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L^p REGULARITY OF AVERAGING OPERATORS WITH HIGHER FOLD SINGULARITIES

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ABSTRACT. In this paper, we give a sharp L^p regularity result of averaging operators along curves in the plane with two-sided $k-$fold singularities.

1. INTRODUCTION

Let Ω_L and Ω_R be open subsets in $\mathbb{R}^2$ containing the origin. Let $\mathcal{M}$ be a hypersurface in $\Omega_L \times \Omega_R$ with conormal bundle $N^*\mathcal{M}$. Set $\Lambda = N^*\mathcal{M} \setminus 0$ where 0 is the zero section. We assume that $\Lambda \subset T^*\Omega_L \setminus 0 \times T^*\Omega_L \setminus 0$. Then $\mathcal{M}$ can be given by

$$\mathcal{M} = \{(x, y) : y_2 = S(x, y_1)\} \tag{1.1}$$

and S_{x_2} does not vanish on $\mathcal{M}$. We consider averaging operators of the form

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} f(y)\chi(x, y)d\sigma_x(y)$$

where $\mathcal{M}_x = \{y \in \Omega_R : (x, y) \in \mathcal{M}\}$, $\chi(x, y)$ is a smooth function with a compact support near the origin and $d\sigma_x$ is a smooth density on $\mathcal{M}_x$ depending smoothly on x. By using (1.1) we can rewrite $\mathcal{R}$ as

$$\mathcal{R}f(x) = \int f(y_1, S(x, y_1))\chi(x, y_1)dy_1.$$

These operators are Fourier integral operators of order $-\frac{1}{2}$. Let $\mathcal{C}$ be a twisted canonical relation where (x, ξ, y, μ) is replaced by $(x, \xi, y, -\mu)$ in Λ. We consider two natural projections π_L and π_R,

$$\pi_L : \mathcal{C} \to T^*(\Omega_L) \setminus 0,$$

$$\pi_R : \mathcal{C} \to T^*(\Omega_R) \setminus 0.$$

π_L (or equivalently π_R) is locally diffeomorphic if and only if the determinant of $d\pi_L$ does not vanish, which is equivalent to saying that

$$\mathcal{J}(x, y_1) := (S_{x_1y_1}S_{x_2} - S_{x_2y_1}S_{x_1})|_{(x,y_1)} \neq 0.$$

$k-$fold singularity of $\mathcal{M}$ has been defined in [PSt2] and [I].

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Definition 1.1. Let $\Sigma = \{c \in \mathcal{C} : \pi_L \text{ is not locally 1-1}\}$. We say that $\mathcal{M}$ has two-sided $k-$fold singularities if the following conditions hold:

(1) Σ is a submanifold of $\mathcal{C}$ of codimension 1.
(2) $\det(d\pi_L)$ and $\det(d\pi_R)$ vanish of order k along Σ.
(3) $T_c(\Sigma) \oplus \mathrm{Ker}(d\pi_L)_c = T_c(\mathcal{C})$.
(4) $T_c(\Sigma) \oplus \mathrm{Ker}(d\pi_R)_c = T_c(\mathcal{C})$.

There have been results on Fourier integral operators with these types of singularities ([C], [PSt2], [S1], [I]). The optimal L^2 regularizing property of $\mathcal{R}$ with $k-$fold singularity has been obtained by Phong and Stein [PSt2]. This result has been extended to a higher dimensional case by Cuccagna [C]. Sharp L^p regularity of $\mathcal{R}$ up to two end-points has been settled by Seeger when $k = 1$ [S1]. L^p boundedness of maximal operators has been obtained by Iosevich [I]. Sharp L^p regularity of the case that $S(x, y_1) = x_2 + P(x_1, y_1)$ and $P(x_1, y_1)$ is a homogeneous polynomial was obtained by Phong and Stein [PSt1]. The case that $\mathcal{M}$ only has 'finite type' conditions has been studied by Seeger with an ϵ loss [S2]. In this paper, we assume that $\mathcal{M}$ has two-sided $k-$fold singularities, that is, if we solve $\partial_{y_1}^{k-1}\mathcal{J}(x, y_1) = 0$ for y_1 and if we let $y_1 = \bar{z}(x_1, x_2)$ be the solution, then we have

