THE BANACH ENVELOPE OF PALEY-WIENER TYPE SPACES

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Abstract. We give an explicit computation of the Banach envelope for the Paley-Wiener type spaces $E^p$, $0 < p < 1$. This answers a question by Joel Shapiro.

1. Introduction

The Paley-Wiener type space $E^p$ (a precise definition is given below) consists of certain band-limited functions [1]. We will show that for $0 < p < 1$ this space can be identified as a complemented subspace of the direct sum of two classical Hardy spaces. Since the Banach envelopes of the Hardy spaces are known, we are able to establish a necessary and sufficient condition for entire functions to belong to the envelope $E^p$.

For an open subset $\Omega \subseteq \mathbb{C}$ let $A(\Omega)$ be the space of holomorphic functions on $\Omega$. An entire function $f$ is of exponential type $\tau > 0$ if for all $\epsilon > 0$ there is a $C_\epsilon > 0$ such that for all $z \in \mathbb{C}$ we have $|f(z)| \leq C_\epsilon e^{(\tau + \epsilon)|z|}$. For $0 < p < \infty$ let $E^{p,\tau}$ be the space of entire functions of exponential type $\tau$ such that their restrictions to the real axis are in $L^p(\mathbb{R})$:

$$E^{p,\tau} = \{ f \in A(\mathbb{C}) : f \text{ has exponential type } \tau, f|_\mathbb{R} \in L^p(\mathbb{R}) \}.$$  

We will only consider $\tau = \pi$ and write $E^p = E^{p,\pi}$ from now on. This causes no loss of generality, because we can simply rescale a function $f \in E^{p,\tau}$ to obtain $\text{supp}(f|_\mathbb{R}) \subseteq [-\pi, \pi]$. The quantity

$$\|f\|_{E^p} = \left( \int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p} = \|f|_\mathbb{R}\|_{L^p}$$

defines a norm on $E^p$ for $1 \leq p < \infty$ and a quasi-norm for $0 < p < 1$.

These spaces $E^p$ are complete and hence can be identified with closed subspaces of $L^p(\mathbb{R})$ (e.g. see [3] or [10]). A classical theorem of Paley and Wiener gives a characterization of $E^2$ as the image of the inverse Fourier transform of $L^2[-\pi, \pi]$. Hence functions in $E^2$ have compactly supported Fourier transform, i.e. they are band-limited. These functions are important to signal processing due to their sampling properties (Shannon sampling theorem). Since $\|f\|_q \leq C_{pq}\|f\|_p$ for $f \in E^p, 0 < q \leq p, p q \leq 1$, we have $E^p \subset E^q$ for $0 < p \leq q$. In particular,
$E^p \subset E^1 \subset E^2$ for $0 < p < 1$. A characterization of $E^p$ for $0 < p < 1$ as a discrete Hardy space is shown in [6].

Let $X$ be a quasi-normed space with separating dual. Then the Banach envelope $X_e$ of $X$ is the completion of $(X, \| \cdot \|_C)$ where $C$ is the convex hull of the closed unit ball $B_X$ and $\| \cdot \|_C$ is the Minkowski functional of $C$. $X_e$ is a Banach space. The Banach envelope is characterized up to isomorphism by $(X_e)^* = X$ and $\overline{X} = X_e$. Every operator $T : X \to Y$ extends uniquely to $\hat{T} : X_e \to Y_e$ ([8], [11], [17] and [20]).

The standard example for finding a Banach envelope is $\ell^p$ for $0 < p < 1$. In this case we have $\ell^p \subset \ell^1$, $\ell^p$ is dense in $\ell^1$ and $(\ell^p)^* = \ell^\infty = (\ell^1)^*$. Therefore, $\ell^p_c = \ell^1$.

