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THE ℓ^1-INDICES OF TSIRELSON TYPE SPACES

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ABSTRACT. If α and β are countable ordinals such that $\beta \neq 0$, denote by $\widetilde{T}_{\alpha,\beta}$ the completion of c_{00} with respect to the implicitly defined norm

$$\|x\| = \max\{\|x\|_{\mathcal{S}_\alpha}, \frac{1}{2}\sup\sum_{i=1}^{j}\|E_i x\|\},$$

where the supremum is taken over all finite subsets $E_1, \dots, E_j$ of $\mathbb{N}$ such that $E_1 < \cdots < E_j$ and $\{\min E_1, \dots, \min E_j\} \in \mathcal{S}_\beta$. It is shown that the Bourgain ℓ^1-index of $\widetilde{T}_{\alpha,\beta}$ is $\omega^{\alpha+\beta\cdot\omega}$. In particular, if $\omega_1 > \alpha = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_n}\cdot m_n$ in Cantor normal form and α_n is not a limit ordinal, then there exists a Banach space whose ℓ^1-index is ω^α.

Let E be a separable Banach space not containing a copy of ℓ^1. There are several measures of the complexity of $\ell^1(n)$'s inside E. These include the Bourgain ℓ^1-index [4], the existence of so-called ℓ^1_α-spreading models ([2] and [9]) and the asymptotic constants as defined by Odell, Tomczak-Jaegermann, and Wagner [11]. In this paper, we concentrate on the first two measures in Tsirelson type spaces. It is easy to see that the existence of ℓ^1_α-spreading models implies a large ℓ^1-index. In general, the implication is not reversible [8, Remark 6.6(i)]. However, suppose that T is the standard Tsirelson space constructed by Figiel and Johnson [6] (the dual of the original Tsirelson space [12]). It is known that there is a constant K such that every normalized block basic sequence in T is K-equivalent to a subsequence of the unit vector basis of T (see e.g., [5]). Using this observation, one can show that the existence of ℓ^1-block trees in T with large indices leads to the existence of large ℓ^1_α-spreading models. The result can be used to calculate the ℓ^1-index of T. In this paper, we show that a similar method can be applied to certain general Tsirelson type spaces. In particular, it is shown that if $\omega_1 > \alpha = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_n}\cdot m_n$ in Cantor normal form and α_n is not a limit ordinal, then there exists a Banach space whose ℓ^1-index is ω^α. This gives a partial answer to Question 1 in [8]. The authors have extended the method to compute the Bourgain ℓ^1-indices of mixed Tsirelson spaces [10]. In particular, it is shown there that if α is not of the form ω^β for some limit ordinal β, then there exists a Banach space whose ℓ^1-index is ω^α.

If M is an infinite subset of $\mathbb{N}$, denote the set of all finite, respectively infinite, subsets of M by $[M]^{<\infty}$, respectively $[M]$. A subset $\mathcal{F}$ of $[\mathbb{N}]^{<\infty}$ is *hereditary* if $G \in \mathcal{F}$ whenever $G \subseteq F \in \mathcal{F}$. $\mathcal{F}$ is *spreading* if whenever $F = \{n_1, \dots, n_k\} \in \mathcal{F}$ with $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ satisfies $m_i \geq n_i$ for $1 \leq i \leq k$, then

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$\{m_1, \ldots, m_k\} \in \mathcal{F}$. $\mathcal{F}$ is *compact* if it is compact in the product topology in $2^{\mathbb{N}}$. A set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called *regular* if it has all three properties. If E and F are finite subsets of $\mathbb{N}$, we write $E < F$, respectively $E \leq F$, to mean $\max E < \min F$, respectively $\max E \leq \min F$ ($\max \emptyset = 0$ and $\min \emptyset = \infty$). We abbreviate $\{n\} < E$ and $\{n\} \leq E$ to $n < E$ and $n \leq E$ respectively. Given $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, a sequence of finite subsets $\{E_1, \ldots, E_n\}$ of $\mathbb{N}$ is said to be $\mathcal{F}$-admissible if $E_1 < \cdots < E_n$ and $\{\min E_1, \ldots, \min E_n\} \in \mathcal{F}$. If $\mathcal{M}$ and $\mathcal{N}$ are regular subsets of $[\mathbb{N}]^{<\infty}$, we let

$$\mathcal{M}[\mathcal{N}] = \{\bigcup_{i=1}^{k} F_i : F_i \in \mathcal{N} \text{ for all } i \text{ and } \{F_1, \ldots, F_k\}\ \mathcal{M}\text{-admissible}\}$$

and

$$(\mathcal{M}, \mathcal{N}) = \{M \cup N : M < N, M \in \mathcal{M} \text{ and } N \in \mathcal{N}\}.$$

We also write $\mathcal{M}^2$ for $(\mathcal{M}, \mathcal{M})$. Of primary importance are the Schreier classes as defined in [1]. Let $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathcal{S}_1 = \{F \subseteq \mathbb{N} : |F| \leq \min F\}$. Here $|F|$ denotes the cardinality of F. The higher Schreier classes are defined inductively as follows. $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$ for all $\alpha < \omega_1$. If α is a countable limit ordinal, choose a sequence (α_n) strictly increasing to α and set

$$\mathcal{S}_\alpha = \{F : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \leq |F|\}.$$

It is clear that $\mathcal{S}_\alpha$ is a regular family for all $\alpha < \omega_1$. If $M = (m_1, m_2, \ldots)$ is a subsequence of $\mathbb{N}$, let $\mathcal{S}_\alpha(M) = \{\{m_i : i \in F\} : F \in \mathcal{S}_\alpha\}$. Since $\mathcal{S}_\alpha$ is spreading, $\mathcal{S}_\alpha(M) \subseteq \mathcal{S}_\alpha$.

Let c_{00} be the space of all finitely supported sequences. If $F \in [\mathbb{N}]^{<\infty}$ and $a = (a_n) \in c_{00}$, let $Fa = (b_n) \in c_{00}$, where $b_n = a_n$ if $n \in F$ and 0 otherwise; also set $\sigma_F((a_n)) = \sum_{n \in F} |a_n|$. Finally, if $\mathcal{S}_0 \subseteq \mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, define an associated norm $\|\cdot\|_{\mathcal{F}}$ on c_{00} by $\|(a_n)\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \sigma_F((a_n))$.

