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# REMARKS ON SMALL SETS OF REALS

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ABSTRACT. We show that the Dual Borel Conjecture implies that $\mathfrak{d} > \aleph_1$ and find some topological characterizations of perfectly meager and universally meager sets.

## 1. INTRODUCTION

For $f, g \in \omega^\omega$ let $f \leq^\star g$ mean that $f(n) \leq g(n)$ for all but finitely many $n$. Let

$$\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega \ \& \ \forall h \in \omega^\omega \ \exists f \in F \ f \not\leq^\star h\},$$

$$\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega \ \& \ \forall h \in \omega^\omega \ \exists f \in F \ h \leq^\star f\}.$$

Let $\mathcal{N}$ be the ideal of measure zero subsets of $2^\omega$ with respect to the standard product measure $\mu$, and let $\mathcal{M}$ be the ideal of meager subsets of $2^\omega$. Let $+$ be the addition mod 2 on $2^\omega$. For $A, B \subseteq 2^\omega$ let $A + B = \{a + b : a \in A, b \in B\}$.

**Definition 1.** Let $X \subseteq 2^\omega$. We say that:
(1) $X$ has strong measure zero if $X + F \neq 2^\omega$ for all $F \in \mathcal{M}$,
(2) $X$ is strongly meager if $X + F \neq 2^\omega$ for all $F \in \mathcal{N}$,
(3) $X$ is meager additive if $X + F \in \mathcal{M}$ for all $F \in \mathcal{M}$,
(4) $X$ is null additive if $X + F \in \mathcal{N}$ for all $F \in \mathcal{N}$.

Let the Borel Conjecture be the statement that there are no uncountable strong measure zero sets and the Dual Borel Conjecture that there are no uncountable strongly meager sets. Both the Borel Conjecture and the Dual Borel Conjecture are (separately) known to be consistent with the ZFC [7], [4], and moreover the Dual Borel Conjecture is consistent with $\mathfrak{b} = \aleph_1$, while Rothberger showed that if $\mathfrak{b} = \aleph_1$, then there is an uncountable strong measure zero set. We will strengthen this result by showing that:

**Theorem 2.** (1) *If $\mathfrak{b} = \aleph_1$, then there exists an uncountable meager additive set.*
(2) *If $\mathfrak{d} = \aleph_1$, then there exists an uncountable null additive set. In particular, the Dual Borel Conjecture implies that $\mathfrak{d} > \aleph_1$.*

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*Proof.* The construction presented here is a modification of a construction invented by Todorcevic. Part (1) was also proved in [5] using different methods.

Let $F = \{f_\alpha : \alpha < \omega_1\}$ be a family of functions in $\omega^\omega$ such that

(1) $f_\alpha$ is strictly increasing for $\alpha < \omega_1$,
(2) $\forall \alpha < \beta \ f_\alpha \leq^\star f_\beta$.

For a perfect tree $p \subseteq 2^{<\omega}$ let $[p]$ denote the set of its branches. Every perfect subset of $2^\omega$ is a set of branches of a perfect tree.

We will build an $\omega_1$-tree $\mathbf{T}$ of perfect trees of $2^{<\omega}$. Let $\mathbf{T}_\alpha$ denote the $\alpha$-th level of $\mathbf{T}$. We require that

(1) $\forall \beta > \alpha \ \forall n \in \omega \ \forall p \in \mathbf{T}_\alpha \ \exists q \in T_\beta \ \ (q \subseteq p \ \& \ q \cap 2^n = p \cap 2^n)$,
(2) $\forall p \in \mathbf{T}_{\alpha+1} \ \forall^\infty n \ |p \cap 2^{f_\alpha(n)}| \leq 2^n$,
(3) $\forall \alpha \ \mathbf{T}_\alpha$ is countable.

Successor step. Suppose that $\mathbf{T}_\alpha$ is given. For each $p \in \mathbf{T}_\alpha$ choose $\{q_n : n \in \omega\}$ such that

(1) $\forall n \ q_n \subseteq p$,
(2) $q_n \cap 2^n = p \cap 2^n$,
(3) $[q_n] \cap [q_m] = \emptyset$ for $n \neq m$,
(4) $\forall n \ \forall^\infty k \ |q_n \cap 2^{f_\alpha(k)}| \leq 2^k$.

Set $\{q_n : n \in \omega\}$ to be the successors of $p$ on level $\mathbf{T}_{\alpha+1}$.

Limit step. Suppose that $\gamma$ is a limit ordinal and $\{\mathbf{T}_\alpha : \alpha < \gamma\}$ are already constructed.