The spaces $E^p$ are nested as well, $E^p \subset E^1$ for $0 < p < 1$. Furthermore, $E^p$ is dense in $E^1$ since it contains all Schwartz functions with Fourier transform supported in $[-\pi, \pi]$. This makes $E^1$ a candidate for the envelope of $E^p$, but it turns out that $E^p_c$ is a certain weighted Bergman space of entire functions different from $E^1$. The proof relies heavily on the theory of Hardy spaces $H^p$. We use a deep result by Duren, Romberg and Shields [2] that the Banach envelope of $H^p$ over the unit disk is a certain weighted $L^1$-Bergman space (see below).

The problem of identifying the Banach envelope of $E^p$ as a space of entire functions was originally posed by Joel Shapiro. I would like to thank Professor Nigel Kalton for communicating this problem and his helpful suggestions. I would also like to thank the referee for some very constructive comments.

2. Preliminaries

We recall the classical Hardy spaces of the disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and the upper half planes $\mathbb{C}_\pm = \{ z \in \mathbb{C} : \pm y \in \mathbb{R} \}$,

$$H^p(\mathbb{D}) = \left\{ f \in \mathcal{A}(\mathbb{D}) : \| f \|_{H^p(\mathbb{D})}^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty \right\},$$

$$H^p(\mathbb{C}_+) = \left\{ f \in \mathcal{A}(\mathbb{C}_+) : \| f \|_{H^p(\mathbb{C}_+)}^p = \sup_{y > 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty \right\}.$$

Analogously we define for the lower half plane

$$H^p(\mathbb{C}_-) = \left\{ f \in \mathcal{A}(\mathbb{C}_-) : \| f \|_{H^p(\mathbb{C}_-)}^p = \sup_{y < 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty \right\}.$$

An isometric isomorphism between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{C}_-)$ is given by $f(z) \mapsto \overline{f(z)}$.

Let $\mathcal{S}$ be the space of Schwartz functions and $\mathcal{S}'$ the space of tempered distributions. Every $f \in H^p(\mathbb{C}_+)$ is uniquely determined by its boundary value distribution $f^b = \lim_{y \to 0} f(x + iy) \in \mathcal{S}'$. Denote the space of these boundary distributions by $H^p_b(\mathbb{R})$, and let $\| f \|_{H^p_b(\mathbb{R})} = \| f^b \|_{H^p(\mathbb{C}_+)}$. In the same way define $H^p_c(\mathbb{R})$. The real and imaginary parts of $f$ have a boundary value distribution in the real Hardy space as defined in [2], [18]. Hence the Fourier transform of $f \in H^p_b(\mathbb{R}), 0 < p < 1$, is a continuous functions and satisfies $\hat{f} \in [-1, 0]$. More precisely, we have $|\hat{f}(\xi)| \leq C|\xi|^{1/p - 1} \| f \|_{H^p_b(\mathbb{R})}$. Transferring to the lower half plane shows that $f \in H^p(\mathbb{R})$ has $\text{supp} \hat{f} \subseteq (-\infty, 0]$. All these results can be found e.g. in [2], [7], [11], [18].
The Bergman spaces over the disc and the upper half plane are defined for $0 < p < \infty$, $\alpha > -1$ as
\[
A^{p,\alpha}(\mathbb{D}) = \left\{ f \in \mathcal{A}(\mathbb{D}) : \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(x + iy)|^p (1 - |z|^2)^\alpha dx \, dy < \infty \right\},
\]
\[
A^{p,\alpha}(\mathbb{C}_+) = \left\{ f \in \mathcal{A}(\mathbb{C}_+) : \|f\|_{p,\alpha}^p = \int_{\mathbb{C}_+} |f(x + iy)|^p y^\alpha dx \, dy < \infty \right\}.
\]

Analogously we define $A^{p,\alpha}(\mathbb{C}_-)$. The Banach envelope of $H^p(\mathbb{D})$ was identified by Duren, Romberg and Shields [5].

**Proposition 2.1.** $H^p(\mathbb{D}) = A^{1,1/p-2}(\mathbb{D})$.

For a different approach in the setting of Besov and Triebel-Lizorkin spaces see [11] and [12]. In particular, it is shown that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $1/2 < p < 1$ we have

**Proposition 2.2.** $H^p(\Omega) = A^{1,1/p-2}(\Omega)$.

We will need the analogous statement for the upper and lower half plane. The following proposition is certainly well known; for completeness we give a proof using a conformal map from $\mathbb{C}_+$ onto $\mathbb{D}$.