Definition 1. Let α, β be countable ordinals such that $\beta \neq 0$. Define $\|\cdot\|_n$ and $\|\cdot\|'_n$, $n \in \mathbb{N}$, inductively as follows. Let $\|\cdot\|_0 = \|\cdot\|'_0 = \|\cdot\|_{\mathcal{S}_\alpha}$. If $x \in c_{00}$, set

$$\|x\|_{n+1} = \max\left\{\|x\|_n, \sup\left\{\frac{1}{2}\sum_{i=1}^{j}\|E_i x\|_n : \{E_1, ..., E_j\}\ \ \mathcal{S}_\beta\text{-admissible}\right\}\right\}$$

and

$$\|x\|'_{n+1} = \max\left\{\|x\|'_n, \sup\left\{\frac{1}{2}\sum_{i=1}^{j}\|E_i x\|'_n : \{E_1, ..., E_j\}\ \ (\mathcal{S}_\beta)^2\text{-admissible}\right\}\right\}.$$

Note that $(\|x\|_n)_{n\in\mathbb{N}}$ and $(\|x\|'_n)_{n\in\mathbb{N}}$ are increasing sequences majorized by the ℓ^1-norm of x. Let $\|x\|_{\widetilde{T}} = \lim_{n\to\infty} \|x\|_n$ and $\|x\|_{\widetilde{\widetilde{T}}} = \lim_{n\to\infty} \|x\|'_n$. Denote by $\widetilde{T}_{\alpha,\beta}$ and $\widetilde{\widetilde{T}}_{\alpha,\beta}$ respectively the completion of c_{00} under the norms $\|\cdot\|_{\widetilde{T}}$ and $\|\cdot\|_{\widetilde{\widetilde{T}}}$. Clearly, $\widetilde{T}_{0,1}$ is the Tsirelson space constructed by Figiel and Johnson [6] and $\widetilde{T}_{0,\beta}$ is the space denoted by $T\left(\mathcal{S}_\beta, \frac{1}{2}\right)$ in [8]. The ℓ^1-index of $\widetilde{T}_{0,\beta}$ is shown to be $\omega^{\beta\cdot\omega}$ in [8]. Here, we use a different argument to compute the ℓ^1-indices of the spaces $\widetilde{T}_{\alpha,\beta}$. The next proposition can be verified immediately.

Proposition 2. *The norms* $\|\cdot\|_{\widetilde{T}}$ *and* $\|\cdot\|_{\widetilde{\widetilde{T}}}$ *satisfy the implicit equations*

$$\|x\|_{\widetilde{T}} = \max\left\{\|x\|_{\mathcal{S}_\alpha}, \sup\left\{\frac{1}{2}\sum_{i=1}^{j}\|E_i x\|_{\widetilde{T}} : \{E_1, ..., E_j\}\ \mathcal{S}_\beta\text{-admissible}\right\}\right\}$$

and

$$\|x\|_{\widetilde{\widetilde{T}}} = \max\left\{\|x\|_{\mathcal{S}_\alpha}, \sup\left\{\frac{1}{2}\sum_{i=1}^{j}\|E_i x\|_{\widetilde{\widetilde{T}}} : \{E_1, ..., E_j\}\ (\mathcal{S}_\beta)^2\text{-admissible}\right\}\right\}$$

for all $x \in c_{00}$.

Proposition 4 is a close relative of Lemma 5 in [5]. It is the key observation that allows us to reduce ℓ^1-block trees on $\widetilde{T}_{\alpha,\beta}$ to subsequences of the unit vector basis (e_n) of $\widetilde{\widetilde{T}}_{\alpha,\beta}$. The following lemma is easily established by induction.

Lemma 3. *Suppose that* $n_1 \le I_1 < n_2 \le I_2 < ... < n_k \le I_k$ *and* $|I_j| \le 2$ *for* $1 \le j \le k$. *If* $\{n_1, n_2, ..., n_k\} \in \mathcal{S}_\beta$ *for some* $\beta < \omega_1$, *then* $\bigcup_{j=1}^{k} I_j \in (\mathcal{S}_\beta)^2$.

Obviously, the sequence of coordinate unit vectors (u_n) forms a normalized 1-unconditional basis of $\widetilde{T}_{\alpha,\beta}$. The *support* of an element $x = \sum a_n u_n \in \widetilde{T}_{\alpha,\beta}$ is the set $\operatorname{supp} x = \{n : a_n \neq 0\}$.

Proposition 4. *For every* $\|\cdot\|_{\widetilde{T}}$*-normalized block basis* $(x_1, x_2, ..., x_p)$ *in* $\widetilde{T}_{\alpha,\beta}$ *and all* $(a_k) \in c_{00}$,

$$\left\|\sum_{k=1}^{p} a_k x_k\right\|_{\widetilde{T}} \le 2\left\|\sum_{k=1}^{p} a_k e_{i_k}\right\|_{\widetilde{\widetilde{T}}},$$

where $i_k = \max\operatorname{supp} x_k$, *and* (e_n) *is the unit vector basis of* $\widetilde{\widetilde{T}}_{\alpha,\beta}$.

Proof. With the notation as above, we prove by induction that $\left\|\sum_{k=1}^{p} a_k x_k\right\|_n \le 2\left\|\sum_{k=1}^{p} a_k e_{i_k}\right\|_n'$ for all $n \in \mathbb{N} \cup \{0\}$, $(a_k) \in c_{00}$. To establish the inequality for the case $n = 0$, let $I \in \mathcal{S}_\alpha$. Define $J = \{k : I \cap \operatorname{supp} x_k \neq \emptyset\}$. Then

$$\begin{aligned}
\sigma_I\left(\sum_{k=1}^{p} a_k x_k\right) &= \sum_{k=1}^{p} |a_k|\,\sigma_I(x_k) \\
&\le \sum_{k\in J} |a_k|\,\|x_k\|_0 \\
&\le \sum_{k\in J} |a_k| = \sigma_L\left(\sum_{k=1}^{p} a_k e_{i_k}\right), \text{ where } L = \{i_k : k \in J\}, \\
&\le \left\|\sum_{k=1}^{p} a_k e_{i_k}\right\|_0', \text{ since } L \in \mathcal{S}_\alpha.
\end{aligned}$$

Suppose the proposition holds for some n. Let $\{E_1, ..., E_q\}$ be $\mathcal{S}_\beta$-admissible. Without loss of generality, we may assume that $E_1, ..., E_q$ are successive integer intervals, that for all j, $E_j \cap \operatorname{supp} x_k \neq \emptyset$ for some k, and that $i_p \le \max E_q$. Also let I_k be the integer interval $[i_{k-1}+1, i_k]$ $(i_0 \equiv 0)$. Let $A = \{j : E_j \nsubseteq I_k \text{ for any } k\}$ and $B = \{j : j \notin A\}$. For $j \in A$, set $H_j = \{k : I_k \subseteq E_j\}$ and $G_j = \{i_k : k \in H_j\}$.

