PERFECT CLIQUES AND $G_\delta$ COLORINGS OF POLISH SPACES

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(Communicated by Carl G. Jockusch, Jr.)

Abstract. A coloring of a set $X$ is any subset $C$ of $[X]^N$, where $N > 1$ is a natural number. We give some sufficient conditions for the existence of a perfect $C$-homogeneous set, in the case where $C$ is $G_\delta$ and $X$ is a Polish space. In particular, we show that it is sufficient that there exist $C$-homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable $k$-clique, where $k$ is a natural number or $\aleph_0$ (a set $K$ is a $k$-clique in $X$ if the convex hull of any of its $k$-element subsets is not contained in $X$).

1. Introduction

For a set $X$ and natural number $N$, $[X]^N$ denotes the collection of all $N$-element subsets of $X$. A (two-color) coloring of $X$ is (represented by) a set $C \subset [X]^N$. We identify $[X]^N$ with a suitable subspace of the product $X^N$. We are interested in the following problem: find sufficient conditions for the existence of a perfect $C$-homogeneous set $P \subset X$, where $X$ is a Polish space and $C \subset [X]^N$ is open (or more generally $G_\delta$). A natural example of this problem is the following: let $X \subset \mathbb{R}^N$ be closed and $C = \{s \in [X]^k : \text{conv } s \not\subset X\}$. Then $C$ is open and a $C$-homogeneous set is called a $k$-clique in $X$. It is known (see [3]) that there exists a closed set $X \subset \mathbb{R}^N$ such that $X$ is not a countable union of convex sets but every $k$-clique in $X$ is countable for every $k < \omega$. On the other hand, it is proved in [3] that if a closed set $X \subset \mathbb{R}^N$ contains an uncountable $k$-clique for some $k$, then it contains a perfect 3-clique.

We prove that if $C$ is a $G_\delta$ coloring of a Polish space and there are no perfect $C$-homogeneous sets, then there is a countable ordinal $\gamma$ such that the Cantor-Bendixson rank of every $C$-homogeneous set is $< \gamma$. In the context of cliques, this strengthens the result of Kojman [2] (see Theorem 3.1(a) below). From our result it follows that if $C$ is a $G_\delta$ coloring of an analytic space, then either there exists a perfect $C$-homogeneous set or all $C$-homogeneous sets are countable. This is not true for $F_\sigma$ colorings: a result of Shelah [4] states that consistently there exist $F_\sigma$ 2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set $X \subset \mathbb{R}^2$ contains an uncountable $\aleph_0$-clique, then $X$ contains also a perfect 3-clique.

Received by the editors August 20, 2001 and, in revised form, October 1, 2001.

2000 Mathematics Subject Classification. Primary 52A37, 54H05; Secondary 03E02, 52A10.

Key words and phrases. Open ($G_\delta$) coloring, perfect homogeneous set, clique.
1.1. **Notation.** Any subset of $[X]^N$ is called a coloring (or an $N$-coloring) of $X$. We write $\neg C$ instead of $[X]^N \setminus C$. A set $S \subset X$ is $C$-homogeneous if $[A]^N \subset C$. We identify $[X]^N$ with the subspace of $X^N$ consisting of all $N$-tuples $(x_0, \ldots, x_{N-1})$ with $x_i \neq x_j$ for $i \neq j$. Thus we may consider topological properties of colorings. If $f : X \rightarrow Y$ is a function, then we write $f[S]$ for the image of a set $S \subset X$ and $f(s)$ for the value at a point $s \in X$. By a perfect set we mean a compact, nonempty, topological space with no isolated points.

2. On Colorings

First we recall a simple result on open 2-colorings of analytic spaces which can be found in Todorcević-Farah's book [5] p. 81].

**Proposition 2.1.** Let $X$ be an analytic space and let $C \subset [X]^2$ be open. Then either there exists a perfect $C$-homogeneous set or else $X$ is a countable union of $\neg C$-homogeneous sets, i.e. $X = \bigcup_{n \in \omega} A_n$ where $[A_n]^2 \cap C = \emptyset$ for every $n \in \omega$.