**Proposition 2.3.** $H^p(\mathbb{C}_+) = A^{1,1/p-2}(\mathbb{C}_+)$.

**Proof.** It is enough to consider the upper half plane. We use the conformal map $w = \phi(z) = \frac{z}{|z|^2}$ from $\mathbb{C}_+$ onto $\mathbb{D}$. Then with $F(z) = f(\phi(z))$, $z = x + iy$, we have $f(w) \in H^p(\mathbb{D})$ if and only if $F(z)/(z + i)^{2/p} \in H^p(\mathbb{C}_+)$ (see [4], [10]). For $\alpha > -1$ we get
\[
\int_{\mathbb{D}} |f(w)|^p (1 - |w|^2)^\alpha dx \, dy = \int_{\mathbb{C}_+} |F(z)|^p (1 - |\phi(z)|^2)^\alpha |\phi'(z)|^2 dx \, dy.
\]

A short computation shows $1 - |\phi(z)|^2 = \frac{4y}{|z|^2}$ and $|\phi'(z)|^2 = \frac{4}{|z|^4}$. Therefore,
\[
\int_{\mathbb{D}} |f(w)|^p (1 - |w|^2)^\alpha dx \, dy = 4^{\alpha+1} \int_{\mathbb{C}_+} \left| \frac{F(z)}{(z + i)^{2\alpha+4}/p} \right|^p y^\alpha dx \, dy.
\]

Hence $F(z) \in H^p_\alpha(\mathbb{C}_+)$ if and only if $f(w)(\phi^{-1}(w) + i)^{2/p} \in A^{1,\alpha}(\mathbb{D})$ where $\alpha = 1/p - 2$. Then $2\alpha + 4 = 2/p$, and mapping back to the upper half plane shows $F(z) \in H^p_\alpha(\mathbb{C}_+)$ if and only if $F(z) \in A^{1,\alpha}(\mathbb{C}_+)$. \qed

### 3. The Banach envelope of $E^p$

The next proposition is due to Plancherel and Pólya [14].

**Proposition 3.1.** Let $0 < p < \infty$ and $f \in E^p$. Then for every $y \in \mathbb{R}$ we have
\[
\int_{-\infty}^{\infty} |f(x + iy)|^p dx \leq e^{p\pi|y|} \int_{-\infty}^{\infty} |f(x)|^p dx.
\]

For an entire function $f$ let $f_{\pm}(z) = e^{\pm inz} f|_{\mathbb{C}_\pm}(z)$ and $j(f) = (f_+, f_-)$. Then from Proposition [14] it follows that $f \mapsto f_{\pm\pi}$ is an isometric isomorphism of $E^p$ into $H^p(\mathbb{C}_\pm)$. Hence $j$ embeds $E^p$ into $H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$.
Crucial to compute the envelope of $E^p$ is the following.

**Lemma 3.2.** $j(E^p)$ is complemented in $H^p(\mathbb{C}_+ \oplus H^p(\mathbb{C}_-)$.  

**Proof.** We construct a bounded projection onto $j(E^p)$. Choose $\phi, \psi \in \mathcal{S}$ such that $\text{supp } \hat{\phi} \subseteq [-2\pi, \pi], \text{supp } \hat{\psi} \subseteq [-\pi, 2\pi]$ and $\phi(x) + \psi(x) = 1$ for all $x \in [-\pi, \pi]$. This can be done by a suitable partition of unity on the Fourier transform side. Then let $T : S' \times S' \to S'$ be defined by  