Then define $F_j = (E_j \cap \{i_1, ..., i_p\}) \setminus G_j$. Note that $F_j < G_j$ for all $j \in A$. If $j \in B$, set $G_j = E_j \cap \{i_1, ..., i_p\}$.

It follows from Lemma 3 that $(F_j)_{j\in A} \cup (G_j)_{j=1}^q$ is $(\mathcal{S}_\beta)^2$-admissible. Finally, let $J = \left\{k : k \notin \bigcup_{j\in A} H_j,\ I_k \cap (\bigcup E_j) \neq \emptyset\right\}$. Now

$$\begin{aligned}
\sum_{j=1}^q \left\| E_j \left(\sum_{k=1}^p a_k x_k \right) \right\|_n &= \sum_{j=1}^q \left\| E_j \left(\sum_{j'\in A} \sum_{k\in H_{j'}} a_k x_k + \sum_{k\in J} a_k x_k \right) \right\|_n \\
&\le \sum_{j=1}^q \left(\left\| E_j \left(\sum_{j'\in A} \sum_{k\in H_{j'}} a_k x_k \right) \right\|_n + \left\| E_j \left(\sum_{k\in J} a_k x_k \right) \right\|_n \right) \\
&= \sum_{j\in A} \left\| E_j \left(\sum_{k\in H_j} a_k x_k \right) \right\|_n + \sum_{j=1}^q \left\| E_j \left(\sum_{k\in J} a_k x_k \right) \right\|_n \\
&\le \sum_{j\in A} \left\| \sum_{k\in H_j} a_k x_k \right\|_n + \sum_{k\in J} |a_k| \sum_{j=1}^q \|E_j x_k\|_n \\
&\le \sum_{j\in A} \left\| \sum_{k\in H_j} a_k x_k \right\|_n + 2 \sum_{k\in J} |a_k| \, \|x_k\|_{n+1} \\
&\le 2 \left(\sum_{j\in A} \left\| \sum_{k\in H_j} a_k e_{i_k} \right\|_n' + \sum_{k\in J} |a_k| \right) \text{ by the inductive hypothesis,} \\
&= 2 \left(\sum_{j\in A} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{k\in J} |a_k| \right).
\end{aligned}$$

If $k \in J$, then either $\{i_k\} = F_j$ for some $j \in A$ or $\{i_k\} = G_j$ for some $j \in B$. Therefore

$$\sum_{k\in J} |a_k| \le \sum_{j\in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{j\in B} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n'.$$

Hence

$$\begin{aligned}
\sum_{j=1}^q \left\| E_j \left(\sum_{k=1}^p a_k x_k \right) \right\|_n &\le 2 \sum_{j\in A} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + 2 \sum_{j\in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \\
&\qquad + 2 \sum_{j\in B} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \\
&= 2 \left(\sum_{j\in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{j=1}^q \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \right) \\
&\le 4 \left\| \sum_{k=1}^p a_k e_{i_k} \right\|_{n+1}', \text{ as } (F_j)_{j\in A} \cup (G_j)_{j=1}^q \text{ is } (\mathcal{S}_\beta)^2\text{-admissible.}
\end{aligned}$$

Thus

$$\frac{1}{2}\sum_{j=1}^{q}\left\|E_j\left(\sum_{k=1}^{p}a_kx_k\right)\right\|_n \leq 2\left\|\sum_{k=1}^{p}a_ke_{i_k}\right\|'_{n+1}$$

whenever $\{E_1,...,E_q\}$ is $\mathcal{S}_\beta$-admissible. It follows that

$$\left\|\sum_{k=1}^{p}a_kx_k\right\|_{n+1} \leq 2\left\|\sum_{k=1}^{p}a_ke_{i_k}\right\|'_{n+1}.$$

This completes the induction. ☐

Let α, β be countable ordinals. Define the families $(\mathcal{F}_n)$, $(\mathcal{F}'_n)$, $(\mathcal{G}_n)$ and $(\mathcal{G}'_n)$ inductively as follows: $\mathcal{F}_0 = \mathcal{F}'_0 = \mathcal{S}_\alpha$, $\mathcal{G}_1 = \mathcal{S}_\beta$, $\mathcal{G}'_1 = (\mathcal{S}_\beta)^2$, for all $n \in \mathbb{N}$,

$$\mathcal{F}_{n+1} = \mathcal{S}_\beta[\mathcal{F}_n], \ \mathcal{F}'_{n+1} = (\mathcal{S}_\beta)^2[\mathcal{F}'_n], \ \mathcal{G}_{n+1} = \mathcal{S}_\beta[\mathcal{G}_n], \text{ and } \mathcal{G}'_{n+1} = (\mathcal{S}_\beta)^2[\mathcal{G}'_n].$$

It is easily verified that $\mathcal{G}_n[\mathcal{S}_\alpha] = \mathcal{F}_n$, $\mathcal{G}'_n[\mathcal{S}_\alpha] = \mathcal{F}'_n$, $\mathcal{G}_n[\mathcal{S}_\beta] = \mathcal{G}_{n+1}$ and $\mathcal{G}'_n\left[(\mathcal{S}_\beta)^2\right] = \mathcal{G}'_{n+1}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, denote the norms $\|\cdot\|_{\mathcal{F}_n}$ and $\|\cdot\|_{\mathcal{F}'_n}$ by ρ_n and ρ'_n respectively.

