GENERALIZED SCHWARZ-PICK ESTIMATES

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Abstract. We obtain higher derivative generalizations of the Schwarz-Pick inequality for analytic self-maps of the unit disk as a consequence of recent characterizations of boundedness and compactness of weighted composition operators between Bloch-type spaces.

1. Introduction

Part of the Schwarz-Pick inequality, sometimes called the invariant Schwarz inequality, says that whenever \( \varphi \) is an analytic self-map of the unit disk \( \mathbb{D} \), then

\[
\frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2} \leq 1
\]

for all \( z \in \mathbb{D} \). If \( C_\varphi \) is the composition operator defined by \( C_\varphi(f) = f \circ \varphi \) for \( f \) analytic in \( \mathbb{D} \), the Schwarz-Pick inequality directly yields the boundedness of all composition operators on the classical Bloch space. We will prove the following generalized Schwarz-Pick estimates.

**Theorem 1.** For \( n \geq 1 \) and \( \varphi \) an analytic self-map of \( \mathbb{D} \),

\[
\sup_{z \in \mathbb{D}} \frac{|\varphi^{(n)}(z)|(1 - |z|^2)^n}{1 - |\varphi(z)|^2} < \infty.
\]

Our proof of this theorem will be an application of boundedness criteria for weighted composition operators between various Bloch-type spaces recently obtained in [3]. These Bloch-type spaces and boundedness criteria for weighted composition operators will be discussed in the next section, which also contains the proof of the above theorem. A natural generalization of the above result is given in Theorem 3 when \( \varphi \) satisfies an additional condition. In Section 3 we give “little-oh” versions of Theorems 1 and 3 and in Section 4 we briefly discuss converses to our main results.

2. Proof of the main theorem

The Bloch-type spaces we consider here are defined by

\[
B^\alpha = \{ f \text{ analytic in } \mathbb{D} : \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty \}.
\]

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These become Banach spaces with norms $|f(0)| + \sup \{ (1-|z|^2)^\alpha |f'(z)| : z \in \mathbb{D} \}$. The range of the parameter $\alpha$ can be taken to be $0 < \alpha < \infty$, although our interest here is restricted to the case $\alpha \geq 1$. Note that $\alpha = 1$ gives the classical Bloch space $B$. A weighted composition operator $uC_\varphi$ is defined for analytic $u$ on $\mathbb{D}$ and analytic self-map $\varphi$ of $\mathbb{D}$ by $uC_\varphi(f) = u(f \circ \varphi)$. A characterization of boundedness of $uC_\varphi$ from $B^\alpha$ to $B^\beta$ is given in Theorem 2.1 of [3]; this characterization depends on whether $0 < \alpha < 1$, $\alpha = 1$, or $\alpha > 1$. Here we will only make use of the $\alpha > 1$ case:

**Theorem 2** ([3]). When $\alpha > 1$ and $\beta > 0$ the weighted composition operator $uC_\varphi$ maps $B^\alpha$ boundedly into $B^\beta$ if and only if

(a) $\sup_{z \in \mathbb{D}} |u(z)| \left( \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| \right) < \infty$ and

(b) $\sup_{z \in \mathbb{D}} |\varphi'(z)| \left( \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} \right) < \infty$.

**Proof of Theorem 2** For $n = 1$, the result is the classical Schwarz-Pick inequality. The rest of the argument proceeds by induction, however it is instructive to look explicitly at the $n = 2$ case. For this, note that $DC_\varphi$ is bounded from $B^1$ to $B^2$ for all $\varphi$, as noted above. We have $DC_\varphi(f) = (f' \circ \varphi)\varphi'$. Thus the weighted composition operator $\varphi'C_\varphi$ is bounded from $B^2$ to $B^2$, since $f \in B^1$ if and only if $f' \in B^2$. In particular by (b) of the boundedness criteria above we have the desired statement for $n = 2$.

Now fix an integer $n \geq 2$ and assume by induction that the generalized Schwarz-Pick estimates hold for all positive integers less than or equal to $n$. We will show that the estimate holds for $n + 1$. Consider the bounded operator $D^nC_\varphi : B^1 \to B^{n+1}$.

