(F^*(N))$ is π-quasinormal in G, then G is supersolvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of smallest order. Then we have:

(1) Every proper normal subgroup of G is supersolvable.

If H is a maximal normal subgroup of G, we have that $H/H \cap N$ is supersolvable, $F^*(H \cap N)$ is contained in $F^*(N)$ by Lemma 2.3. So $H, H \cap N$ satisfy the hypotheses of the theorem. The minimal choice of G implies that H is supersolvable.

(2) $N = G$, and $F^*(G) = F(G) < G$.

If $N < G$, then N is supersolvable by (1), in particular, N is solvable, so $F^*(N) = F(N)$, then G is supersolvable by applying Lemma 2.5, a contradiction.

If $F^*(G) = G$, then G is supersolvable by Lemma 2.4, a contradiction. Thus $F^*(G) < G$ is supersolvable. Since $F^*(G)$ is solvable, $F^*(G) = F(G)$ by Lemma 2.3(3).

(3) Contradiction.

(3.1) Now G has a unique maximal normal subgroup, say M.

Suppose not; then there are maximal normal subgroups M, N. It follows that $G = NM$ and $G/M \cong N/(M \cap N)$ is supersolvable, as N is. Now since M is supersolvable, G must be solvable. Hence, G is supersolvable by Lemma 2.5, contrary to our assumption.

(3.2) G/M is non-abelian simple and $G' = G$.

By (3.1) G/M is simple. If G/M is cyclic group of prime order, then G is solvable. This implies that G is supersolvable by Lemma 2.5, which implies a contradiction. Thus G/M is non-abelian simple; it follows that $G' \not\leq M$, so $G' = G$.

(3.3) $\forall q \in \pi(G)$, $O^q(G) = G$.

If not, $O^q(G) < G$, then $O^q(G) \leq M$ is supersolvable. On the other hand, $G/O^q(G)$ is a q-group, so G is solvable, a contradiction as in the proof of (3.2).

Since $F^*(G) = F(G)$ is not the identity group, we may choose a minimal prime divisor q of $\mid F(G) \mid$, now that the Sylow q-subgroup Q of $F(G)$ is a non-trivial normal subgroup of G. For every element x of $\mathcal{P}^*(Q)$, x is π-quasinormal in G, thus $N_G(\langle x \rangle) \geq O^q(G) = G$ by Lemma 2.2. Thus $\langle x \rangle$ is normal in G. Since $G/C_G(\langle x \rangle)$ is abelian, we have that $G = G' \leq C_G(\langle x \rangle)$, and therefore G centralizes every element of $\mathcal{P}^*(Q)$. By Huppert's result ([4, Satz IV 5.12]), $G/C_G(Q)$ is a q-group. Since $G' = G$, we have that $C_G(Q) = G$, i.e., $Q \leq Z(G)$. Hence by Lemma 2.3(4)

$F^*(G/Q) = F^*(G)/Q$. Consider the group $\overline{G} = G/Q$. For any cyclic subgroup $\overline{L}$ of prime order r of $F^*(\overline{G})$, since Q is a Sylow q-subgroup of $F^*(G)$ with the minimal prime divisor, we have that $r > q$ (in particular, $r \neq 2$). Applying the Schur-Zassenhaus Theorem ([3]), we can denote $\overline{L} = LQ/Q$, where L is a cyclic subgroup of order r of $F^*(G)$. Since L is π-quasinormal in G by hypotheses, so $\overline{L}$ is π-quasinormal in $\overline{G}$. It follows that $\overline{G}$ satisfies the hypotheses of the theorem, and from the minimal choice of G it follows that G/Q is supersolvable and so G is solvable, thus $G' < G$, contrary to (3.2). The final contradiction. Thus, G must be supersolvable, as required.

Denote the supersolvable residual of G by $G^{\mathcal{U}}$. Using terminology of formations, it is easy to see that the above theorem can be expressed as the following:

Theorem 3.1′. *Suppose G is a group. Then G is supersolvable if and only if every element of $\mathcal{P}^*(F^*(G^{\mathcal{U}}))$ is π-quasinormal in G.*

Corollary 3.2. *Let G be a group. If every element of prime order or order 4 of $F^*(G)$ is π-quasinormal in G, then G is supersolvable.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathcal{F}$ and every element of $\mathcal{P}^*(F^*(H))$ is π-quasinormal in G.*

Proof. We only need to prove the "if " part.

By hypotheses every element of $\mathcal{P}^*(F^*(H))$ is π-quasinormal in G, thus is π-quasinormal in H by Lemma 2.1(2). Corollary 3.2 implies that H is supersolvable, so $F^*(H) = F(H)$ and hence $G \in \mathcal{F}$ by applying Lemma 2.5.

Using a proof similar to [1, Corollary 4], we also have a generalization of it.

Corollary 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ under either of the following assumptions:*

(a) *G is 2-nilpotent and every element of odd prime order of $F^*(H)$ is π-quasinormal in G.*

(b) *The Sylow$_2$-subgroups of G are abelian and every element of $F^*(H)$ of prime order is π-quasinormal in G.*

Remark. Theorem 3.3 is not true for saturated formations which do not contain $\mathcal{U}$. For example, if $\mathcal{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

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