Proposition 5. *For all $a \in c_{00}$ and all $n \in \mathbb{N}\cup\{0\}$, $\|a\|_{\widetilde{T}} \geq \frac{1}{2^n}\rho_n(a)$.*

Proof. The proof is by induction on n. The case $n = 0$ is clearly true by definition. Suppose the result holds for some n. Let $E \in \mathcal{F}_{n+1}$. Then $E = \bigcup_{i=1}^{j} E_i$, where $E_1,...,E_j \in \mathcal{F}_n$, $E_1 < ... < E_j$, and $\{E_1,...,E_j\}$ is $\mathcal{S}_\beta$-admissible. For any $a = (a_k) \in c_{00}$,

$$\sum_{k\in E}|a_k| = \sum_{i=1}^{j}\sum_{k\in E_i}|a_k| \leq \sum_{i=1}^{j}\rho_n(E_ia) \leq 2^n\sum_{i=1}^{j}\|E_ia\|_{\widetilde{T}} \leq 2^{n+1}\|a\|_{\widetilde{T}}.$$

Since $E \in \mathcal{F}_{n+1}$ is arbitrary, the result follows. ☐

Proposition 6. *For all $a \in c_{00}$ and all $n \in \mathbb{N}\cup\{0\}$,*

$$\|a\|_{\approx T} \leq \sum_{i=0}^{n}\frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+1}}\sup\left\{\sum_{i=1}^{j}\|E_ia\|_{\approx T} : \{E_1,...,E_j\}\ \mathcal{G}'_{n+1}\text{-admissible}\right\}.$$

Proof. The proof is by induction on n. The case $n = 0$ follows immediately from the definition of $\|\cdot\|_{\approx T}$. Assume the result holds for some n. Let $a \in c_{00}$. Suppose $\{E_1,...,E_j\}$ is $\mathcal{G}'_{n+1}$-admissible. Let $I = \left\{i : \|E_ia\|_{\approx T} = \rho'_0(E_ia)\right\}$ and $J = \{1,2,...,j\}\setminus I$. For each $i \in I$, choose $D_i \subseteq E_i$, $D_i \in \mathcal{S}_\alpha$, such that $\rho'_0(E_ia) = \sum_{k\in D_i}|a_k|$. Now $D = \bigcup_{i\in I} D_i \in \mathcal{G}'_{n+1}[\mathcal{S}_\alpha] = \mathcal{F}'_{n+1}$. Hence

$$\sum_{i\in I}\|E_ia\|_{\approx T} = \sum_{k\in D}|a_k| \leq \rho'_{n+1}(a). \tag{1}$$

On the other hand, for each $i \in J$, there exist $(\mathcal{S}_\beta)^2$-admissible sets $\{E_1^i,...,E_{k_i}^i\}$, $E_1^i\cup...\cup E_{k_i}^i \subseteq E_i$ such that

$$\|E_ia\|_{\approx T} = \frac{1}{2}\sum_{p=1}^{k_i}\|E_p^ia\|_{\approx T}.$$

Now

$$\left\{\min E_p^i : i \in J,\ 1 \leq p \leq k_i\right\} \in \mathcal{G}'_{n+1}\left[\left(\mathcal{S}_\beta\right)^2\right] = \mathcal{G}'_{n+2}.$$

Hence $\left(E_p^i\right)_{i\in J,\ 1\leq p\leq k_i}$ is $\mathcal{G}'_{n+2}$-admissible. Thus

$$\begin{aligned}(2)\qquad \sum_{i\in J}\|E_i a\|_{\widetilde{\widetilde{T}}} &= \frac{1}{2}\sum_{i\in J}\sum_{p=1}^{k_i}\left\|E_p^i a\right\|_{\widetilde{\widetilde{T}}} \\ &\leq \frac{1}{2}\sup\left\{\sum_{i=1}^{\ell}\|F_i a\|_{\widetilde{\widetilde{T}}} : \{F_1, ..., F_\ell\}\ \ \mathcal{G}'_{n+2}\text{-admissible}\right\}.\end{aligned}$$

From the inductive hypothesis and inequalities (1) and (2) we get

$$\begin{aligned}\|a\|_{\widetilde{\widetilde{T}}} &\leq \sum_{i=0}^{n}\frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+1}}\left(\rho'_{n+1}(a) + \frac{1}{2}\sup\left\{\sum_{i=1}^{\ell}\|F_i a\|_{\widetilde{\widetilde{T}}}\right\}\right) \\ &= \sum_{i=0}^{n+1}\frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+2}}\sup\left\{\sum_{i=1}^{\ell}\|F_i a\|_{\widetilde{\widetilde{T}}}\right\},\end{aligned}$$

where both suprema are taken over all $\mathcal{G}'_{n+2}$-admissible sets $\{F_1, ..., F_\ell\}$. This completes the induction. $\square$

Endow $[\mathbb{N}]^{<\infty}$ with the product topology inherited from $2^{\mathbb{N}}$. If $\mathcal{F}$ is a closed subset of $[\mathbb{N}]^{<\infty}$, let $\mathcal{F}'$ be the set of all limit points of $\mathcal{F}$. Define a transfinite sequence of sets $\left(\mathcal{F}^{(\alpha)}\right)_{\alpha<\omega_1}$ as follows: $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(\alpha+1)} = \left(\mathcal{F}^{(\alpha)}\right)'$ for all $\alpha < \omega_1$; $\mathcal{F}^{(\alpha)} = \bigcap_{\beta<\alpha}\mathcal{F}^{(\beta)}$ if α is a countable limit ordinal.

Definition 7 ([11])**.** Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be regular. Define $\iota(\mathcal{F})$ to be the unique countable ordinal α such that $\mathcal{F}^{(\alpha)} = \{\emptyset\}$.

Let $F \in \mathcal{M}[\mathcal{N}]$, $F \neq \emptyset$, where $\mathcal{M}$ and $\mathcal{N}$ are regular families. There exists a largest $k \in F$ such that $F \cap [1, k] \in \mathcal{N}$. Set $F_1 = F \cap [1, k]$. If $F_1, F_2, ..., F_{n-1}$ have been defined and $F \setminus \bigcup_{i=1}^{n-1} F_i \neq \emptyset$, set $F_n = \left(F \setminus \bigcup_{i=1}^{n-1} F_i\right) \cap [1, k']$, where k' is the largest integer in F such that $\left(F \setminus \bigcup_{i=1}^{n-1} F_i\right) \cap [1, k'] \in \mathcal{N}$. Since F is finite, there exists an n such that $F = \bigcup_{i=1}^{n} F_i$. Now $F \in \mathcal{M}[\mathcal{N}]$ implies that there exists an $\mathcal{M}$-admissible collection $\{G_1, ..., G_m\}$ such that $F = \bigcup_{j=1}^{m} G_j$ and $G_j \in \mathcal{N}$, $1 \leq j \leq m$. By the choice of F_i and the fact that $\mathcal{N}$ is hereditary, it is easy to see that $\min G_i \leq \min F_i$, $1 \leq i \leq n$. Thus $\{F_1, ..., F_n\}$ is $\mathcal{M}$-admissible, as $\mathcal{M}$ is spreading. We call $(F_i)_{i=1}^n$ the standard representation of F (as an element of $\mathcal{M}[\mathcal{N}]$).

