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NILPOTENCY DEGREE OF COHOMOLOGY RINGS IN CHARACTERISTIC p

PHAM ANH MINH

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Dedicated to Professor Huỳnh Mùi on his sixtieth birthday

ABSTRACT. Let p be an odd prime number. The purpose of this paper is to provide a p-group $\mathcal{G}$ whose mod-p cohomology ring has a nilpotent element $\xi \in H^*(\mathcal{G})$ satisfying $\xi^p \neq 0$.

1. STATEMENT OF THE MAIN RESULT

For every p-group $\mathcal{G}$, denote by $H^*(\mathcal{G})$ the mod-p cohomology algebra of $\mathcal{G}$. We are now interested in the nilpotency degrees of elements of $H^*(\mathcal{G})$. For the case $p = 2$, in [1], [4], it was shown that, given any positive integer n, there exists a 2-group whose cohomology ring has elements of nilpotency degree $n + 1$. It was also noted in [1] that, for p odd, we did not have any example of elements of $H^*(\mathcal{G})$ having nilpotency degrees greater than p.

Recently, such an example was given in [8] for $p = 3$. In this case, $\mathcal{G}$ is chosen to be an extension of $\mathbb{E} \times \mathfrak{A}$ by $\mathbb{Z}/p$, where $\mathbb{E}$ is the extraspecial 3-group of order 3^3 and of exponent 3, and $\mathfrak{A}$ the elementary abelian 3-group of rank 4.

The purpose of this paper is to generalize the result in [8] to the case of any odd prime p. Let $\mathfrak{S}_{p^2,p}$ be the Sylow subgroup of the symmetric group on p^2 letters (so $\mathfrak{S}_{p^2,p} = \mathbb{Z}/p \wr \mathbb{Z}/p$, the wreath product of $\mathbb{Z}/p$ and $\mathbb{Z}/p$) and let Z be the center of $\mathfrak{S}_{p^2,p}$. Define $\mathfrak{R} = \mathfrak{S}_{p^2,p}/Z$ (so $\mathfrak{R} = \mathbb{E}$ for $p = 3$) and let $\mathfrak{A}_k$ be the elementary abelian p-group of rank $2k$. We shall prove

Theorem. *For $k \geq 2p - 2$, there exists a p-group $\mathcal{G}$ given by a central extension*

$$0 \to \mathbb{Z}/p \to \mathcal{G} \to \mathfrak{R} \times \mathfrak{A}_k \to 1$$

whose mod-p *cohomology ring has a nilpotent element $\xi \in H^2(\mathcal{G})$ satisfying $\xi^p \neq 0$.*

We shall use the following notation. For homogeneous elements $X, Y, \dots$ of a graded ring R, $|X|$ denotes the degree of X and $(X, Y, \dots)$ the ideal of R generated by $X, Y, \dots$. p is assumed from now on to be an arbitrary odd prime number. If S is a subset of a group G, then $\langle S \rangle$ denotes the subgroup of G generated by S. With some abuse of notation, for every $\zeta \in H^*(G)$ and for every subgroup K of G, we consider ζ as an element of $H^*(K)$ via the restriction map; also, for every extension T of G, ζ is considered as an element of $H^*(T)$ via the inflation maps.

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2. The groups $\mathfrak{R}$ and $\mathfrak{U}$

Let $\mathfrak{U}$ be the p-group of order p^{p+2} given by

$$\begin{aligned}\mathfrak{U} = \langle a, a_1, \dots, a_{p+1} | a^p = a_3^p = \cdots = a_{p+1}^p = [a, a_{p+1}] = [a_i, a_j] = a_1^p a_2^{p(p-1)/2} a_p \\ = a_2^p a_{p+1} = 1, [a, a_\ell] = a_{\ell+1}, 1 \le i, j \le p+1, 1 \le \ell \le p\rangle.\end{aligned}$$

