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PERFECT CLIQUES AND G_δ COLORINGS OF POLISH SPACES

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ABSTRACT. A *coloring* of a set X is any subset C of $[X]^N$, where $N > 1$ is a natural number. We give some sufficient conditions for the existence of a perfect C-homogeneous set, in the case where C is G_δ and X is a Polish space. In particular, we show that it is sufficient that there exist C-homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable k-clique, where k is a natural number or $\aleph_0$ (a set K is a *k-clique* in X if the convex hull of any of its k-element subsets is not contained in X).

1. Introduction

For a set X and natural number N, $[X]^N$ denotes the collection of all N-element subsets of X. A (two-color) *coloring* of X is (represented by) a set $C \subset [X]^N$. We identify $[X]^N$ with a suitable subspace of the product X^N. We are interested in the following problem: find sufficient conditions for the existence of a perfect C-homogeneous set $P \subset X$, where X is a Polish space and $C \subset [X]^N$ is open (or more generally G_δ). A natural example of this problem is the following: let $X \subset \mathbb{R}^N$ be closed and $C = \{s \in [X]^k\colon \operatorname{conv} s \not\subset X\}$. Then C is open and a C-homogeneous set is called a k-clique in X. It is known (see [3]) that there exists a closed set $X \subset \mathbb{R}^N$ such that X is not a countable union of convex sets but every k-clique in X is countable for every $k < \omega$. On the other hand, it is proved in [3] that if a closed set $X \subset \mathbb{R}^N$ contains an uncountable k-clique for some k, then it contains a perfect 3-clique.

We prove that if C is a G_δ coloring of a Polish space and there are no perfect C-homogeneous sets, then there is a countable ordinal γ such that the Cantor-Bendixson rank of every C-homogeneous set is $< \gamma$. In the context of cliques, this strengthens the result of Kojman [2] (see Theorem 3.1(a) below). From our result it follows that if C is a G_δ coloring of an analytic space, then either there exists a perfect C-homogeneous set or all C-homogeneous sets are countable. This is not true for F_σ colorings: a result of Shelah [4] states that consistently there exist F_σ 2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set $X \subset \mathbb{R}^2$ contains an uncountable $\aleph_0$-clique, then X contains also a perfect 3-clique.

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1.1. **Notation.** Any subset of $[X]^N$ is called a *coloring* (or *an N-coloring*) of X. We write $\neg C$ instead of $[X]^N \setminus C$. A set $S \subset X$ is *C-homogeneous* if $[A]^N \subset C$. We identify $[X]^N$ with the subspace of X^N consisting of all N-tuples $(x_0, \dots, x_{N-1})$ with $x_i \neq x_j$ for $i \neq j$. Thus we may consider topological properties of colorings. If $f\colon X \to Y$ is a function, then we write $f[S]$ for the image of a set $S \subset X$ and $f(s)$ for the value at a point $s \in X$. By a *perfect set* we mean a compact, nonempty, topological space with no isolated points.

2. On colorings

First we recall a simple result on open 2-colorings of analytic spaces which can be found in Todorčević-Farah's book [5, p. 81].

Proposition 2.1. *Let X be an analytic space and let $C \subset [X]^2$ be open. Then either there exists a perfect C-homogeneous set or else X is a countable union of $\neg C$-homogeneous sets, i.e. $X = \bigcup_{n\in\omega} A_n$ where $[A_n]^2 \cap C = \emptyset$ for every $n \in \omega$.*

The above result is no longer valid when we replace the word “open” with “closed”; see [5, p. 83]. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of 2^ω such that there are no uncountable homogeneous sets either of this color or of its complement; see Blass' example [1]. In Blass' example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for G_δ colorings.

For a topological space Y and an ordinal α we denote by $Y^{(\alpha)}$ the α-derivative of Y; the *Cantor-Bendixson rank* of Y is the minimal ordinal γ such that $Y^{(\gamma+1)}$ is empty.

Theorem 2.2. *Let C be a G_δ N-coloring of a Polish space X. If for every countable ordinal γ there exists a C-homogeneous set of the Cantor-Bendixson rank $\geq \gamma$, then X contains a perfect C-homogeneous set.*

Proof. Fix a countable base $\mathcal{B}$ in X and fix a complete metric on X. Let $C = \bigcap_{n\in\omega} C_n$, where each C_n is open and $C_{n+1} \subset C_n$. We will construct a tree of open sets

$$T = \{u_s \colon s \in 2^{<\omega}\}$$

with the following properties:

(i) $\operatorname{cl} u_{s^\frown i} \subset u_s$, $\operatorname{cl} u_s \cap \operatorname{cl} u_t = \emptyset$ if s, t are incompatible and $\operatorname{diam}(u_s) < 2^{-\operatorname{length}(s)}$;
(ii) if $k < \omega$ and $s_0, \dots, s_{N-1} \in 2^k$ are pairwise distinct, then $\{x_0, \dots, x_{N-1}\} \in C_k$ whenever $x_i \in u_{s_i}$, $i < N$;
(iii) if $k < \omega$, then for each $\gamma < \omega_1$ there exists a C-homogeneous set $P = P_{k,\gamma}$ such that $P^{(\gamma)} \cap u_s \neq \emptyset$ for each $s \in 2^k$.