$$\partial_{y_1}^{i}\mathcal{J}(x, \bar{z}(x_1, x_2)) = 0 \quad \text{if } 0 \le i \le k-1$$

and

$$\partial_{y_1}^{k}\mathcal{J}(x, \bar{z}(x_1, x_2)) \neq 0.$$

Since we assume two-sided $k-$fold singularities, this is true for x_1 in a symmetrical way. Hence we have $\partial_{x_1}\bar{z}(x_1, x_2) \neq 0$. By using the change of variables in x we may assume that $\bar{z}(x_1, x_2) = x_1$. In this paper, we prove L^p regularity of $\mathcal{R}$ excluding two end-points. More precisely, we prove

Theorem 1.2. *If $\mathcal{M}$ has two-sided k-fold singularities, then for all $\alpha \in \mathbb{R}$, $\mathcal{R}$ is bounded from L^p_α into $L^p_{\alpha+\frac{1}{k+2}}$ when $(k+2)/(k+1) < p < k+2$.*

This result is sharp by [S2] and the result for the critical exponents $(k+2)/(k+1)$ and $k+2$ breaks down by the translation invariant counterexample of M. Christ [Ch]. The proof of this theorem is dependent on Seeger's argument in [S1]. He used an interpolation argument between the L^2 estimate and an estimate in a space of bounded mean oscillation associated to a family of nonisotropic balls. Let $\chi_0 \in C_0^\infty(\mathbb{R})$ be supported in $(-1, 1)$ and $\chi_0(s) = 1$ if $|s| \le 1/2$. Set $\chi_l(t) = \chi_0(2^l t) - \chi_0(2^{l-1}t)$ for $l > 0$. Now we define $\mathcal{R}^l_m$ by

$$\mathcal{R}^l_m f(x) = \int\int e^{i\tau(y_2 - S(x,y_1))}\chi_l(y_1 - x_1)a_m(x, y, \tau)f(y)d\tau dy$$

where $a_m(x, y, \tau) = \chi_m(\tau)\chi(x, y_1)$. An estimate of $\mathcal{R}^l_m$ was obtained by Cuccagna [C].

Proposition 1.3. *If $\mathcal{R}^l_m$ is as above, then we have*

$$||\mathcal{R}^l_m f||_{L^2} \le C2^{kl/2}2^{-m/2}||f||_{L^2}. \tag{1.2}$$

Now we define $\mathcal{R}_{m,n}$ by

$$\mathcal{R}_{m,n}f(x_1, x_2) = \int\int e^{i\tau(y_2 - S(x_1,x_2,y_1))}\chi_{m,n}(y_1 - x_1)a_m(x, y, \tau)f(y)d\tau dy,$$

where

$$\begin{aligned}\chi_{m,0}(u) &= \chi_0(2^{m/(k+2)}u),\\ \chi_{m,n}(u) &= \chi_0(2^{m/(k+2)-n}u) - \chi_0(2^{m/(k+2)-n+1}u), \quad n \geq 1.\end{aligned}$$

Since $R_{m,n}$ is essentially R_m^l with $l \approx m/(k+2) - n$, (1.2) gives

$$||2^{-m/(k+2)}\mathcal{R}_{m,n}f||_{L^2} \leq C2^{-kn/2}||f||_{L^2}. \tag{1.3}$$

In section 2, we shall define a family of nonisotropic balls and prove some properties of the balls. We define a $BMO_{\mathfrak{h}}$ space associated to the family of nonisotropic balls. In section 3, we shall prove the following proposition.

Proposition 1.4.

$$||\{2^{m/(k+2)}R_{m,n}f_m\}||_{BMO_{\mathfrak{h}}(\ell^2)} \leq Cn2^n \,||\{f_m\}||_{L^\infty(\ell^2)}\,.$$

We remark that the same inequality for $k = 1$ is stated in [S1, Proposition 7.1], however 2^n in [S1] has to be replaced by $n2^n$ as well. By the argument in [S1], Proposition 1.4 and (1.3) yield

$$\left|\left|\left(\sum_m |2^{m/(k+2)}R_{m,n}f_m|^2\right)^{1/2}\right|\right|_{L^q} \leq Cn^{1-2/q}2^{n(-(k+2)/q+1)}\left|\left|\left(\sum_m |f_m|^2\right)^{1/2}\right|\right|_{L^q},$$

where $2 \leq q < \infty$. This proves Theorem 1.2 when $2 \leq q < k+2$ and the duality argument finishes the proof of Theorem 1.2.