Then $\mathfrak{U}$ is a 2-generator p-group of maximal class and $\mathfrak{R} = \mathfrak{U}/\langle a_p, a_{p+1}\rangle$ (see e.g. [2, Section 4]). Set $\mathfrak{T} = \mathfrak{U}/\langle a_{p+1}\rangle, g_i = a_i\langle a_p, a_{p+1}\rangle \in \mathfrak{R}, 1 \le i \le p-1$, and $K = \langle g_1, \dots, g_{p-1}\rangle$. We then have extensions of groups

$$0 \to \mathbb{Z}/p \to \mathfrak{U} \xrightarrow{p_1} \mathfrak{T} \to 1, \tag{$\mathfrak{U}$}$$

$$0 \to \mathbb{Z}/p \xrightarrow{j} \mathfrak{T} \xrightarrow{p_2} \mathfrak{R} \to 1, \tag{$\mathfrak{T}$}$$

$$0 \to (\mathbb{Z}/p)^{p-1} \cong K \to \mathfrak{R} \to \langle a\rangle \to 1. \tag{$\mathfrak{R}$}$$

Let $x \in H^1(\langle a\rangle)$ be the dual of a and consider x as an element of $H^1(\mathfrak{R})$ via the inflation map. Define $x_1 \in H^1(\mathfrak{R})$ satisfying $x_1(g_1) = 1, x_1(a) = 0$ (x_1 is well-defined, as $\mathfrak{R}$ is a 2-generator p-group and $\mathfrak{R} = \langle a, g_1\rangle$). x, x_1 is then a basis of $H^1(\mathfrak{R})$. Set $y = \beta x, y_1 = \beta x_1$ with β the Bockstein homomorphism. Let w be the element of $H^2(\mathfrak{R})$ classifying the extension ($\mathfrak{T}$) and set $U = w + y_1$. We have

Lemma 1. (i) $U|_K = 0$*;*

(ii) $U^p + xy^{p-2} \cdot \beta U - y^{p-1} \cdot U = 0$*;*
(iii) $x \cdot U = xy_1$*;*
(iv) $U^p = y^{p-1}y_1$.

Proof. Set $f = p_1(a_1)$. Since $f^p = j(-1)$ and $p_2^{-1}(K)$ is abelian, $w|_K = -y_1$. Hence $U|_K = 0$. (ii) follows from [6, Remark 1] (see also [7, Theorem 1.1]) and the fact that $U|_K = 0$.

Note that the term $E_2(\mathfrak{T})$ of the Hochschild-Serre spectral sequence corresponding to the extension ($\mathfrak{T}$) is of form

$$E_2(\mathfrak{T}) = H^*(\mathfrak{R}) \otimes H^*(j(\mathbb{Z}/p)).$$

Let w' be the element of $H^2(\mathfrak{T})$ classifying the extension ($\mathfrak{U}$) and let u be the element of $H^1(j(\mathbb{Z}/p))$ satisfying $d_2(u) = w$. Since $\langle a_p, a_{p+1}\rangle = p_1^{-1}(\mathrm{Im}\ j)$ is of exponent p, w' restricts trivially on $j(\mathbb{Z}/p)$. So w' represents a non-zero element of $E_\infty^{2,0}(\mathfrak{T}) \oplus E_\infty^{1,1}(\mathfrak{T})$. Note that $X|_{\langle a, j(\mathbb{Z}/p)\rangle}$ (resp. $X|_{\langle a_1, j(\mathbb{Z}/p)\rangle}$) is a scalar multiple of y (resp. y_1), for any $X \in E_\infty^{2,0}(\mathfrak{T}) \subset \mathrm{Im\ Inf}_{\mathfrak{T}}^{\mathfrak{R}}$. As $[a, a_p] = a_{p+1}$ and $[a_1, a_p] = 1$, it follows that w' represents a non-zero element θ of $E_2^{1,1}(\mathfrak{T})$ and $w'|_{\langle a, j(\mathbb{Z}/p)\rangle} \ne 0, w'|_{\langle f, j(\mathbb{Z}/p)\rangle} = 0$. So θ is a non-zero scalar multiple of $x \otimes u$. This means that $xd_2(u) = xw = 0$. Hence $x \cdot U = x(w + y_1) = xy_1$.