We start with $u_\emptyset = X$. Suppose that u_s has been defined for all $s \in 2^{\leqslant k}$. Fix $\gamma < \omega_1$ and consider $P = P_{k,\gamma+1}$, as in (iii). Then for each $s \in 2^k$ the set $P^{(\gamma)} \cap u_s$ is infinite. Fix $S \subset P^{(\gamma)}$ such that $|S \cap u_s| = 2$ for each $s \in 2^k$. Next, enlarge each $x \in S \cap u_s$ to a small open set $v_x \in \mathcal{B}$, contained in u_s, such that $\{y_0, \dots, y_{N-1}\} \in C_{k+1}$ whenever y_i are taken from pairwise distinct v_x's. This is possible, because C_{k+1} is open. Let $\varphi(\gamma) = \{v_x \colon x \in S\}$. This defines a mapping $\varphi\colon \omega_1 \to [\mathcal{B}]^{<\omega}$. As $\mathcal{B}$ is countable, there is unbounded $F \subset \omega_1$ such that $\varphi \restriction F$ is constant, say $\{v_{s^\frown i} \colon s \in 2^k, i < 2\}$, where $v_{s^\frown i} \subset u_s$. Set $u_{s^\frown i} = v_{s^\frown i}$. Observe that (i) holds if we let v_x's be small enough. Also (ii) holds, by the definition of v_x's. Finally, (iii) holds, because $P^{(\gamma)}_{k,\gamma+1} \cap u_t \neq \emptyset$ for $t \in 2^{k+1}$ whenever $\gamma \in F$. By (ii) the perfect set obtained from this construction is C-homogeneous. □

Using the above theorem we obtain the following corollary which, for the case of 2-colorings of Polish spaces, was mentioned by Shelah [4, Remark 1.14]:

Corollary 2.3. *Let $1 \leqslant N < \omega$ and let C be a G_δ N-coloring of an analytic space X. If there exists an uncountable C-homogeneous set, then there exists also a perfect one.*

Proof. Let $f \colon \omega^\omega \to X$ be a continuous surjection and define $C' = \{s \in [\omega^\omega]^2 \colon f[s] \in C\}$. If $K \subset X$ is C-homogeneous and $K = f[K']$ where $f \restriction K'$ is one-to-one, then K' is C'-homogeneous. If K is uncountable then so is K' and by Theorem 2.2 we get a perfect set $P \subset \omega^\omega$ which is C'-homogeneous. Then $f \restriction P$ is one-to-one and hence $f[P]$ is a perfect C-homogeneous set. □

3. Applications to convexity

Let $X \subset E$, where E is a real vector space. A subset K of X is a *k-clique* (k can be a cardinal or just a natural number; we will use this notion for $k < \omega$ and $k = \aleph_0$) if $\operatorname{conv} S \not\subset X$ whenever $S \in [K]^k$. If E is finite-dimensional and $k > \dim E$, then we can define the notion of a *strong k-clique* replacing $\operatorname{conv} S$ by $\operatorname{int}\operatorname{conv} S$ in the definition. A finite set $S \subset X$ is (*strongly*) *defected* in X if $\operatorname{conv} S \not\subset X$ ($\operatorname{int}\operatorname{conv} S \not\subset X$). It is clear that the relation of strong defectedness is open and defectedness is open provided that X is closed.

Applying the results of the previous section we get the following:

Theorem 3.1. (a) *Let X be a closed set in a Polish linear space and let $N < \omega$. If X does not contain a perfect N-clique, then all N-cliques in X are countable. Moreover, there exists an ordinal $\gamma < \omega_1$ which bounds the Cantor-Bendixson ranks of all N-cliques in X.*

(b) *Let X be an analytic subset of $\mathbb{R}^m$. If $m < N < \omega$ and X contains an uncountable strong N-clique, then X contains also a perfect one.*

Theorem 3.1(a) was proved, under the stronger assumption that X is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

Theorem 3.2. *Let $X \subset \mathbb{R}^2$ be analytic and assume that X contains an uncountable $\aleph_0$-clique. Then either X contains a perfect strong 3-clique or else, for some line L, $X \cap L$ contains a perfect 2-clique. In particular X contains a perfect 3-clique.*

Proof. Fix a continuous function $f \colon \omega^\omega \to X$ onto X and fix an uncountable $\aleph_0$-clique $K \subset X$. We may assume that every line contains only countably many points of L: otherwise, for some line L, $X \cap L$ contains an uncountable $\aleph_0$-clique, so it contains a perfect 2-clique (Proposition 2.1). Fix uncountable $K' \subset \omega^\omega$ such that $f \restriction K'$ is a bijection onto K.