2. Nonisotropic balls

In this section we shall define a family of nonisotropic balls and investigate some properties of the family of the balls which allow us to apply Calderon-Zygmund theory. Our situation is really a special case of the one considered by Greenblatt but we prefer to give the direct and elementary proof in this specific case (cf. Remark 2.6 below). In what follows, we assume that

$$S(x, x_1) = x_2. \tag{2.1}$$

Motivated by [S1], we define a nonisotropic metric $\rho(x, y)$ by

$$\rho(x, y) = |x_1 - y_1|^{k+2} + |S(x, y_1) - y_2|.$$

Now we define a ball $B(x;\delta)$ by

$$B(x;\delta) = \{y\ ;\ \rho(x, y) < \delta\}$$

and we define $BMO_{\mathfrak{h}}$ as the space of bounded mean oscillation associated to this family of balls.

We solve $S(z, y_1) = S(x, y_1)$ for z_2 and let $v = v(x, z_1, y_1)$ be the solution. By the definition of v, we have

$$\partial_{y_1}v(x, z_1, y_1) = (\partial_{y_1}S(x, y_1) - \partial_{y_1}S(z_1, v, y_1))/\partial_{x_2}S(z_1, v, y_1), \tag{2.2}$$

$$\partial_{z_1}v(x, z_1, y_1) = -\partial_{x_1}S(z_1, v, y_1)/\partial_{x_2}S(z_1, v, y_1), \tag{2.3}$$

and

$$
\begin{aligned}
&\partial_{y_1}\partial_{z_1}v(x,z_1,y_1)\\
&\quad= (\partial_{x_1}\partial_{y_1}S\partial_{x_2}S-\partial_{x_2}\partial_{y_1}S\partial_{x_1}S)/(\partial_{x_2}S)^2(z_1,v,y_1)\\
&\qquad+(\partial_{x_1}\partial_{x_2}S(z_1,v,y_1)+\partial^2_{x_2}S(z_1,v,y_1)\partial_{z_1}v(x,z_1,y_1))\partial_{y_1}v(x,z_1,y_1\\
&\quad=\mathcal{J}(z_1,v,y_1)/(\partial_{x_2}S)^2(z_1,v,y_1)+F(x,z_1,y_1)\partial_{y_1}v(x,z_1,y_1),
\end{aligned}
\tag{2.4}
$$

where $F(x,z_1,y_1)=\partial_{x_1}\partial_{x_2}S(z_1,v,y_1)+\partial^2_{x_2}S(z_1,v,y_1)\partial_{z_1}v(x,z_1,y_1)$. First we shall prove the following lemma.

Lemma 2.1. *For $\alpha\geq 0$ and $\beta\geq 1$, we have*

$$\partial^{\beta}_{y_1}\partial^{\alpha}_{z_1}v(x,x_1,x_1)=0 \quad \text{if} \quad \alpha+\beta<k+2 \tag{2.5}$$

and

$$\partial^{\beta}_{y_1}\partial^{\alpha}_{z_1}v(x,x_1,x_1)\neq 0 \quad \text{if} \quad \alpha+\beta=k+2. \tag{2.6}$$

Proof. First, we show that for $\beta\geq 1$,

$$\partial^{\beta}_{y_1}v(x,x_1,x_1)=0.$$

To show this we use the induction on β. This is clear when $\beta=1$ by (2.2) and the fact that $v(x,x_1,x_1)=x_2$. By using (2.2) it is easy to see that $\partial^{\beta}_{y_1}v(x,z_1,y_1)$ is in the ideal of the ring of smooth functions in $\mathbb{R}^4$ generated by (a) $\partial^{i}_{y_1}S(x,y_1)-\partial^{i}_{y_1}S(z_1,v,y_1)$ for $1\leq i\leq\beta$ and (b) $\partial^{j}_{y_1}v(x,z_1,y_1)$ for $1\leq j\leq\beta-1$. Since $v(x,x_1,x_1)=x_2$, functions of the form (a) vanish if we replace (z_1,y_1) by (x_1,x_1). By the induction hypothesis functions of the form (b) vanish if we replace (z_1,y_1) by (x_1,x_1). Thus the assertion is proved if $\alpha=0$. Now we use the induction on $\alpha+\beta$ to show (2.5). Since the case $\alpha+\beta=1$ has been done, it suffices to treat the case $\alpha+\beta=l\leq k+1$ assuming the first statement is true for $\alpha+\beta\leq l-1$. By using (2.4) and the fact that $\mathcal{J}(x_1,x_2,y_1)$ vanishes along $y_1=x_1$ with order k, we have