The above result is no longer valid when we replace the word “open” with “closed”; see [3] p. 83]. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of $\omega$ such that no uncountable homogeneous sets either of this color or of its complement; see Blass' example [1]. In Blass' example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for $G_\delta$ colorings.

For a topological space $Y$ and an ordinal $\alpha$ we denote by $Y^{(\alpha)}$ the $\alpha$-derivative of $Y$; the Cantor-Bendixson rank of $Y$ is the minimal ordinal $\gamma$ such that $Y^{(\gamma+1)}$ is empty.

**Theorem 2.2.** Let $C$ be a $G_\delta$ $N$-coloring of a Polish space $X$. If for every countable ordinal $\gamma$ there exists a $C$-homogeneous set of the Cantor-Bendixson rank $\geq \gamma$, then $X$ contains a perfect $C$-homogeneous set.

**Proof.** Fix a countable base $\mathcal{B}$ in $X$ and fix a complete metric on $X$. Let $C = \bigcap_{n \in \omega} C_n$, where each $C_n$ is open and $C_{n+1} \subset C_n$. We will construct a tree of open sets

$$T = \{u_s : s \in 2^{<\omega}\}$$

with the following properties:

(i) $cl u_{s \upharpoonright i} \subset u_s$, $cl u_s \cap cl u_t = \emptyset$ if $s, t$ are incompatible and $diam(u_s) < 2^{-\text{length}(s)}$;

(ii) if $k < \omega$ and $s_0, \ldots, s_{N-1} \in 2^k$ are pairwise distinct, then $\{x_0, \ldots, x_{N-1}\} \in C_k$ whenever $x_i \in u_{s_i}$, $i \in N$;

(iii) if $k < \omega$, then for each $\gamma < \omega_1$ there exists a $C$-homogeneous set $P = P_k, \gamma$ such that $P(\gamma) \cap u_s \neq \emptyset$ for each $s \in 2^k$.

We start with $u_0 = X$. Suppose that $u_s$ has been defined for all $s \in 2^{\leq k}$. Fix $\gamma < \omega_1$ and consider $P = P_{k, \gamma+1}$, as in (iii). Then for each $s \in 2^k$ the set $P(\gamma) \cap u_s$ is infinite. Fix $S \subset P(\gamma)$ such that $|S|/|u_s| = 2$ for each $s \in 2^k$. Next, enlarge each $x \in S \cap u_s$ to a small open set $v_x \in \mathcal{B}$, contained in $u_s$, such that $\{y_0, \ldots, y_{N-1}\} \in C_{k+1}$ whenever $y_i$ are taken from pairwise distinct $v_x$'s. This is possible, because $C_{k+1}$ is open. Let $\varphi(\gamma) = \{v_x : x \in S\}$. This defines a mapping $\varphi : \omega_1 \rightarrow |\mathcal{B}|^{<\omega}$. As $\mathcal{B}$ is countable,
there is unbounded $F \subset \omega_1$ such that $\varphi \upharpoonright F$ is constant, say $\{v_{\alpha^{-1}} : s \in 2^k, i < 2\}$, where $v_{\alpha^{-1}} \in u_\varphi$. Set $v_{\alpha^{-1}} = v_{\alpha^{-1}}$. Observe that (i) holds if we let $v_x$'s be small enough. Also (ii) holds, by the definition of $v_x$'s. Finally, (iii) holds, because $P_{k,\gamma+1} \cap u_\varphi \neq \emptyset$ for $t \in 2^{k+1}$ whenever $\gamma \in F$. By (ii) the perfect set obtained from this construction is $C$-homogeneous.

Using the above theorem we obtain the following corollary which, for the case of 2-colorings of Polish spaces, was mentioned by Shelah [3, Remark 1.14]:

**Corollary 2.3.** Let $1 \leq N < \omega$ and let $C$ be a $G_\delta$ $N$-coloring of an analytic space $X$. If there exists an uncountable $C$-homogeneous set, then there exists also a perfect one.