Finally, as

$$\begin{aligned} y \cdot U - x \cdot \beta U &= \beta(x \cdot U) \\ &= \beta(xy_1) \quad \text{by (iii)} \\ &= yy_1, \end{aligned}$$

we have, by (ii),

$$\begin{aligned} 0 &= U^p + xy^{p-2} \cdot \beta U - y^{p-1} \cdot U \\ &= U^p + y^{p-2}(x \cdot \beta U - y \cdot U) \\ &= U^p - y^{p-2} \cdot yy_1 \\ &= U^p - y^{p-1} y_1. \end{aligned}$$

The lemma is proved. □

Remarks. 1. It is known that $\mathrm{Inf}_{\mathfrak{S}_{p^2,p}}^{\mathfrak{R}} : H^2(\mathfrak{R}) \to H^2(\mathfrak{S}_{p^2,p})$ is surjective (this comes from the fact that $\mathfrak{S}_{p^2,p}$ is a terminal p-group; see [3]). Furthermore, as $H^*(\mathfrak{S}_{p^2,p})$ is detected by elementary abelian subgroups ([9]) and $U|_{\langle a \rangle} = 0, U|_K = 0$, it follows that $\mathrm{Inf}_{\mathfrak{S}_{p^2,p}}^{\mathfrak{R}}(U) = 0$. Hence U is nothing other than the cohomology class classifying the central extension $1 \to Z \to \mathfrak{S}_{p^2,p} \to \mathfrak{R} \to 1$.

2. The group $\mathfrak{U}$ can be obtained from algebraic number theory, as follows. Let ϑ be a primitive pth root of unity and let $\mathbb{Z}_p$ be the ring of p-adic integers. Then

$$\mathbb{Z}_p[\vartheta] = \bigoplus_{j=0}^{p-2} \vartheta^j \mathbb{Z}_p.$$

Let $\langle a \rangle$ be the cyclic group of order p. Multiplication by ϑ induces an action of $\langle a \rangle$ on $\mathbb{Z}_p[\vartheta]$, hence on $\mathfrak{Z} = \mathbb{Z}_p[\vartheta]/((\vartheta - 1)^{p+1})$. We can check that the map

$$\{a, a_1, \ldots, a_{p+1}\} \to \langle a \rangle \ltimes \mathfrak{Z}$$

which maps a to a, a_i to $(\vartheta - 1)^{i-1}, 1 \le i \le p+1$, can be extended to a homomorphism of groups from $\mathfrak{U}$ to $\langle a \rangle \ltimes \mathfrak{Z}$ which is bijective. So $\mathfrak{U} \cong \langle a \rangle \ltimes \mathfrak{Z}$.

3. Proof of the Theorem

Pick a basis $c_1, \ldots, c_{2k}$ of the elementary abelian p-group $\mathfrak{A}_k$ of rank $2k$. Let $u_1, \ldots, u_{2k}$ be the basis of $H^1(\mathfrak{A}_k)$, dual to that of $\mathfrak{A}_k$. Define $v_i = \beta u_i, 1 \le i \le 2k$. Set $G = \mathfrak{R} \times \mathfrak{A}_k$. By Künneth formula,

$$\begin{aligned} H^*(G) &= H^*(\mathfrak{R}) \otimes H^*(\mathfrak{A}_k) \\ &= H^*(\mathfrak{R}) \otimes \Lambda[u_1, \ldots, u_{2k}] \otimes \mathbb{F}_p[v_1, \ldots, v_{2k}]. \end{aligned}$$