Remark 8. Suppose that $(F_i)_{i=1}^n$ *and* $(G_i)_{i=1}^m$ *are the standard representations of* F *and* G *respectively. If* $\ell, k \in \mathcal{N}$ *are such that* $F \cap [1, \ell] = G \cap [1, \ell]$ *and* $\max F_k \leq \ell$, *then by construction,* $F_i = G_i$, $1 \leq i \leq k$.

Lemma 9. *Let* $\mathcal{M}$, $\mathcal{N} \subseteq [\mathbb{N}]^{<\infty}$ *be regular. Suppose that* $\iota(\mathcal{N}) = \alpha$. *Then*

$$(\mathcal{M}[\mathcal{N}])^{(\alpha)} = \left(\mathcal{M}^{(1)}\right)[\mathcal{N}].$$

Proof. Let $F \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$. Then F can be written as $F = \bigcup_{i=1}^n F_i$, where $F_1 < F_2 < ... < F_n$, $F_1, ..., F_n \in \mathcal{N}$ and $\{\min F_1, ..., \min F_n\} \in \mathcal{M}^{(1)}$. In particular, there exists $k_0 > \max F$ such that $\{\min F_1, ..., \min F_n, k\} \in \mathcal{M}$ for all $k \geq k_0$. Therefore,

$$\text{for all } G \in \mathcal{N}, \min G \geq k_0,\ F \cup G \in \mathcal{M}[\mathcal{N}]. \tag{3}$$

Note that as $\mathcal{N}$ is spreading,

$$\iota\left(\{G \in \mathcal{N} : \min G \geq k_0\}\right) = \iota\left(\mathcal{N}\right) = \alpha. \tag{4}$$

From (3),

$$\left(\{F \cup G : G \in \mathcal{N},\, \min G \geq k_0\}\right)^{(\alpha)} \subseteq \left(\mathcal{M}[\mathcal{N}]\right)^{(\alpha)}.$$

But from (4), $F \in \left(\{F \cup G : G \in \mathcal{N},\, \min G \geq k_0\}\right)^{(\alpha)}$. Hence $F \in \left(\mathcal{M}[\mathcal{N}]\right)^{(\alpha)}$.

Conversely, we prove by induction that $\left(\mathcal{M}[\mathcal{N}]\right)^{(\gamma)} \subseteq \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\gamma)}\right)$ for all $\gamma \leq \alpha$. The cases where $\gamma = 0$ is clear.

Suppose the claim is true for some $\gamma < \alpha$. Let $F \in \left(\mathcal{M}[\mathcal{N}]\right)^{(\gamma+1)}$ with standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. Choose a sequence (G_k) in $\left(\mathcal{M}[\mathcal{N}]\right)^{(\gamma)} \subseteq \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\gamma)}\right)$ that converges nontrivially to F. We may assume that $G_k \cap [1, \min F_n] = F \cap [1, \min F_n]$ for all k. Now we may write $G_k = P_k \cup Q_k$, where $P_k < Q_k$, $P_k \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$ and $Q_k \in \mathcal{N}^{(\gamma)}$. Let $P = \bigcup_{i=1}^{n-1} F_i$. Note that $P \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$. We consider two cases.

Case 1. There exists k such that $\min F_n \in P_k$.

In this case, $P \cap [1, \max F_{n-1}] = F \cap [1, \max F_{n-1}] = G_k \cap [1, \max F_{n-1}] = P_k \cap [1, \max F_{n-1}]$. It is clear that $(F_i)_{i=1}^{n-1}$ is the standard representation of P as an element of $\left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$. By Remark 8, the standard representation of P_k as an element of $\left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$ has the form $(F_1, ..., F_{n-1}, P_k^n, ..., P_k^m)$. In particular,

$$\{\min F_1, ..., \min F_{n-1}, \min F_n\} = \{\min F_1, ..., \min F_{n-1}, \min P_k^n\} \in \mathcal{M}^{(1)}.$$

Thus $F = \bigcup_{i=1}^n F_i \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}] \subseteq \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\gamma+1)}\right)$, as required.

Case 2. Suppose $\min F_n \notin P_k$ for all $k \in \mathbb{N}$.

In this case, $G_k \cap [\min F_n, \infty) \subseteq Q_k$ for all k. Hence $G_k \cap [\min F_n, \infty) \in \mathcal{N}^{(\gamma)}$ for all k. Furthermore, $G_k \cap [\min F_n, \infty)$ converges to $F \cap [\min F_n, \infty) = F_n$ nontrivially. Thus $F_n \in \mathcal{N}^{(\gamma+1)}$. Therefore $F = P \cup F_n \in \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\gamma+1)}\right)$, as required.

Suppose $\gamma \leq \alpha$ is a limit ordinal and the result holds for all $\eta < \gamma$. Let $F \in \left(\mathcal{M}[\mathcal{N}]\right)^{(\gamma)}$ have standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. By the inductive hypothesis, for each $\eta < \gamma$, $F = P_\eta \cup Q_\eta$, where $P_\eta < Q_\eta$, $P_\eta \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}]$ and $Q_\eta \in \mathcal{N}^{(\eta)}$. By the argument in Case 1 above, if there exists η such that $\min F_n \in P_\eta$, then $F \in \left(\mathcal{M}^{(1)}\right)[\mathcal{N}] \subseteq \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\eta)}\right)$. Otherwise, $F_n \subseteq Q_\eta \in \mathcal{N}^{(\eta)}$ for all $\eta < \gamma$. Hence $F = \left(\bigcup_{i=1}^{n-1} F_i\right) \cup F_n \in \left(\left(\mathcal{M}^{(1)}\right)[\mathcal{N}], \mathcal{N}^{(\gamma)}\right)$. This completes the induction. □