$$T(u, v) = u * \phi + v * \psi.$$  

We have $\text{supp } T(u, v) \subseteq [-2\pi, 2\pi]$, and if $u$ has $\text{supp } \hat{u} \subseteq [-\pi, \pi]$, then $T(u, u) = u$. Write $u_{\pm \pi} = e^{\pm i\pi x} u$ for $u \in S'$. Suppose $(u, v) \in H^p_\sigma(\mathbb{R}) \oplus H^p_\sigma(\mathbb{R})$. Then $T(u_{-\pi}, v_{\pi})$ is continuous and supported in $[-\pi, \pi]$. This shows that $T(u_{-\pi}, v_{\pi})$ has an extension to a function in $E^2$. Non-tangential (distributional) boundary values of functions in $H^p(\mathbb{C}_+)$ are in $L^p(\mathbb{R})$, and we have $\|u * \Phi\|_{H^p_\sigma(\mathbb{R})} \leq C\|u\|_{H^p_\sigma(\mathbb{R})}$ for $u \in H^p_\sigma(\mathbb{R}), \Phi \in S' [13]$. Hence $T(u_{-\pi}, v_{\pi}) \in L^p(\mathbb{R})$, and $T(u_{-\pi}, v_{\pi})$ extends to a function in $E^p$. This extension has the explicit form $Q : H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-) \to E^p$,  

$$Q(f, g)(z) = \langle T(f^b_{-\pi}, g^b_{\pi}), e^{itz} \rangle = \int_{-\pi}^{\pi} T(f^b_{-\pi}, g^b_{\pi}) e^{itz} dt.$$  

By choice of $\phi, \psi$ we have $Q(j(f)) = f$, and hence $P = jQ$ is the desired projection. $\square$  

Now we arrive at our characterization of $E^p_{\epsilon q}$.

**Theorem 3.3.** An entire function $f$ belongs to $E^p_{\epsilon q}$ if and only if  

$$\|f\| = \int_{\mathbb{C}} e^{-\pi|y|}|y|^{1/p - 2} |f(x + iy)| dx \, dy < \infty.$$  

Moreover, $\|\cdot\|$ is equivalent to the norm of $E^p_{\epsilon q}$.

**Proof.** Let $\alpha = 1/p - 2$. Define  

$$Z = \{ f \in \mathcal{A}(\mathbb{C}) : f_\pi \in A^{1, \alpha}(\mathbb{C}_+), f_{-\pi} \in A^{1, \alpha}(\mathbb{C}_-) \} \subset A^{1, \alpha}(\mathbb{C}_+) \oplus A^{1, \alpha}(\mathbb{C}_-).$$  

We will show $E^p_{\epsilon q} = Z$ with equivalence of norms. We have $H^p(\mathbb{C}_+) \subset A^{1, \alpha}(\mathbb{C}_+)$, and hence $E^p \subset Z$. It is crucial to observe that the operator $Q$ from the previous proof extends to $Q : A^{1, \alpha}(\mathbb{C}_+) \oplus A^{1, \alpha}(\mathbb{C}_-) \to E^p_{\epsilon q}$ while preserving the defining equation, i.e. $\hat{Q}(f, g)(z) = \langle T(f^b_{-\pi}, g^b_{\pi}), e^{itz} \rangle$. This follows from the characterization of boundary value distributions for functions in $A^{1, \alpha}(\mathbb{C}_+) [13]$. These distributions are uniquely determined by their values on $\mathcal{S}$, and passing from $f \in A^{q, \alpha}(\mathbb{C}_+)$ to $f^b \in A^{1, \alpha}(\mathbb{R})$ is a continuous operation. Therefore, we have $E^p_{\epsilon q} \subset E^2$ and $Q(j(f)) = f$ for $f \in Z$. We conclude that $j(E^p_{\epsilon q})$ is a complemented subspace of $H^p_\sigma(\mathbb{C}_+) \oplus H^p_\sigma(\mathbb{C}_-) = A^{1, \alpha}(\mathbb{C}_+) \oplus A^{1, \alpha}(\mathbb{C}_-)$. Hence $E^p_{\epsilon q} \subseteq Z$, and on $j(E^p_{\epsilon q})$ the envelope-norm is equivalent to the norm of $A^{1, \alpha}(\mathbb{C}_+) \oplus A^{1, \alpha}(\mathbb{C}_-)$. It remains to establish density of $j(E^p)$ in $j(Z)$. Pick $f \in Z$. Then there are $(h_n, g_n) \in H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-) \subset E^p_{\epsilon q}$ such that $(h_n, g_n) \to (f_\pi, f_{-\pi}) = j(f) \in A^{1, \alpha}(\mathbb{C}_+) \oplus A^{1, \alpha}(\mathbb{C}_-)$. Then $Q(h_n, g_n) \in E^p_{\epsilon q}$ and $Q(h_n, g_n) \to Q(j(f)) = f$. $\square$  