Let

$$\begin{aligned} z &= y + u_1 u_2 + \cdots + u_{2k-1} u_{2k}, \\ \alpha &= u_1 u_2 + \cdots + u_{2k-1} u_{2k}, \\ \gamma &= u_3 u_4 + \cdots + u_{2k-1} u_{2k} \end{aligned}$$

be elements of $H^*(G)$. Set $\Lambda = \Lambda[u_1, \ldots, u_{2k}]$ and $\Lambda' = \Lambda[u_3, \ldots, u_{2k}]$. For $i \ge 0$, let Λ_i be the degree i-part of Λ. We have

Lemma 2. *Let $0 \ne X$ be a homogeneous element of $H^*(G)$ and let n, k be positive integers.*

(i) *If $n \le \min(p-1, k)$ and $X \cdot \alpha^n = 0$, then $|X| \ge \min(2p - 2n, k - n + 1)$.*
(ii) *If $X \cdot z = 0$:*
  (iia) *$|X| \ge \min(2p - 2, k)$;*
  (iib) *$|X| > 2p - 1$, provided that $k \ge 2p - 2$ and $X \in (v_1, \ldots, v_{2k})$.*

Proof. (i) By induction on k. It obviously holds for $k = 1$. Assume that it holds for $k-1$. Without loss of generality, we may assume that $X \in \Lambda$. If $n = k \le p-1$, then $0 = X \cdot \alpha^n = k!X \cdot u_1 \dots u_{2k}$ implies $|X| \ge 1$. We may then suppose that $n \le k-1$.

Write $X = X_1 + X_2 \cdot u_1u_2 + X_3 \cdot u_1 + X_4 \cdot u_2$ with X_i homogeneous in Λ'. So $0 = X \cdot \alpha^n = X \cdot (\gamma + u_1u_2)^n$ implies

$$X_1 \cdot \gamma^n = 0, \tag{1}$$

$$(nX_1 + X_2 \cdot \gamma) \cdot \gamma^{n-1} = 0, \tag{2}$$

$$X_3 \cdot \gamma^n = X_4 \cdot \gamma^n = 0. \tag{3}$$

If $X_3 \neq 0$ or $X_4 \neq 0$, it follows from (3) and the inductive hypothesis that $\min(|X_3|, |X_4|) \ge \min(2p-2n, k-n)$, hence

$$|X| = \min(|X_3|, |X_4|) + 1 \ge \min(2p-2n+1, k-n+1).$$

So we may suppose that $X_3 = X_4 = 0$. By (2) and by the inductive hypothesis, $|nX_1 + X_2 \cdot \gamma| \ge \min(2p-2n+2, k-n+1)$. Hence $|X| = |nX_1 + X_2 \cdot \gamma| \ge \min(2p-2n, k-n+1)$ if $nX_1 + X_2 \cdot \gamma \neq 0$. If $nX_1 + X_2 \cdot \gamma = 0$, then $nX_1 = -X_2 \cdot \gamma$. By (1), $X_2 \cdot \gamma^{n+1} = 0$. By the inductive hypothesis, $|X_2| \ge \min(2p-2n-2, k-n-1)$. Hence $|X| = |X_2| + 2 \ge \min(2p-2n, k-n+1)$. (i) is then proved.

(ii) Write $X = \sum_{i=0}^{m} X_i \cdot Y_i$ with $Y_i \in \Lambda_i, X_i \in H^*(\mathfrak{R}) \otimes \mathbb{F}_p[v_1, \dots, v_{2k}]$ and $X_m \cdot Y_m \neq 0$. Then $X_m \cdot Y_m \cdot \alpha = 0$. By (i), it follows that $|X| = |X_m \cdot Y_m| \ge \min(2p-2, k)$. (iia) is proved.