$$\partial_{y_1}\partial_{z_1}v(x,z_1,y_1)=G(z_1,v(x,z_1,y_1),y_1)(y_1-z_1)^k+F(x,z_1,y_1)\partial_{y_1}v(x,z_1,y_1). \tag{2.7}$$

Since the first term of the right-hand side of (2.7) satisfies (2.5), it suffices to show that

$$\partial^{\beta-1}_{y_1}\partial^{\alpha-1}_{z_1}(F\partial_{y_1}v)(x,x_1,x_1)=0 \quad \text{if} \quad \alpha+\beta=l.$$

Using the product rule it is easy to see that $\partial^{\beta}_{y_1}\partial^{\alpha}_{z_1}(F\partial_{y_1}v)$ is in the ideal of the ring of smooth functions in $\mathbb{R}^4$ generated by $\partial^{\beta'}_{y_1}\partial^{\alpha'}_{z_1}v$ for $\alpha'+\beta'\leq l-1$. Now it is easy to see that (2.5) is followed by induction hypothesis. (2.6) is followed by two easy observations that the first term of the right-hand side of (2.7) satisfies (2.6) and that by using (2.5) we have

$$\partial^{\beta-1}_{y_1}\partial^{\alpha-1}_{z_1}(F\partial_{y_1}v)(x,x_1,x_1)=0 \quad \text{if} \quad \alpha+\beta=k+2.$$

This completes the proof. □

To make a neighborhood of the origin a space of homogeneous type, we have to show that the family of balls satisfies two important properties called the doubling property and the overlapping property. More precisely, if m denotes Lebesgue measure in $\mathbb{R}^2$, then we shall prove:

Lemma 2.2. *There exists $C > 0$ such that*

(i) $$m(B(x, 2\delta)) \leq Cm(B(x, \delta))$$

and

(ii) $$\text{if} \quad B(x_1, \delta) \cap B(x_2, \delta), \quad \text{then} \quad B(x_1, \delta) \subset B(x_2, C\delta).$$

To prove this lemma, we need the following lemmas.

Lemma 2.3.

$$|y_2 - S(x, y_1)| \leq C(|x_2 - S(y, x_1)| + |y_1 - x_1|^{k+2}).$$

Proof. If we solve $S(x, x_1) = S(y, x_1)$ for x_2, we have

$$S(x_1, v(y, x_1, x_1), x_1) = S(y, x_1)$$

by the definition of v. By (2.1), we have

$$v(y, x_1, x_1) = S(y, x_1).$$

By using this we have

$$\begin{aligned} y_2 - S(x, y_1) &= y_2 - S(x_1, S(y, x_1), y_1) + (S(x_1, S(y, x_1), y_1) - S(x, y_1)) \\ &= y_2 - S(x_1, v(y, x_1, x_1), y_1) + F(x, y)(x_2 - S(y, x_1)), \end{aligned}$$

where

$$F(x, y) = \int_0^1 \partial_{x_2} S(x_1, s(S(y, x_1) - x_2) + x_2, y_1) ds.$$

Since $F(x, y)$ is bounded in a small neighborhood of the origin, it suffices to show that

$$|y_2 - S(x_1, v(y, x_1, x_1), y_1)| \leq C|y_1 - x_1|^{k+2}.$$

By solving $S(x, y_1) = S(y, y_1)$ for x_2 and using (2.1), we have

$$S(x_1, v(y, x_1, y_1), y_1) = S(y, y_1) = y_2.$$

Hence we have

$$\begin{aligned} y_2 - S(x_1, v(y, x_1, x_1), y_1) &= S(x_1, v(y, x_1, y_1), y_1) - S(x_1, v(y, x_1, x_1), y_1) \\ &= F_1(x_1, y)(v(y, x_1, y_1) - v(y, x_1, x_1)) \\ &= F_1(x_1, y)(y_1 - x_1) \int_0^1 \partial_{y_1} v(y, x_1, y_s) ds, \end{aligned}$$

where $y_s = sy_1 + (1 - s)x_1$ and

$$F_1(x_1, y) = \int_0^1 \partial_{x_2} S(x_1, sv(y, x_1, y_1) + (1 - s)v(y, x_1, x_1), y_1) ds.$$