A finite collection $\{u_0, \ldots, u_{k-1}\}$ of open subsets of ω^ω will be called *relevant* if each u_i contains uncountably many points of K', $\operatorname{cl} u_i \cap \operatorname{cl} u_j = \emptyset$ whenever $i < j < k$ and $\operatorname{int}\operatorname{conv}\{f(x_0), f(x_1), f(x_2)\} \not\subset X$ whenever x_0, x_1, x_2 are taken from pairwise distinct u_i's. To find a perfect strong 3-clique in X, it suffices to construct a perfect tree of open sets in ω^ω with relevant levels. If P is a perfect set obtained from such a tree, then $f \restriction P$ is one-to-one and $f[P]$ is a perfect strong 3-clique.

Suppose that we have a relevant collection $\{u_0, \ldots, u_k\}$. We have to show that it is possible to split each u_i to obtain again a relevant collection. We will split u_k. Let $L = K' \cap u_k$ and pick $y_i \in u_i$ for $i < k$. Define $c_i \colon [L]^2 \to 2$ by letting $c_i(x_0, x_1) = 1$ iff $\operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$. Observe that there are no infinite c_i-homogeneous sets of color 0: if $S \subset L$ is infinite, then, by Carathéodory's theorem, there is $s \in [S]^3$ such that $f[s]$ is defected in X (because $f[S]$ is defected) and hence for some $x_0, x_1 \in s$ we have $\operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$, because $\operatorname{conv} T \subset \bigcup_{x,y\in T} \operatorname{conv}\{x, y, p\}$ for $T \subset \mathbb{R}^2$, $p \in \mathbb{R}^2$. Using k times the theorem of Dushnik-Miller we obtain uncountable $L' \subset L$ which is c_i-homogeneous of color 1 for $i < k$. Shrinking L' we may assume that each nonempty open subset of L' is uncountable. Now choose disjoint open sets v_0, v_1 with $\operatorname{cl} v_j \subset u_k$ and $v_j \cap L' \neq \emptyset$ for $j < 2$. To finish the proof we need the following geometric property of the plane:

Claim 3.3. *Let $A, B \subset X \subset \mathbb{R}^2$ and $c \in \mathbb{R}^2$ be such that A, B are uncountable, each line contains countably many points of $A \cup B$ and $\operatorname{conv}\{a, b, c\} \not\subset X$ whenever $a \in A$, $b \in B$. Then there are $a_0 \in A$, $b_0 \in B$ such that $\operatorname{int}\operatorname{conv}\{a_0, b_0, c\} \not\subset X$.*

Proof. Suppose this is not true. Observe that, replacing A and B if necessary, we may assume that for some $b_0 \in B$, $[a, b_0] \cup [a, c] \not\subset X$ whenever $a \in A$. Indeed, if $[b, c] \subset X$ for some $b \in B$, then we take $b_0 = b$, otherwise we take any $a_0 \in A$ and we replace A and B. Now, without loss of generality, we may assume that $b_0 = (-1, 0)$, $c = (1, 0)$ and A is contained in $(-1, 1) \times (0, 1)$. Now, if some vertical line contains two elements of A, then we are done: we take $a_0 \in A$ such that some $a_1 \in A$ is below a_0; then the relative interiors of segments $[b_0, a_1]$, $[c, a_1]$ are contained in the interior of $\operatorname{conv}\{a_0, b_0, c\}$.

Assume that each vertical line contains at most one element of A. As A is uncountable, there is $a_1 \in A$ such that arbitrarily close to a_1 there are uncountably many points both on the left and the right side of a_1. Suppose now that e.g. $\{b_0, a_1\}$ is defected in X. As $[b_0, a_1]$ contains only countably many points of A, we can find $a_2 \in A$ which is close enough to a_1, on the left side of a_1 and not in $[b_0, a_1]$. If a_2 is below $[b_0, a_1]$, then we can set $a_0 = a_1$, otherwise we can set $a_0 = a_2$. □

Let $i = 0$. Using Claim 3.3 for $A = f[v_0 \cap L']$, $B = f[v_1 \cap L']$ and $c = f(y_i)$ we get $x_j \in v_j$ such that $\operatorname{int}\operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$. By continuity, shrink v_0, v_1 and enlarge y_i to an open set $u_i' \subset u_i$ such that each triple selected from $f[v_0] \times f[v_1] \times f[u_i']$ is strongly defected in X. Repeat the same argument for each $i < k$, obtaining a relevant collection $\{u_0', \ldots, u_{k-1}', v_0', v_1'\}$ which realizes the splitting of u_k. This completes the proof. □

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