**Proof.** Let $f : \omega^\omega \to X$ be a continuous surjection and define $C' = \{s \in [\omega^\omega]^2 : f[s] \in C\}$. If $K \subset X$ is $C$-homogeneous and $K = f[K']$ where $f \upharpoonright K'$ is one-to-one, then $K'$ is $C'$-homogeneous. If $K$ is uncountable then so is $K'$ and by Theorem 2.2 we get a perfect set $P \subset \omega^\omega$ which is $C'$-homogeneous. Then $f \upharpoonright P$ is one-to-one and hence $f[P]$ is a perfect $C$-homogeneous set.

### 3. Applications to Convexity

Let $X \subset E$, where $E$ is a real vector space. A subset $K$ of $X$ is a $k$-clique ($k$ can be a cardinal or just a natural number; we will use this notion for $k < \omega$ and $k = \aleph_0$) if $\text{conv} S \not\subset X$ whenever $S \in [K]^k$. If $E$ is finite-dimensional and $k > \dim E$, then we can define the notion of a strong $k$-clique replacing $\text{conv} S$ by $\text{int} \text{conv} S$ in the definition. A finite set $S \subset X$ is (strongly) defected in $X$ if $\text{conv} S \not\subset X$ ($\text{int} \text{conv} S \not\subset X$). It is clear that the relation of strong defectedness is open and defectedness is open provided that $X$ is closed.

Applying the results of the previous section we get the following:

**Theorem 3.1.** (a) Let $X$ be a closed set in a Polish linear space and let $N < \omega$. If $X$ does not contain a perfect $N$-clique, then all $N$-cliques in $X$ are countable. Moreover, there exists an ordinal $\gamma < \omega_1$ which bounds the Cantor-Bendixson ranks of all $N$-cliques in $X$.

(b) Let $X$ be an analytic subset of $\mathbb{R}^m$. If $m < N < \omega$ and $X$ contains an uncountable strong $N$-clique, then $X$ contains also a perfect one.

Theorem 3.1(a) was proved, under the stronger assumption that $X$ is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

**Theorem 3.2.** Let $X \subset \mathbb{R}^2$ be analytic and assume that $X$ contains an uncountable $\aleph_0$-clique. Then either $X$ contains a perfect strong 3-clique or else, for some line $L$, $X \cap L$ contains a perfect 2-clique. In particular $X$ contains a perfect 3-clique.

**Proof.** Fix a continuous function $f : \omega^\omega \to X$ onto $X$ and fix an uncountable $\aleph_0$-clique $K \subset X$. We may assume that every line contains only countably many points of $L$; otherwise, for some line $L$, $X \cap L$ contains an uncountable $\aleph_0$-clique, so it contains a perfect 2-clique (Proposition 2.4). Fix uncountable $K' \subset \omega^\omega$ such that $f \upharpoonright K'$ is a bijection onto $K$. 

A finite collection \( \{u_0, \ldots, u_{k-1}\} \) of open subsets of \( \omega^\omega \) will be called relevant if each \( u_i \) contains uncountably many points of \( K' \), \( \operatorname{cl} u_i \cap \operatorname{cl} u_j = \emptyset \) whenever \( i < j < k \) and int conv\{\( f(x_0), f(x_1), f(x_2) \)\} \( \subseteq X \) whenever \( x_0, x_1, x_2 \) are taken from pairwise distinct \( u_i \)'s. To find a perfect strong 3-clique in \( X \), it suffices to construct a perfect tree of open sets in \( \omega^\omega \) with relevant levels. If \( P \) is a perfect set obtained from such a tree, then \( f \upharpoonright P \) is one-to-one and \( f[P] \) is a perfect strong 3-clique.