Let us see that $E^p_{\epsilon q} \neq E^1$. For $f \in E^1, y > 0$ let $f_y(x) = f(x + iy)$. Then by Proposition [13] we have $\|f_y\|_{L^1} \leq e^{\pi y}\|f_0\|_{L^1} = \|f\|_{E^1}$. For a given $y > 0$ we choose a function that satisfies the converse inequality up to a constant as follows: Take $\phi_\epsilon \in \mathcal{S}$ such that $\text{supp } \phi_\epsilon \subseteq [-\pi, -\pi + \epsilon]$, and let $f_\epsilon(z) = \langle \phi_\epsilon, e^{itz} \rangle$. Then for fixed
$y_0 > 0$ we choose $\epsilon > 0$ small enough to give $\|f_y^+\|_{L^1} \geq C e^{\pi y} \|f^+\|_{L^1}$ for all $0 \leq y \leq y_0$. We obtain $\|f^+\|_{L^p} = \int_{-\infty}^{\infty} e^{-\pi y} |y|^{1/p-2} \|f_y^+\|_{L^1} dy \geq C \int_{y_0}^{\infty} y^{1/p-2} \|f^+\|_{L^1} dy = (1/p-1)^{-1} y_0^{1/p-1} \|f^+\|_{L^1}$. Since $1/p - 1 > 0$ we conclude that the $E^1$-norm and the $E^p_c$-norm are not equivalent.

A result from [9] (see also [20]) is $A^{1,\alpha}(\mathbb{D}) \approx \ell^1$, and therefore we use a result of Pelczynski [13] that every complemented subspace of $\ell^1$ is isomorphic to $\ell^1$.

**Corollary 4.3.** $E^p_c \approx \ell^1$.

4. The $q$-envelope of $E^p$

The result from the previous section can be generalized to the $q$-envelopes. Let $X$ be a quasi-normed space with separating dual and $0 < q \leq 1$. Then the Banach $q$-envelope $X_{c,q}$ of $X$ is the completion of $(X, \| \cdot \|_{C_q})$ where $C_q$ is the $q$-convex hull of the closed unit ball $B_X$ and $\| \cdot \|_{C_q}$ is the Minkowski functional of $C_q$. $X_{c,q}$ is a complete quasi-normed space.

The results in [3] and [20] give the $q$-envelopes of the Hardy spaces.

**Proposition 4.1.** For $0 < p < q \leq 1$ we have $H^p_{c,q}(\mathbb{D}) = A^{q,p-2}(\mathbb{D})$.

The proofs of Theorem 3.3 and Proposition 2.3 work analogously for the $q$-envelope, $0 < q < 1$, if we use $\alpha = q/p - 2$ and $A^{q,\alpha}(C_\pm)$. Hence

**Theorem 4.2.** An entire function $f$ belongs to $E^p_{c,q}$ if and only if

$$\|f\| = \left( \int_{\mathbb{C}} e^{-\pi |y|} |y|^{q/p-2} |f(x+iy)|^q dx dy \right)^{1/q} < \infty.$$ Moreover, $\| \cdot \|$ is equivalent to the quasi-norm of $E^p_{c,q}$.

In the case of [9] (see also [20]) it is shown that $A^{q,\alpha}(\mathbb{D}) \approx \ell^q$, and therefore we use a result of Stiles [19] or [8] for $q < 1$ that every complemented subspace of $\ell^q$ is isomorphic to $\ell^q$.

**Corollary 4.3.** $E^q_{c,q} \approx \ell^q$.

**References**


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