For each $p \in \bigcup_{\alpha<\gamma} \mathbf{T}_\alpha$ and $n \in \omega$ we will construct an element $q = q(p, n)$ belonging to the level $\mathbf{T}_\gamma$ as follows.

Suppose $p = p_0 \in \mathbf{T}_{\alpha_0}$, $n = n_0$ and construct sequences $\langle \alpha_k : k \in \omega \rangle$, $\langle n_k : k \in \omega \rangle$ and $\langle p_k : k \in \omega \rangle$ such that

(1) $p_k \in \mathbf{T}_{\alpha_k}$,
(2) $\sup_k \alpha_k = \gamma$, $\lim_k n_k = \infty$,
(3) $p_{k+1} \subseteq p_k$ for all $k$,
(4) $p_{k+1} \cap 2^{n_k} = p_k \cap 2^{n_k}$,
(5) $q = \bigcap_k p_k$ is a perfect tree.

The last condition is guaranteed by the careful choice of the sequence $\langle n_k : k \in \omega \rangle$.

Let $X$ be the set obtained by selecting one element out of every tree $p \in \mathbf{T}$.

The following lemma gives the first part of the theorem.

**Lemma 3.** *If $F$ is an unbounded family in $\omega^\omega$, then $X$ is meager additive.*

*Proof.* Suppose that $H \subseteq 2^\omega$ is a meager set. It is well known (see [2], Theorem 2.2.4) that there exists $x_H \in 2^\omega$ and a strictly increasing function $f_H \in \omega^\omega$ such that

$$H \subseteq \{x \in 2^\omega : \forall^\infty n \ \exists j \in \big[f_H(n), f_H(n+1)\big) \ x(j) \neq x_H(j)\}.$$

Since we work with translations without loss of generality we can assume that $x_H(k) = 0$ for all $k$. As $F$ is unbounded there is $\alpha_0 < \omega_1$ such that

$$\exists^\infty n \ \exists k \ f_{\alpha_0}(n) < f_H(k) < f_H(k+1) < \cdots < f_H(k+2^n) < f_{\alpha_0}(n+1).$$

Fix sequences $\langle u_n, k_n : n \in \omega \rangle$ such that

$$\forall n \ \ f_{\alpha_0}(u_n) < f_H(k_n) < f_H(k_n+1) < \cdots < f_H(k_n + 2^{u_n}) < f_{\alpha_0}(u_n+1).$$

Fix $p \in \mathbf{T}_{\alpha_0+1}$ and let $z_p \in 2^\omega$ be defined as follows: given $n \in \omega$, let $\{s_1, \dots, s_{2^{u_n}}\}$ be an enumeration of $p \cap 2^{f_{\alpha_0}(u_n+1)}$. Define $z_p \restriction [f_H(k_n+i), f_H(k_n+i+1)) = s_i \restriction [f_H(k_n+i), f_H(k_n+i+1))$ for $i \leq 2^{u_n}$. This defines $z_p$ on an infinite subset of $\omega$; extend it arbitrarily to a total function. Let

$$G = \{x \in 2^\omega : \forall^\infty n \; x \restriction [f_{\alpha_0}(u_n), f_{\alpha_0}(u_n+1)) \neq z_p[f_{\alpha_0}(u_n), f_{\alpha_0}(u_n+1))\}.$$

We claim that $[p] + H \subseteq G$. Note that if $x \in [p] + H$, then there exists $y \in [p]$ such that

$$\forall^\infty n \; x \restriction [f_H(n), f_H(n+1)) \neq y \restriction [f_H(n), f_H(n+1)).$$

Fix one such $y$ and note that for sufficiently large $n$ there is an $i$ such that

$$\begin{aligned} y \restriction [f_H(k_n+i), f_H(k_n+i+1)) &= s_i \restriction [f_H(k_n+i), f_H(k_n+i+1)) \\ &= z_p \restriction [f_H(k_n+i), f_H(k_n+i+1)), \end{aligned}$$

which implies that $x \restriction [f_H(k_n+i), f_H(k_n+i+1)) \neq z_p \restriction [f_H(k_n+i), f_H(k_n+i+1))$. It follows that $x \restriction [f_{\alpha_0}(u_n), f_{\alpha_0}(u_n+1)) \neq z_p \restriction [f_{\alpha_0}(u_n), f_{\alpha_0}(u_n+1))$, which means that $x \in G$.