Suppose that $k \ge 2p-2, |X| \le 2p-1$ and $X \in (v_1, \dots, v_{2k})$. Fix i with $1 \le i \le 2k$ and write $X = v_iX' + X''$ with X'' free of v_i. Then $X' \cdot z = 0$. As $|X'| \le 2p-3$, it follows from (iia) that $X' = 0$. X is then free of v_i. Hence by repeating the argument, we see that X is free of $v_1, \dots, v_{2k}$, a contradiction.

The lemma is proved. □

Suppose from now on that $k \ge 2p-2$. We have

Lemma 3. *If X is homogeneous in $H^*(G)$ of degree $\le 2p-5$ and $X \cdot \beta z \in (z)$, then $X \in (z, \beta z)$.*

Proof. Write $X \cdot \beta z = A \cdot z$ with A homogeneous in $H^*(G)$. Then $A \cdot \beta z \cdot z = 0$. As $|A \cdot \beta z| = |X| + 4 \le 2p-1$ and $A \cdot \beta z \in (v_1, \dots, v_{2k})$, it follows from Lemma 2(ii) that $A \cdot \beta z = 0$. By [5, Lemma 2.1], $A \in (\beta z, u_1 \dots u_{2k})$. So $A \in (\beta z)$, as $|A| \le 2p-4 < 2k$. Write $A = B \cdot \beta z$. We then have $(X - B \cdot z)\beta z = 0$. Again, by [5, Lemma 2.1], $X - B \cdot z \in (\beta z)$. Therefore $X \in (z, \beta z)$. The lemma is proved. □

For every $1 \le j \le p-1$, let B_j be a set of elements of Λ_{2j} such that the disjoint union $B_j \sqcup \{\alpha^j\}$ forms a basis of Λ_{2j}. We then have a decomposition

$$H^*(\mathfrak{R}) \otimes \Lambda_{2j} = H^*(\mathfrak{R}) \otimes (\langle B_j \rangle \oplus \langle \alpha^j \rangle), 1 \le j \le p-1. \tag{4}$$

For $1 \le i < p-1$ and for $0 \neq b \in \langle B_i \rangle, \mu \in \mathbb{Z}/p$, as $|b - \mu\alpha^i| = 2i < 2p-2$, it follows from Lemma 2 that $(b - \mu\alpha^i) \cdot \alpha \neq 0$. The B_j's can then be chosen such that, for every $1 \le i < p-1$,

$$\{b \cdot \alpha | 0 \neq b \in \langle B_i \rangle\} \subset \langle B_{i+1} \rangle. \tag{5}$$

Lemma 4. $y^{p-1}y_1 \notin (z, \beta z)$.

Proof. Note that, as $\mathfrak{R}$ is of exponent p, $y^{p-1}y_1|_{\langle ag_1\rangle} \neq 0$. So $y^{p-1}y_1 \neq 0$.

Suppose that $y^{p-1}y_1 = A \cdot z + B \cdot \beta z$ with A, B homogeneous in $H^*(G)$. Write $A = A' + A''$ with $A' \in H^*(\mathfrak{R}) \otimes \Lambda$, $A'' \in (v_1, \ldots, v_{2k})$. It follows that $y^{p-1}y_1 = A' \cdot z$ and $A'' \cdot z + B \cdot \beta z = 0$. Hence, without loss of generality, we may assume that $y^{p-1}y_1 = A \cdot z$ with $A \in H^*(\mathfrak{R}) \otimes \Lambda$. By the decomposition (4), we have

$$A = A_{2p-2} + \sum_{i=1}^{p-1}(A_{2p-2i-2} \cdot \alpha^i + \sum_{b \in B_i} A_{2p-2i-2,b} \cdot b)$$

with $A_j, A_{j,b} \in H^*(\mathfrak{R})$. By (5), it follows that

$$\begin{aligned} A_{2p-2}y &= y^{p-1}y_1, \\ (A_{2p-2} + A_{2p-4}y) \cdot \alpha &= 0, \\ &\cdots \\ (A_2 + A_0 y) \cdot \alpha^{p-1} &= 0. \end{aligned}$$