By Taylor expansion of $\partial_{y_1} v(y, x_1, y_s)$ at $(x_1, y_s) = (y_1, y_1)$ and using Lemma 2.1, we obtain

$$|\partial_{y_1} v(y, x_1, y_s)| \leq C|y_1 - x_1|^{k+1},$$

which proves the lemma. □

Lemma 2.4.

$$|S(z,y_1)-S(x,y_1)| \le C(|z_2-S(x,z_1)|+\sum_{i=1}^{k+1}|y_1-x_1|^i|z_1-x_1|^{k+2-i}).$$

Proof. By solving $S(z,y_1)=S(x,y_1)$ for z_2, we have

$$S(x,y_1)=S(z_1,v(x,z_1,y_1),y_1).$$

If we replace y_1 with z_1 and use (2.1), then we have

$$v(x,z_1,z_1)=S(x,z_1).$$

By using this, we have

$$\begin{aligned} S(z,y_1)-S(x,y_1) &= S(z,y_1)-S(z_1,S(x,z_1),y_1) \\ &\quad +S(z_1,v(x,z_1,z_1),y_1)-S(z_1,v(x,z_1,y_1),y_1) \\ &= F_2(x,z,y_1)(z_2-S(x,z_1)) \\ &\quad +F_3(x,z,y_1)(v(x,z_1,z_1)-v(x,z_1,y_1)) \\ &= F_2(x,z,y_1)(z_2-S(x,z_1)) \\ &\quad +F_3(x,z,y_1)\int_0^1 \partial_{y_1}v(x,z_1,z_s)ds(y_1-z_1) \end{aligned}$$

where

$$\begin{aligned} F_2(x,z,y_1) &= \int_0^1 \partial_{x_2}(z_1,sz_2+(1-s)S(x,z_1),y_1)ds, \\ F_3(x,z,y_1) &= \int_0^1 \partial_{x_2}(z_1,sv(x,z_1,z_1)+(1-s)v(x,z_1,y_1),y_1)ds, \end{aligned}$$

and $z_s=sz_1+(1-s)y_1$. Now we consider the Taylor expansion of $\partial_{y_1}v(x,z_1,z_s)$ at $(z_1,z_s)=(x_1,x_1)$:

$$\begin{aligned} &\partial_{y_1}v(x,z_1,z_s) \\ &\quad = \sum_{0\le p+q\le k}\frac{(p+q)!}{p!q!}\partial_{y_1}^{q+1}\partial_{z_1}^{p}v(x,x_1,x_1)(z_1-x_1)^p(z_s-x_1)^q \\ &\qquad +O(\sum_{i=1}^{k}|z_s-x_1|^i|z_1-x_1|^{k+1-i}). \end{aligned}$$

By Lemma 2.1, the first term of the right-hand side vanishes and by using the fact $|z_s-x_1|\le|y_1-x_1|+|z_1-x_1|$ it is easy to see that the second term gives the desired estimate. □

Lemma 2.5.

$$|S(x,y_1)-y_2|\le C(|S(x,z_1)-z_2|+|S(z,y_1)-y_2|+|x_1-z_1|^{k+2}+|z_1-y_1|^{k+2}).$$

Proof.

$$|S(x,y_1)-y_2|\le|S(z,y_1)-y_2|+|S(x,y_1)-S(z,y_1)|.$$

By Lemma 2.4, we have

$$\begin{aligned}|S(z,y_1)-S(x,y_1)| &\leq C(|z_2-S(x,z_1)|+\sum_{i=1}^{k+1}|y_1-x_1|^i|z_1-x_1|^{k+2-i})\\ &\leq C'(|z_2-S(x,z_1)|+\sum_{i=0}^{k+2}|y_1-z_1|^i|z_1-x_1|^{k+2-i}).\end{aligned}$$