Let $X_{\alpha_0}$ be the collection of points selected from levels $\bigcup_{\alpha<\alpha_0} \mathbf{T}_\alpha$. We have

$$X + G \subseteq (X_{\alpha_0} + G) \cup \bigcup_{p \in \mathbf{T}_{\alpha_0}} [p] + G \in \mathcal{M},$$

which finishes the proof. □

To prove the second part of the theorem, we will use the following lemma:

**Lemma 4.** *If $F$ is a dominating family, then $X$ is null additive.*

*Proof.* The following is well known: □

**Lemma 5.** *Suppose that $H \subseteq 2^\omega$ is a null set. There exists a sequence of clopen sets $\{C_n : n \in \omega\}$ such that that for all $n$,*

(1) $\mu(C_n) < 4^{-n}$,
(2) $\forall x \in H \; \exists^\infty n \; x \in C_n$.

*Proof.* Since $H$ has measure zero, there are open sets $\langle U_n : n \in \omega \rangle$ covering $H$ such that $\mu(U_n) < 4^{-n-2}$, for $n \in \omega$. Write each set $U_n$ as a disjoint union of open basic intervals, $U_n = \bigcup_{m=1}^\infty [s^n_m]$ for $n \in \omega$, and order these sequences lexicographically in a single sequence $\{t_n : n \in \omega\}$. Put $k_0 = 0$ and for $n > 0$ let $k_n = \min\{k : \sum_{j \geq k} \mu([t_j]) < 4^{-n-1}\}$. Let

$$C_n = \bigcup_{k \in [k_n, k_{n+1})} [t_k].$$

Clearly $\mu(C_n) < 4^{-n}$ and, since all basic sets have been accounted for,

$$\forall x \in H \; \exists^\infty n \; x \in C_n.$$

□

Let $H \subseteq 2^\omega$ be a measure zero set and $\{C_n : n \in \omega\}$ the sequence given by the lemma. Since each clopen set is a union of finitely many basic sets, we can find a function $f_H \in \omega^\omega$ such that for every $n$, $C_n \subseteq 2^{f_H(n)}$. Let $\alpha_0$ and $n_0$ be such that

$$\forall n > n_0 \; f_H(n) < f_{\alpha_0}(n).$$

By modifying finitely many $C_n$'s, we can assume that $n_0 = 0$. Suppose that $p \in \mathbf{T}_{\alpha_0+1}$. Note that

$$[p] + C_n \subseteq \left[p \cap 2^{f_H(n)}\right] + C_n = D_n.$$

Since $|p \cap 2^{f_H(n)}| \le |p \cap 2^{f_{\alpha_0}(n)}|$, it follows that

$$\mu\left([p] + C_n\right) = \mu(D_n) \le 2^n \cdot 4^{-n} \le 2^{-n}.$$

Therefore, $[p] + H \subseteq G = \{x \in 2^\omega : \exists^\infty n \; x \in D_n\}$, and $\mu(G) = 0$. The rest of the proof is identical to the proof of the first part. □

## 2. Perfectly meager and universally meager sets

A set $X \subseteq 2^\omega$ is perfectly meager ($X \in \mathbf{PM}$) if $P \cap X$ is meager in $P$ for every perfect set $P$.

A set $X \subseteq 2^\omega$ is universally meager ($X \in \mathbf{UM}$, [10]) if for every Borel isomorphism $F : 2^\omega \longrightarrow 2^\omega$, $F''(X) \in \mathcal{M}$.

A set $X \subseteq 2^\omega$ is universally null ($X \in \mathbf{UN}$) if for every Borel isomorphism $F : 2^\omega \longrightarrow 2^\omega$, $F''(X) \in \mathcal{N}$.

In [10] it is shown that the universally meager sets are a category analog of universally null sets. Yet, for quite a while perfectly meager sets were viewed in this role. Theorems 6 and 7 explain this phenomenon, which has to do with the fact that the families $\mathbf{PM}$ and $\mathbf{UM}$ are not *very* different. It is easy to see that $\mathbf{UM} \subseteq \mathbf{PM}$. The Continuum Hypothesis or Martin's Axiom imply that $\mathbf{UM} \neq \mathbf{PM}$, but it is also consistent that $\mathbf{UM} = \mathbf{PM}$, [1].

The first theorem gives a simple proof of a characterization of perfectly meager sets found in [3].