Proposition 10. *Let $\mathcal{M}$, $\mathcal{N} \subseteq [\mathbb{N}]^{<\infty}$ be regular. Suppose that $\iota(\mathcal{N}) = \alpha$. Then for all $\beta < \omega_1$, $\left(\mathcal{M}[\mathcal{N}]\right)^{(\alpha\cdot\beta)} = \left(\mathcal{M}^{(\beta)}\right)[\mathcal{N}]$.*

Proof. The proof is by induction on β. The case $\beta = 0$ is clear. Suppose the result is true for some β. Then

$$\begin{aligned}(\mathcal{M}[\mathcal{N}])^{(\alpha\cdot(\beta+1))} &= (\mathcal{M}[\mathcal{N}])^{(\alpha\cdot\beta+\alpha)} \\ &= \left((\mathcal{M}[\mathcal{N}])^{(\alpha\cdot\beta)}\right)^{(\alpha)} \\ &= \left(\left(\mathcal{M}^{(\beta)}\right)[\mathcal{N}]\right)^{(\alpha)} \text{ by the inductive hypothesis,} \\ &= \left(\left(\mathcal{M}^{(\beta)}\right)^{(1)}[\mathcal{N}]\right) \text{ by Lemma 9,} \\ &= \left(\mathcal{M}^{(\beta+1)}\right)[\mathcal{N}].\end{aligned}$$

Suppose the proposition is true for all $\beta < \beta_0$, where $\beta_0 < \omega_1$ is some limit ordinal. Clearly,

$$\left(\mathcal{M}^{(\beta_0)}\right)[\mathcal{N}] \subseteq \bigcap_{\beta<\beta_0}\left(\mathcal{M}^{(\beta)}\right)[\mathcal{N}] = \bigcap_{\beta<\beta_0}(\mathcal{M}[\mathcal{N}])^{(\alpha\cdot\beta)} = (\mathcal{M}[\mathcal{N}])^{(\alpha\cdot\beta_0)}.$$

On the other hand, let $F \in \bigcap_{\beta<\beta_0}\left(\mathcal{M}^{(\beta)}\right)[\mathcal{N}]$ have standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. It is clear that $(F_i)_{i=1}^n$ is also the standard representation of F as an element of $\left(\mathcal{M}^{(\beta)}\right)[\mathcal{N}]$ for any $\beta < \beta_0$. In particular, $\{\min F_i : 1 \le i \le n\} \in \mathcal{M}^{(\beta)}$ for all $\beta < \beta_0$. Hence $\{\min F_i : 1 \le i \le n\} \in \mathcal{M}^{(\beta_0)}$. It follows that $F \in \left(\mathcal{M}^{(\beta_0)}\right)[\mathcal{N}]$. This completes the proof. □

It is well known that $\iota(\mathcal{S}_\gamma) = \omega^\gamma$ for all $\gamma < \omega_1$ ([1, Proposition 4.10]). The indices of $\mathcal{F}_n$ and $\mathcal{F}'_n$ can be computed readily with the help of Proposition 10.

Corollary 11. *$\iota(\mathcal{F}_n) = \omega^{\alpha+\beta\cdot n}$ and $\iota(\mathcal{F}'_n) = \omega^{\alpha+\beta\cdot n}\cdot 2$.*

Before proceeding further, let us recall the relevant terminology concerning trees. A *tree* on a set X is a subset T of $\bigcup_{n=1}^\infty X^n$ such that $(x_1,\dots,x_n) \in T$ whenever $n \in \mathbb{N}$ and $(x_1,\dots,x_{n+1}) \in T$. These are the only kind of trees we will consider. A tree T is *well-founded* if there is no infinite sequence (x_n) in X such that $(x_1,\dots,x_n) \in T$ for all n. Given a well-founded tree T, we define the *derived tree* $D(T)$ to be the set of all $(x_1,\dots,x_n) \in T$ such that $(x_1,\dots,x_n,x) \in T$ for some $x \in X$. Inductively, we let $D^0(T) = T$, $D^{\alpha+1}(T) = D(D^\alpha(T))$, and $D^\alpha(T) = \bigcap_{\beta<\alpha} D^\beta(T)$ if α is a limit ordinal. The *order* of a well-founded tree T is the smallest ordinal $o(T)$ such that $D^{o(T)}(T) = \emptyset$. If E is a Banach space and $1 \le K < \infty$, an ℓ^1-K tree on E is a tree T on $S(E) = \{x \in E : \|x\| = 1\}$ such that $\|\sum_{i=1}^n a_i x_i\| \ge K^{-1}\sum_{i=1}^n |a_i|$ whenever $(x_1,\dots,x_n) \in T$ and $(a_i) \subseteq \mathbb{R}$. If E has a basis (e_i), a *block tree* on E is a tree T on E so that every $(x_1,\dots,x_n) \in T$ is a finite block basis of (e_i). An ℓ^1-K-block tree on E is a block tree that is also an ℓ^1-K tree. The index $I(E,K)$ is defined to be $\sup\{o(T) : T \text{ is an } \ell^1\text{-}K \text{ tree on } E\}$. If E has a basis (e_i), the index $I_b(E,K)$ is defined similarly, with the supremum taken over all ℓ^1-K-block trees. The Bourgain ℓ^1-*index* of E is the ordinal $I(E) = \sup\{I(E,K) : 1 \le K < \infty\}$. The index $I_b(E)$ is defined similarly. Bourgain proved that if E is a separable Banach space not containing a copy of ℓ^1, then $I(E) < \omega_1$ [4]. More on these and related indices can be found in [8] and [3].

Proposition 12. *Let T be a well-founded block tree on some basis (e_i). Define*

$$\mathcal{F}(T) = \{\{\max\operatorname{supp} x_i : i = 1,\dots,n\} : (x_1, x_2, \dots, x_n) \in T\}$$

and

$$\mathcal{G}(T) = \{G : \exists F \in \mathcal{F}(T), f : \mathbb{N} \to \mathbb{N} \text{ strictly increasing, such that } G \subseteq f(F)\}.$$

Then $\iota(\mathcal{G}(T)) \geq o(T)$.

Proof. Let $\xi = o(T)$. The proof is by induction on ξ. If $o(T) = 1$, then $\mathcal{G}(T) \supseteq \{\{k\} : k \geq n\}$ for some $n \in \mathbb{N}$. Therefore $(\mathcal{G}(T))^{(1)} \supseteq \{\emptyset\}$ and hence $\iota(\mathcal{G}(T)) \geq 1 = o(T)$.