Therefore, for every $i \geq 1, A_{2i} = -A_{2i-2}y$. So $y^{p-1}y_1 = A_0 y^p$. As $y^{p-1}y_1|_{\langle a\rangle} = 0$, this implies $A_0 = 0$. Hence $y^{p-1}y_1 = 0$, a contradiction. Thus $y^{p-1}y_1 \notin (z, \beta z)$. The lemma is proved. □

Let $\{E_r, d_r\}$ be the Hochschild-Serre spectral sequence corresponding to the central extension

$$0 \to \mathbb{Z}/p \xrightarrow{i} \mathcal{G} \to G \to 1$$

classified by $z \in H^2(G)$. Pick s (resp. t) in $H^1(i(\mathbb{Z}/p))$ (resp. $H^2(i(\mathbb{Z}/p))$ satisfying $d_2(s) = z$ (resp. $d_3(t) = \beta z$). It is known that

$$\begin{aligned} E_2 &= H^*(G) \otimes H^*(i(\mathbb{Z}/p)), \\ E_3 &= H^*(G)/(z) \otimes \mathbb{F}_p[t] \oplus \operatorname{Ann}_{H^*(G)}(z) \otimes \mathbb{F}_p[t]s. \end{aligned}$$

We have

Lemma 5. *There exists no element $\eta \in E_n^{2p-n,n-1}, n \geq 3$, satisfying $d_n(\eta) = y^{p-1}y_1$.*

Proof. Suppose that η is an element of $E_{2r}^{2p-2r,2r-1}, r \geq 2$, represented by $X \otimes t^{r-1}s \in E_2^{2p-2r,2r-1}$. Then $0 = d_2(X \otimes t^{r-1}s) = X \cdot z \otimes t^{r-1}$. So $X \cdot z = 0$. Since $|X| = 2p - 2r \leq 2p - 4$, $X = 0$ by Lemma 2. Hence $0 = d_{2r}(\eta) \neq y^{p-1}y_1$.

Suppose now that $\eta \in E_{2r+1}^{2p-2r-1,2r}$ is represented by $X \otimes t^r \in E_3^{2p-2r-1,2r}, p-1 \geq r \geq 1$. If $r = 1$, then $y_1^{p-1}y_2 \neq X \cdot \beta z = -d_3(X \otimes t)$ by Lemma 4. If $r \geq 2$, then $-rX \cdot \beta z \otimes t^{r-1} = d_3(X \otimes t^r) = 0$ in E_3. So $X \cdot \beta z \in (z)$. As $|X| = 2p-2r-1 \leq 2p-5$, $X \in (z, \beta z)$ by Lemma 3. This implies $X \otimes t^r = 0$ in E_4 unless $r = p-1$ and $X \in (\beta z)$. But $r = p-1$ also implies $|X| = 2p - 2r - 1 = 1$ and $X \in (\beta z)$, hence $X = 0$. So $0 = d_{2r+1}(\eta) \neq y^{p-1}y_1$. The lemma is proved. □

Proof of the Theorem. Set $\xi = \operatorname{Inf}_{\mathcal{G}}^G(U)$. By Lemma 1, $\xi^p = y^{p-1}y_1$. Since z vanishes in $H^*(\mathcal{G})$, $y^p = -\alpha^p = 0$ in $H^*(\mathcal{G})$. So $\xi^{2p} = y^{2p-2}y_1^2 = 0$. Hence ξ is nilpotent. Besides, Lemmas 4 and 5 show that $y^{p-1}y_1 \notin \operatorname{Ker} \operatorname{Inf}_{\mathcal{G}}^G$, which means that $\xi^p = y^{p-1}y_1 \neq 0$. The Theorem is proved. □

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Department of Mathematics, College of Sciences, University of Hue, Dai hoc Khoa hoc, Hue, Vietnam

E-mail address: `paminh@dng.vnn.vn`