**Theorem 6.** *The following are equivalent:*

1. $X \in \mathbf{PM}$,
2. *for every countable dense-in-itself set $A$ there exists a set $B \subseteq A$ such that* $\mathsf{cl}(A) = \mathsf{cl}(B)$ *and $B$ is a $G_\delta$-set relative to $X \cup A$,*
3. *for every countable set $A$ there exists a set $B \subseteq A$ such that* $\mathsf{cl}(A) = \mathsf{cl}(B)$ *and $B$ is a $G_\delta$-set relative to $X \cup A$,*
4. *for every perfect set $P$ there exists an $F_\sigma$-set $F$ such that $X \subseteq F$ and $F$ is meager in $P$.*

*Proof.* (2) → (1). Let $P$ be a perfect set and $Q \subseteq P$ a countable dense set. By (2) without loss of generality we can assume that $Q$ is a $G_\delta$ relative to $X \cup Q$. In other words $Q = (X \cup Q) \cap \bigcap G_n$, where $G_n$ are open sets. Clearly $G_n$'s are dense in $P$. It follows that $X \cap P \subseteq Q \cup P \setminus \bigcap G_n$.

(1) → (4). Since $X \in \mathbf{PM}$, there exists an $F_\sigma$-set $F_1$ such that $X \subseteq F_1$, and there exists an $F_\sigma$-set $F_2$ such that $F_2 \cap P$ is meager in $P$ and $X \cap P \subseteq F_2$. Now the set $F = (F_1 \setminus P) \cup (F_2 \cap P)$ is the set we are looking for.

(4) → (3). Suppose that $A$ is countable. By the Cantor-Bendixson Theorem, [6], $\mathsf{cl}(A) = P \dot{\cup} C$, where $P$ is perfect, $C$ is countable and open relative to $P$ (that is, $C$ is contained in an open set disjoint with $P$). If $P = \emptyset$, then $C$ is countable and closed and $A = C \setminus (C \setminus A)$ is a $G_\delta$-set in $2^\omega$. Thus assume that $P \neq \emptyset$ and let $\langle F'_n : n \in \omega\rangle$ be closed nowhere dense sets such that $X \subseteq \bigcup_n F'_n$ and $F'_n \cap P$ is closed nowhere dense in $P$ for each $n$. Let $\{F_n : n \in \omega\}$ be closed sets such that $\bigcup_n F_n = \left(\bigcup_{n\in\omega} F'_n\right) \setminus C$. Let $A' = A \setminus C$ and consider sets $A_n = A' \setminus F_n$. Since the

family $\langle A_n : n \in \omega \rangle$ has the finite intersection property, we can find a set $B' \subseteq A'$ such that

(1) $B' \setminus A_n$ is finite for all $n$,
(2) $B'$ is dense in $P$.

Let $F_n^\star = F_n \setminus B'$. Note that each set $F_n^\star$ is $F_\sigma$ since it differs from $F_n$ by a finite set. Put $B = (X \cup A) \cap \bigcap_n (2^\omega \setminus F_n^\star)$. It follows that $B' \cup (A \cap C) \subseteq B \subseteq A$, which finishes the proof.

(3) $\rightarrow$ (2) is obvious. □

**Theorem 7.** *The following are equivalent:*

(1) $X \in \mathbf{UM}$.
(2) *For every sequence of countable dense-in-itself sets* $\{A_n : n \in \omega\}$ *there exists a sequence* $B_n \subseteq A_n$ *such that* $\mathsf{cl}(A_n) = \mathsf{cl}(B_n)$ *and* $\bigcup_n B_n$ *is a* $G_\delta$*-set relative to* $X \cup \bigcup_n A_n$.
(3) *For every sequence of perfect sets* $\{P_n : n \in \omega\}$*, there exists an* $F_\sigma$*-set* $F$ *such that* $X \subseteq F$ *and* $F$ *is meager in* $P_n$ *for every* $n$.

*Proof.* (2) $\rightarrow$ (1). The following argument is a small modification of a proof from [9]. Suppose that $X \notin \mathbf{UM}$. Let $G : X \longrightarrow Y'$ be a Borel isomorphism onto a non-meager set. By Kuratowski's Theorem, $G^{-1}$ is continuous on a dense $G_\delta$-set. It follows that there exists a continuous one-to-one function $F : Y \longrightarrow X$, where $Y \notin \mathcal{M}$. Let $\{U_n : n \in \omega\}$ be enumeration of clopen subsets of $2^\omega$ such that $U_n \cap Y$ is uncountable. For each $n$, choose a countable dense-in-itself set $A_n \subseteq F''(Y \cap U_n)$. We will show that every $G_\delta$-set which is disjoint from $X \setminus \bigcup_n A_n$ is also disjoint from one of the $A_n$'s. Let $F = \bigcup_n F_n$ be an $F_\sigma$-set containing $X \setminus \bigcup_n A_n$. For every $n$ let $H_n$ be a closed set such that $H_n \cap Y = G^{-1}(F_n)$. If for every $n$, $Y_n = \text{interior}(H_n) \cap Y$ is countable, then $Y \subseteq \bigcup_n Y_n \cup \bigcup_n (H_n \setminus \text{interior}(H_n)) \in \mathcal{M}$, which is impossible. Thus there exists $m, n \in \omega$ such that $U_m \subseteq H_N$ and therefore $A_m \subseteq F\text{"}(U_m \cap Y) \subseteq F_n$.