By using

$$|y_1-z_1|^i|z_1-x_1|^{k+2-i}\leq|y_1-z_1|^{k+2}+|z_1-x_1|^{k+2},$$

we immediately obtain the inequality. □

Proof of Lemma 2.2. It is easy to see that (i) follows from

$$m(B(x;\delta))\approx\int_{|y_1-x_1|\leq\delta^{1/(k+2)}}\int_{|y_2-S(x,y_1)|\leq\delta}dy_2dy_1=\delta^{(k+3)/(k+2)}$$

and (ii) follows from Lemma 2.3 and Lemma 2.5.

□

Remark 2.6. In [G], a family of nonisotropic balls has been defined in general cases to treat singular Radon transforms by

$$\mathcal{B}(x,r)=\{y;|y_1-x_1|<r,|y_2-S(x,y_1)|<\sigma(x;r)\},$$

where

$$\sigma(x;r)=\left(\sum_{1\leq\alpha,\beta,\alpha+\beta\leq k+2}\left(\partial_{z_1}^{\alpha}\partial_{y_1}^{\beta}v(x,x_1,x_1)(cr)^{\alpha+\beta}\right)^2\right)^{1/2}.$$

Using Lemma 2.1, it is easy to check that $\mathcal{B}(x,\delta^{1/(m+2)})\approx B(x,\delta)$.

3. $BMO_{\mathfrak{h}}$ ESTIMATES

In this section, we prove Proposition 1.4. To do this, we may follow the argument of the proof of Proposition 7.1 in [S1] with suitable modifications and some simplifications.

Proof of Proposition 1.4. We define an exceptional set $\mathcal{E}_n(B(x;\delta))$ by

$$\mathcal{E}_n(B(x;\delta))=\{y\ ;\ |y_1-x_1|\leq 2^{n+3}\delta^{1/(k+2)},\ |y_2-S(x,y_1)|\leq C_0 2^{n+kn}\delta\},$$

where C_0 is a constant large enough to make the following arguments hold true. We decompose f_m into g_m+h_m, where $g_m=f_m\chi_{\mathcal{E}_n(B)}$ and $h_m=f_m(1-\chi_{\mathcal{E}_n(B)})$. It suffices to show that

$$\frac{1}{|B|}\int_B\left(\sum_m|2^{m/(k+2)}\mathcal{R}_{m,n}g_m(z)|^2\right)^{1/2}dz\leq C2^n\left|\left|\left(\sum_m|f_m|^2\right)^{1/2}\right|\right|_{L^\infty},\tag{3.1}$$

$$\sum_{2^{-(1+k)n}\leq 2^m\delta\leq 1}\frac{1}{|B|}\int_B|2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z)|dz\leq Cn2^n||f_m||_{L^\infty},\tag{3.2}$$

$$\frac{1}{|B|}\int_B|2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z)|dz\leq C2^n(2^m\delta)^{-1}||f_m||_{L^\infty}\tag{3.3}$$

if $2^m\delta \geq 1$,

$$\sum_{2^{-Mn}\leq 2^m\delta\leq 2^{-(k+1)n}} \frac{1}{|B|}\int_B |2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z)|dz \leq Cn2^n||f_m||_{L^\infty}, \tag{3.4}$$

and

$$\frac{1}{|B|}\int_B |2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z) - \nu_m|dz \leq C2^n(2^{m+Mn}\delta)^{1/(k+2)}||f_m||_{L^\infty} \tag{3.5}$$

if $2^{m+Mn}\delta < 1$, where $M = (k+1)(k+2)$ and

$$\nu_m = \frac{1}{|B|}\int_B 2^{m/(k+2)}\mathcal{R}_{m,n}h_m(w)dw.$$

By using (1.3), we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B (\sum_m |2^{m/(k+2)}\mathcal{R}_{m,n}g_m(z)|^2)^{1/2}dz \\
&\leq \quad (\frac{1}{|B|}\sum_m \int_B |2^{m/(k+2)}\mathcal{R}_{m,n}g_m(z)|^2dz)^{1/2} \\
&\leq \quad C2^{-kn/2}(\frac{1}{|B|}\sum_m \int_B |g_m(z)|^2dz)^{1/2} \\
&\leq \quad C2^{-kn/2}\frac{|\mathcal{E}_n(B)|}{|B|}||(\sum_m |f_m|^2)^{1/2}||_\infty \\
&\leq \quad C2^n||(\sum_m |f_m|^2)^{1/2}||_\infty
\end{aligned}$$