Suppose the proposition holds for some $\xi < \omega_1$. Let T be a well-founded block tree with $o(T) = \xi + 1$ such that $\mathcal{G}(T)$ is compact. For each $(x) \in T$, let

$$T_x = \bigcup_{n=1}^{\infty} \{(x_1, ..., x_n) : (x, x_1, ..., x_n) \in T\}.$$

According to [4, Proposition 4], $o(T) = \sup_{(x)\in T} \{o(T_x) + 1\}$. Therefore, there exists $(x_0) \in T$ such that $o(T_{x_0}) = \xi$. By the inductive hypothesis, $\iota(\mathcal{G}(T_{x_0})) \geq \xi$. Let $k_0 = \max \operatorname{supp} x_0$. Then $G \in \mathcal{G}(T_{x_0})$ implies $\{k_0\} \cup G \in \mathcal{G}(T)$. Thus $\{k_0\} \in (\mathcal{G}(T))^{(\xi)}$. Since $(\mathcal{G}(T))^{(\xi)}$ is spreading, $\{k\} \in (\mathcal{G}(T))^{(\xi)}$ for all $k \geq k_0$. It follows that $\emptyset \in (\mathcal{G}(T))^{(\xi+1)}$. Hence $\iota(\mathcal{G}(T)) \geq \xi + 1 = o(T)$.

Suppose $o(T) = \xi_0$, where ξ_0 is a countable limit ordinal and the proposition holds for all $\xi < \xi_0$. Since $o(T) > \xi$ for all $\xi < \xi_0$, by the inductive hypothesis, $\iota(\mathcal{G}(T)) > \xi$ for all $\xi < \xi_0$. Hence $\iota(\mathcal{G}(T)) \geq \xi_0 = o(T)$. This completes the induction. □

If $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ are sequences in possibly different normed spaces and $0 < K < \infty$, we write $(x_k)_{k=1}^n \overset{K}{\succeq} (y_k)_{k=1}^n$ to mean $K \left\|\sum_{k=1}^n a_k x_k\right\| \geq \left\|\sum_{k=1}^n a_k y_k\right\|$ for all $(a_k) \subseteq \mathbb{R}$.

Theorem 13. $I\left(\widetilde{T}_{\alpha,\beta}\right) = I_b\left(\widetilde{T}_{\alpha,\beta}\right) = \omega^{\alpha+\beta\cdot\omega}$.

Proof. If $I_b\left(\widetilde{T}_{\alpha,\beta}\right) > \omega^{\alpha+\beta\cdot\omega}$, then $I_b\left(\widetilde{T}_{\alpha,\beta}, K\right) > \omega^{\alpha+\beta\cdot\omega}$ for some $K > 1$. Hence there exists an ℓ^1-K-block tree T on $\widetilde{T}_{\alpha,\beta}$ such that $o(T) = \xi > \omega^{\alpha+\beta\cdot\omega}$. Given $F \in \mathcal{F}(T)$, there exists $(x_1, x_2, ..., x_n) \in T$ such that $F = \{\max \operatorname{supp} x_i\}_{i=1}^n$. According to Proposition 4, $(e_k)_{k\in F} \overset{2}{\succeq} (x_1, x_2, ..., x_n)$, where $(e_k)_{k=1}^\infty$ is the unit vector basis of $\widetilde{\widetilde{T}}_{\alpha,\beta}$. Since $(x_1, x_2, ..., x_n) \in T$, $(x_1, x_2, ..., x_n) \overset{K}{\succeq} \ell^1(|F|)$-basis. Therefore, $(e_k)_{k\in F} \overset{2K}{\succeq} \ell^1(|F|)$-basis for all $F \in \mathcal{F}(T)$. Since it is clear that $\left\|\sum a_k e_{f(k)}\right\|_{\widetilde{\widetilde{T}}} \geq \left\|\sum a_k e_k\right\|_{\widetilde{\widetilde{T}}}$ for all $(a_k) \in c_{00}$ whenever $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing, it follows that $(e_k)_{k\in G} \overset{2K}{\succeq} \ell^1(|G|)$-basis for all $G \in \mathcal{G}(T)$. By Proposition 12, $(\mathcal{G}(T))^{(\omega^{\alpha+\beta\cdot\omega}+1)} \neq \emptyset$. Thus by [7, Corollary 1.2], there exists $L \in [\mathbb{N}]$ such that $\mathcal{S}_{\alpha+\beta\cdot\omega} \cap [L]^{<\infty} \subseteq \mathcal{G}(T)$. Hence, for all $(a_k) \in c_{00}$ and all $G \in \mathcal{S}_{\alpha+\beta\cdot\omega} \cap [L]^{<\infty}$,

$$\left\|\sum_{k\in G} a_k e_k\right\|_{\widetilde{\widetilde{T}}} \geq \frac{1}{2K} \sum_{k\in G} |a_k|. \tag{5}$$

Choose $m \in \mathbb{N}$ such that $2^m > 2K$. According to Corollary 11, $\iota(\mathcal{F}_i') = \omega^{\alpha+\beta\cdot i} \cdot 2$ for all $i = 1, 2, ..., m$. Applying [7, Corollary 1.2], we obtain $M \in [L]$ such that $\mathcal{F}_i' \cap [M]^{<\infty} \subseteq \mathcal{S}_{\alpha+\beta\cdot m+1}$ for all $i = 1, 2, ..., m$. By [11, Proposition 3.6], there exists