(1) $\rightarrow$ (3). Let $\mathbb{C}$ denote the Cohen algebra. The following is a (small) fragment of Theorem 2.1 from [10].

**Theorem 8.** *For a subset* $X$ *of a perfect Polish space* $\mathbf{X}$*, the following are equivalent:*

(1) $X \in \mathbf{UM}$.
(2) *For every* $\sigma$*-ideal* $\mathcal{J}$ *in* $\mathsf{Borel}(\mathbf{X})$ *such that* $\mathsf{Borel}(\mathbf{X})/\mathcal{J} \cong \mathbb{C}$ *there is a Borel set* $B \in \mathcal{J}$ *such that* $X \subseteq B$.

*Proof.* The implication (1) $\rightarrow$ (2) that is of interest to us is a consequence of Sikorski's Theorem ([6], 15.10): if $\mathcal{J}$ is a $\sigma$-ideal in $\mathsf{Borel}(\mathbf{X})$ such that $\mathsf{Borel}(\mathbf{X})/\mathcal{J} \cong \mathbb{C}$, then there is a Borel automorphism $F : \mathbf{X} \longrightarrow \mathbf{X}$ such that

$$\forall X \in \mathsf{Borel}(\mathbf{X}) \; X \in \mathcal{M} \iff F''(X) \in \mathcal{J}. \quad \square$$

Suppose that $\{P_n : n \in \omega\}$ are given. Consider the ideal

$$\mathcal{J} = \{A : A \text{ is Borel and } \forall n \; A \cap P_n \text{ is meager in } P_n\}.$$

It is easy to see that $\mathsf{Borel}(\mathbf{X})/\mathcal{J} \cong \mathbb{C}$ (as $\mathsf{Borel}(\mathbf{X})/\mathcal{J}$ is atomless and has a countable dense subset). Therefore there exists a set $F \in \mathcal{J}$ such that $X \subseteq F$.

(3) $\rightarrow$ (2). Let $P_n = \mathsf{cl}(A_n)$. Let $F = \bigcup_n F_n$ be an $F_\sigma$-set containing $X$ such that $F$ is meager in each $P_n$. As in the proof of Theorem 6, build by induction a

set $B' \subseteq \bigcup_n A_n$ such that for $n \in \omega$,

(1) $\mathsf{cl}(B' \cap A_n) = \mathsf{cl}(A_n)$,
(2) $B' \setminus F_n$ is finite.

As before $F^\star_n = F_n \setminus B'$ is an $F_\sigma$-set and let $B = (X \cup \bigcup_n A_n) \cap \bigcap_n (2^\omega \setminus F^\star_n)$. The sets $B_n = B \cap A_n$ for $n \in \omega$ are as required. □

The assumption in Theorem 7(2) that the sets $A_n$ are dense-in-itself is necessary since we have the following:

**Theorem 9.** *There exists a set $X \in \mathbf{UM}$ and a family $\{A_n : n \in \omega\}$ of countable subsets of $X$ such that if $B_n \subseteq A_n$ is such that $\mathsf{cl}(B_n) = \mathsf{cl}(A_n)$ for each $n$, then $\bigcup_n B_n$ is not a $G_\delta$ relative to $X$.*

*Proof.* Recall that a set of reals is called a $\lambda$-set if all of its countable subsets are relative $G_\delta$-sets. Theorem 7 implies readily that all $\lambda$-sets and unions of countable sets with $\lambda$-sets are universally meager. Rothberger showed (5.6 of [8]) that there exists a set $Y \subseteq 2^\omega$ and a countable set $C \subseteq 2^\omega$ such that

(1) $Y$ is a $\lambda$-set,
(2) $X = Y \cup C$ is not a $\lambda$-set, that is $C$ is not a $G_\delta$-set relative to $Y$.

Let $Y$ and $C$ be as above. Write $C = \bigcup_n A_n$, where each $A_n$ is infinite and discrete, that is $x \notin \mathsf{cl}(A_n \setminus \{x\})$ for $x \in A_n$. It follows that if $B_n \subseteq A_n$ and $\mathsf{cl}(B_n) = \mathsf{cl}(A_n)$, then $A_n = B_n$, which finishes the proof.

□

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