which shows (3.1). To obtain (3.2) and (3.4), we observe the kernel $K_{m,n}$ of $\mathcal{R}_{m,n}$. After integration by parts with respect to τ, we have

$$|K_{m,n}(z,y)| \leq C_N\frac{2^m}{(1+2^m|y_2 - S(z,y_1)|)^N}|\chi_{m,n}(y_1 - z_1)|. \tag{3.6}$$

By using this estimate, we have

$$||\mathcal{R}_{m,n}f||_{L^\infty} \leq C2^n||f||_{L^\infty}.$$

Since the number of m's in $2^{-(1+k)n} \leq 2^m\delta \leq 1$ or $2^{-(k+1)(k+2)n} \leq 2^m\delta \leq 2^{-(k+1)n}$ is $O(n)$, we obtain (3.2) and (3.4). To prove (3.3), we observe that Lemma 2.4 yields

$$|S(z,y_1) - S(x,y_1)| \leq C\delta \quad \text{if } z \in B(x,\delta) \text{ and } |y_1 - x_1| \leq \delta^{1/(k+2)}.$$

If we choose C_0 in the definition of $\mathcal{E}_n(B(x;\delta))$ large enough, then we have

$$|y_2 - S(z,y_1)| \geq |y_2 - S(x,y_1)| - |S(z,y_1) - S(x,y_1)| \geq c|y_2 - S(x,y_1)|.$$

Using this with the pointwise estimate of the kernel (3.6), we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B |2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z)|dz \\
&\leq \quad C_N\int_{\substack{|y_2 - S(x,y_1)|\geq c2^{2n}\delta \\ |x_1-y_1|\leq 2^{n-m/(k+2)}}} 2^{m/(k+2)}\frac{2^m}{(1+2^m|y_2 - S(z,y_1)|)^N}dy||f_m||_{L^\infty} \\
&\leq \quad C_N2^n(2^m\delta)^{1-N}||f_m||_{L^\infty}.
\end{aligned}$$

Now it remains to show (3.5). To do this, we may write

$$\begin{aligned}&\frac{1}{|B|}\int_B |2^{m/(k+2)}\mathcal{R}_{m,n}h_m(z)-\nu_m|dz\\ &\quad\leq \frac{1}{|B|^2}\int_B\int_B 2^{m/(k+2)}|\mathcal{R}_{m,n}h_m(z)-\mathcal{R}_{m,n}h_m(w)|dzdw.\end{aligned}$$

To obtain the desired estimate, we may take into account of the size of the integrand. First, we split the integrand as

$$\begin{aligned}&\mathcal{R}_{m,n}h_m(z)-\mathcal{R}_{m,n}h_m(w)\\ &\quad= \int\int e^{i\tau(y_2-S(w,y_1))}\mathcal{F}(y,z,w,\tau)h_m(y)d\tau dy\\ &\qquad+\int\int\left(e^{i\tau(y_2-S(z,y_1))}-e^{i\tau(y_2-S(w,y_1))}\right)\chi_{m,n}(y_1-z_1)a_m(z,y,\tau)h_m(y)d\tau dy\\ &\quad= \int I_{m,n}(z,w,y)dy+\int II_{m,n}(z,w,y)dy\end{aligned}$$

where

$$\mathcal{F}(y,z,w,\tau)=\chi_{m,n}(y_1-z_1)a_m(z,y,\tau)-\chi_{m,n}(y_1-w_1)a_m(w,y,\tau).$$