Suppose that we have a relevant collection \( \{u_0, \ldots, u_k\} \). We have to show that it is possible to split each \( u_i \) to obtain again a relevant collection. We will split \( u_k \). Let \( L = K' \cap u_k \) and pick \( y_i \in u_i \) for \( i < k \). Define \( c_i : [L]^2 \to 2 \) by letting \( c_i(x_0, x_1) = 1 \) iff \( \operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subseteq X \). Observe that there are no infinite \( c_i \)-homogeneous sets of color 0: if \( S \subseteq L \) is infinite, then, by Carathéodory’s theorem, there is \( s \in [S]^3 \) such that \( f[S] \) is defected in \( X \) (because \( f[S] \) is defected) and hence for some \( x_0, x_1, x_2 \in S \) we have \( \operatorname{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subseteq X \), because \( \operatorname{conv} T \subseteq \bigcup_{x, y \in T} \operatorname{conv}\{x, y, p\} \) for \( T \subseteq \mathbb{R}^2, p \in \mathbb{R}^2 \). Using \( k \) times the theorem of Dushnik-Miller we obtain uncountable \( L' \subseteq L \) which is \( c_i \)-homogeneous of color 1 for \( i < k \). Shrinking \( L' \) we may assume that each nonempty open subset of \( L' \) is uncountable. Now choose disjoint open sets \( v_0, v_1 \) with \( \operatorname{cl} v_j \cap u_k \) and \( v_j \cap L' \neq \emptyset \) for \( j < 2 \). To finish the proof we need the following geometric property of the plane:

**Claim 3.3.** Let \( A, B \subseteq X \subseteq \mathbb{R}^2 \) and \( c \subseteq \mathbb{R}^2 \) be such that \( A, B \) are uncountable, each line contains countably many points of \( A \cup B \) and \( \operatorname{conv}\{a, b, c\} \subseteq X \) whenever \( a \in A, b \in B \). Then there are \( a_0 \in A, b_0 \in B \) such that \( \operatorname{int conv}\{a_0, b_0, c\} \not\subseteq X \).

**Proof.** Suppose that this is not true. Observe that, replacing \( A \) and \( B \) if necessary, we may assume that for some \( b_0 \in B, [a, b_0] \cup [a, c] \not\subseteq X \) whenever \( a \in A \). Indeed, if \( [b, c] \subseteq X \) for some \( b \in B \), then we take \( b_0 = b \), otherwise we take any \( a_0 \in A \) and we replace \( A \) and \( B \). Now, without loss of generality, we may assume that \( b_0 = (-1, 0), c = (1, 0) \) and \( A \) is contained in \((-1, 1) \times (0, 1) \). Now, if some vertical line contains two elements of \( A \), then we are done: we take \( a_0 \in A \) such that some \( a_1 \in A \) is below \( a_0 \); then the relative interiors of segments \([b_0, a_1], [c, a_1]\) are contained in the interior of \( \operatorname{conv}\{a_0, b_0, c\} \).

Assume that each vertical line contains at most one element of \( A \). As \( A \) is uncountable, there is \( a_1 \in A \) such that arbitrarily close to \( a_1 \) there are uncountably many points both on the left and the right side of \( a_1 \). Suppose now that e.g. \( \{b_0, a_1\} \) is defected in \( X \). As \([b_0, a_1]\) contains only countably many points of \( A \), we can find \( a_2 \in A \) which is close enough to \( a_1 \), on the left side of \( a_1 \) and not in \([b_0, a_1]\). If \( a_2 \) is below \([b_0, a_1]\), then we can set \( a_0 = a_1 \), otherwise we can set \( a_0 = a_2 \).

Let \( i = 0 \). Using Claim 3.3 for \( A = f[v_0 \cap L'], B = f[v_1 \cap L'] \) and \( c = f(y_j) \) we get \( x_j \in v_j \) such that \( \operatorname{int conv}\{f(x_0), f(x_1), f(y_i)\} \not\subseteq X \). By continuity, shrink \( v_0, v_1 \) and enlarge \( y_i \) to an open set \( u_i' \subset u_i \) such that each triple selected from \( f[v_0] \times f[v_1] \times f(u_i') \) is strongly defected in \( X \). Repeat the same argument for each \( i < k \), obtaining a relevant collection \( \{u_0', \ldots, u_{k-1}', v_0', v_1'\} \) which realizes the splitting of \( u_k \). This completes the proof.

**References**


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