$F \in \mathcal{S}_{\alpha+\beta\cdot\omega}(M) \subseteq \mathcal{S}_{\alpha+\beta\cdot\omega} \cap [M]^{<\infty}$ and $(a_j)_{j\in F} \subseteq \mathbb{R}^+$ such that $\sum a_j = 1$ and if $G \subseteq F$ with $G \in \mathcal{S}_{\alpha+\beta\cdot m+1}$, then $\sum_{j\in G} a_j < \frac{1}{8K}$. Consider $x = \sum_{j\in F} a_j e_j \in \widetilde{\widetilde{T}}_{\alpha,\beta}$. If $1 \le i \le m$ and $I \in \mathcal{F}'_i$, then $I \cap F \in \mathcal{F}'_i \cap [M]^{<\infty} \subseteq \mathcal{S}_{\alpha+\beta\cdot m+1}$. Hence $\sigma_I(x) = \sigma_{I\cap F}(x) < \frac{1}{8K}$. It follows that $\rho'_i(x) \le \frac{1}{8K}$ for $1 \le i \le m$. By Proposition 6,

$$\begin{aligned}\|x\|_{\widetilde{\widetilde{T}}} &\le \sum_{i=0}^{m} \frac{\rho'_i(x)}{2^i} + \frac{1}{2^{m+1}} \sup\left\{\sum_{i=1}^{j} \|E_i x\|_{\widetilde{\widetilde{T}}} : \{E_1,...,E_j\}\ \mathcal{G}'_{m+1}\text{-admissible}\right\} \\ &\le \sum_{i=0}^{m} \frac{\frac{1}{8K}}{2^i} + \frac{1}{2^{m+1}} \|x\|_{\ell^1} < \frac{1}{2K},\end{aligned}$$

contrary to (5). This proves that $I_b\left(\widetilde{T}_{\alpha,\beta}\right) \le \omega^{\alpha+\beta\cdot\omega}$. On the other hand, according to Proposition 5, for any $n \in \mathbb{N}$, $\|a\|_{\widetilde{T}} \ge \frac{1}{2^n}\|a\|_{\mathcal{F}_n}$ for any $a \in c_{00}$. By Corollary 11, $\iota(\mathcal{F}_n) = \omega^{\alpha+\beta\cdot n}$. Therefore, there exists an ℓ^1-2^n-block basis tree T_n on $\widetilde{T}_{\alpha,\beta}$ with $o(T_n) = \omega^{\alpha+\beta\cdot n}$. Hence $I_b\left(\widetilde{T}_{\alpha,\beta}, 2^n\right) \ge \omega^{\alpha+\beta\cdot n}$. Thus $I_b\left(\widetilde{T}_{\alpha,\beta}\right) = \sup_K I_b\left(\widetilde{T}_{\alpha,\beta}, K\right) \ge \omega^{\alpha+\beta\cdot\omega}$. We conclude that $I_b\left(\widetilde{T}_{\alpha,\beta}\right) = \omega^{\alpha+\beta\cdot\omega}$. As $I\left(\widetilde{T}_{\alpha,\beta}\right) \ge I_b\left(\widetilde{T}_{\alpha,\beta}\right) \ge \omega^\omega$, it follows from [8, Corollary 5.13] that $I\left(\widetilde{T}_{\alpha,\beta}\right) = I_b\left(\widetilde{T}_{\alpha,\beta}\right)$. □

For the final corollary, recall that the Schreier space X_α, $\alpha < \omega_1$, is the completion of c_{00} with respect to the norm $\|\cdot\|_{\mathcal{S}_\alpha}$.

Corollary 14. *Suppose α is a countable ordinal whose Cantor normal form is $\omega^{\alpha_1} \cdot m_1 + ... + \omega^{\alpha_k} \cdot m_k$. If α_k is not a limit ordinal, then there exists a Banach space X such that $I(X) = \omega^\alpha$.*

Proof. If $\alpha_k = 0$, then α is a successor ordinal and $\iota(X_{\alpha-1}) = \omega^\alpha$ ([3]). If α_k is a successor ordinal, let $\gamma = \omega^{\alpha_1} \cdot m_1 + ... + \omega^{\alpha_k} \cdot (m_k - 1)$ and $\eta = \omega^{\alpha_k - 1}$. By Theorem 13, $I\left(\widetilde{T}_{\gamma,\eta}\right) = \omega^{\gamma+\eta\cdot\omega} = \omega^\alpha$. □

Remark 15. The following analog of Proposition 4 for the space X_α, $\alpha < \omega_1$, holds obviously: If $(x_i)_{i=1}^p$ is a normalized block basis of the unit vector basis (e_k) of X_α and $k_i = \max \operatorname{supp} x_i$, $1 \le i \le p$, then $\|\sum_{i=1}^p a_i x_i\| \le \|\sum_{i=1}^p a_i e_{k_i}\|$ for all $(a_i) \in c_{00}$. Therefore the arguments in Proposition 12 and Theorem 13 can be used to compute $I_b(X_\alpha)$ (with respect to the basis (e_k)).

References

[1] D. E. Alspach and S. Argyros, Complexity of weakly null sequences, Diss. Math., 321 (1992), 1-44. MR **93j:**46014

[2] S. A. Argyros, S. Mercourakis and A. Tsarpalias, Convex unconditionality and summability of weakly null sequences, Israel J. Math. **107** (1998), 157-193. MR **99m:**46021

[3] D. E. Alspach, R. Judd and E. Odell, The Szlenk index and local ℓ_1-indices, preprint.

[4] J. Bourgain, On convergent sequences of continuous functions, Bull. Soc. Math. Bel., 32 (1980), 235-249. MR **84e:**46018

[5] P. G. Casazza, W. B. Johnson and L. Tzafriri, On Tsirelson's space, Israel J. Math. 47 (1984), 81-98. MR **85m:**46013

[6] T. Figiel and W. B. Johnson, A uniformly convex Banach space which contains no ℓ_p, Compositio Math. 29 (1974), 179-190. MR **50:**8011

[7] I. Gasparis, A dichotomy theorem for subsets of the power set of the natural numbers, Proc. Amer. Math. Soc. 129 (2001), 759-764. MR **2001f:**03089

[8] R. Judd, E. Odell, Concerning the Bourgain ℓ_1 index of a Banach space, Israel J. Math. 108 (1998), 145–171. MR **2000k:**46013

[9] P. Kiriakouli, Characterizations of spreading models of ℓ_1, Comment. Math. Univ. Carolinae, 41(2000), 79-95. MR **2001e:**46040

[10] D. H. Leung and W.-K. Tang, The Bourgain ℓ^1-indices of mixed Tsirelson spaces, in preparation.

[11] Edward Odell, Nicole Tomczak-Jaegermann, and Roy Wagner, Proximity to ℓ_1 and distortion in asymptotic ℓ_1 spaces, Journal of Functional Analysis 150(1997), 101-145. MR **2000c:**46027

[12] B. S. Tsirelson, Not every Banach space contains an embedding of ℓ_p or c_0, Functional Anal. Appl. 8 (1974), 138-141.

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