Then we have

$$\begin{aligned}&I_{m,n}(z,w,y)\\ &\quad= \int\int e^{i\tau(y_2-S(w,y_1))}\left\langle z-w,\int_0^1\nabla[\chi_{m,n}(y_1-\cdot)a_m(\cdot,y,\tau)]\right\rangle\Big|_{w+s(z-w)}dsd\tau,\end{aligned}$$

$$\begin{aligned}&II_{m,n}(z,w,y)\\ &\quad= \int[e^{i\tau(y_2-S(z,y_1))}-e^{i\tau(y_2-S(w,y_1))}]\chi_{m,n}(y_1-z_1)a_m(z,y,\tau)d\tau\\ &\quad= \int e^{i\tau(y_2-S(z,y_1))}(1-e^{i\tau(S(z,y_1)-S(w,y_1))})\chi_{m,n}(y_1-z_1)a_m(z,y,\tau)d\tau.\end{aligned}$$

For $I_{m,n}$, we may directly apply integration by parts to get

$$|I_{m,n}(z,w,y)|\leq C2^{m/(k+2)-n}\delta^{1/(k+2)}\frac{2^m}{(1+2^m|y_2-S(w,y_1)|)^N}$$

because $z,w\in B(x;\delta)$ and $|y_1-w_1|\leq 2^{n-m/(k+2)}$. This gives

$$\begin{aligned}&\frac{1}{|B|^2}\iint_{B\times B}\left|2^{m/(k+2)}\int I_{m,n}(z,w,y)h_m(y)dy\right|dzdw\\ &\quad\leq C2^n(2^m\delta)^{1/(k+2)}||f_m||_{L^\infty}.\end{aligned}$$

The treatment of $II_{m,n}$ is similar to that of $I_{m,n}$ but we need more careful consideration to get a proper estimate. To perform an integration by parts, we observe the following estimates:

$$\begin{aligned}A_{m,n,N} &:= |\partial_\tau^N((1-e^{i\tau(S(z,y_1)-S(w,y_1))})a_m(z,y,\tau))| \\ &= |\sum_{j=0}^{N} C_j \partial_\tau^j (1-e^{i\tau(S(z,y_1)-S(w,y_1))})\partial_\tau^{N-j} a_m(z,y,\tau)| \\ &\le C2^{-Nm}|1-e^{i\tau(S(z,y_1)-S(w,y_1))}|\chi_{[2^{m-1},2^{m+1}]}(\tau) \\ &\quad +C\sum_{j=1}^{N}|S(z,y_1)-S(w,y_1)|^j 2^{-m(N-j)}\chi_{[2^{m-1},2^{m+1}]}(\tau).\end{aligned}$$

By using Lemma 2.4 and the assumption $\delta \le 2^{-(k+1)(k+2)n-m}$, we have

$$\begin{aligned}|1-e^{i\tau(S(z,y_1)-S(w,y_1))}| &\le |\tau||S(z,y_1)-S(w,y_1)| \\ &\le C2^m(|z_2-S(w,z_1)|+\sum_{i=1}^{k+1}|y_1-w_1|^i|z_1-w_1|^{k+2-i}) \\ &\le C(2^{n(k+1)(k+2)}2^m\delta)^{1/(k+1)}\end{aligned}$$

and

$$\begin{aligned}&|S(z,y_1)-S(w,y_1)|^j 2^{-m(N-j)} \\ &\le C(|z_2-S(w,z_1)|^j+\sum_{i=1}^{k+1}|y_1-w_1|^{ij}|z_1-w_1|^{(k+2-i)j})2^{-m(N-j)} \\ &\le C2^{-Nm}(2^{(k+1)(k+2)n}2^m\delta)^{1/(k+2)}.\end{aligned}$$

Hence we have

$$A_{m,n,N} \le C2^{-Nm}(2^{n(k+2)(k+1)}2^m\delta)^{1/(k+2)}\chi_{[2^{m-1},2^{m+1}]}(\tau).$$

Therefore, the integration by parts with respect to τ yields

$$|II_{m,n}| \le \frac{C2^m}{(1+2^m|y_2-S(z,y_1)|)^N}(2^{(k+1)(k+2)n}2^m\delta)^{1/(k+2)}.$$

Finally we obtain

$$\begin{aligned}&\frac{1}{|B|^2}\iint_{B\times B}\left|2^{m/(k+2)}\int II_{m,n}(z,w,y)h_m(y)dy\right|dzdw \\ &\le C2^n(2^{(k+1)(k+2)n}2^m\delta)^{1/(k+2)}||f_m||_{L^\infty}\end{aligned}$$

and this completes the proof. □

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