If we can show that $\varphi^nC_\varphi$ is bounded from $B^2$ to $B^{n+1}$, then again part (b) of the boundedness criteria above will yield the generalized Schwarz-Pick estimate for $n + 1$. To see why the boundedness of $\varphi^nC_\varphi : B^2 \to B^{n+1}$ follows from the boundedness of $D^nC_\varphi : B^1 \to B^{n+1}$ we consider the expansion of $D^n(f \circ \varphi) = (f \circ \varphi)^{(n)}$ by Faà di Bruno’s formula (see, for example, [4]):

$$
(f \circ \varphi)^{(n)}(z) = \sum_{k_1 + k_2 + \cdots + k_n = n} \frac{n!}{k_1! k_2! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \left( \frac{\varphi^{(j)}(z)}{j!} \right)^{k_j},
$$

where $k = k_1 + k_2 + \cdots + k_n$ and this sum is over all non-negative integers $k_1, k_2, \ldots, k_n$ satisfying $k_1 + 2k_2 + \cdots + nk_n = n$. In particular, one of the terms of this sum is $f' \circ \varphi(\varphi(z)) (\varphi^{(n)}(z))$ and the remaining terms involve products of $f^{(k)} \circ \varphi(z)$ ($1 < k \leq n$) with products of derivatives of $\varphi$. Writing Faà di Bruno’s formula in operator notation we have

$$
D^nC_\varphi = \sum_{k_1 + k_2 + \cdots + k_n = n} \frac{n!}{k_1! k_2! \cdots k_n!} \prod_{j=1}^{n} \left( \frac{D^j \varphi}{j!} \right)^{k_j} C_\varphi D^k.
$$
With $k_n = 1$ (and therefore also $k_1 = k_2 = \cdots = k_{n-1} = 0$) we obtain on the right the term $\varphi^{(n)} C_\varphi D$. If $k_n = 0$ we obtain (constant multiples of) the terms
\[ \prod_{j=1}^{n-1} (\varphi^{(j)}) k_j C_\varphi D^k, \]
where $k = k_1 + \cdots + k_{n-1}$, and $k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n$. Set
\[ u(z) = \prod_{j=1}^{n-1} (\varphi^{(j)}(z)) k_j, \]
where the non-negative integers $k_1, \ldots, k_{n-1}$ are as just described. Our goal is to show that each weighted composition operator $uC_\varphi$ is bounded from $B^{k+1}$ to $B^{n+1}$; this together with the boundedness of $D^n C_\varphi : B^1 \to B^{n+1}$ will imply the boundedness of $\varphi^{(n)} C_\varphi$ from $B^2$ to $B^{n+1}$. To show that $uC_\varphi$ is bounded from $B^{k+1}$ to $B^{n+1}$ we must verify conditions (a) and (b) of Theorem 2.

For condition (a) we observe that the product
\[ \sup_{z \in D} \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k+1}} |u(z)| |\varphi'(z)| \]
can be written as
\[ \left( \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} \right)^{k+1} \prod_{j=2}^{n-1} \left( \frac{(1 - |z|^2)|\varphi^{(j)}(z)|}{1 - |\varphi(z)|^2} \right)^{k_j}, \]
since $n+1 = (k_1 + 1) + 2k_2 + \cdots + (n-1)k_{n-1}$. Using the induction hypothesis we see that
\[ \sup_{z \in D} \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k+1}} |u(z)||\varphi'(z)| < \infty. \]

For condition (b) of Theorem 2 notice that
\[ u'(z) = \sum_{i=1}^{n-1} k_i (\varphi^{(i)}(z))^{k_i} \varphi^{(i+1)}(z) \prod_{j=1, j \neq i}^{n-1} (\varphi^{(j)}(z))^{k_j}. \]

We claim that
\[ \sup_{z \in D} \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k}} |u'(z)| < \infty. \]

To see this, note that when $k_i \neq 0$ we see by the induction hypothesis that
\[ \left| \varphi^{(i)}(z) \right|^{k_i} \left| \varphi^{(i+1)}(z) \right| \prod_{j=1, j \neq i}^{n-1} \left| \varphi^{(j)}(z) \right|^{k_j} \]
is bounded above by a constant multiple of
\[ \frac{(1 - |\varphi(z)|^2)^{k_i-1}}{(1 - |z|^2)^{k_i}} \left( 1 - |\varphi(z)|^2 \right)^{n-1} \prod_{j=1, j \neq i}^{n-1} \left( \frac{1 - |\varphi(z)|^2}{(1 - |z|^2)^2} \right)^{k_j} = \frac{(1 - |\varphi(z)|^2)^k}{(1 - |z|^2)^{n+1}}, \]
and our claim follows.
Conditions (a) and (b) in Theorem 2 are satisfied and the operator $uC_{\varphi}$ maps $B^{k+1}$ boundedly into $B^{n+1}$. The operator $D^k$ maps $B^1$ boundedly onto $B^{k+1}$. Thus, for each $k$ as above with $k_n = 0$, the operator

$$\prod_{j=1}^{n-1} \left( \varphi^{(j)} \right)^{k_j} C_{\varphi} D^k$$

maps $B^1$ boundedly into $B^{n+1}$. We conclude that $\varphi^{(n)}C_{\varphi}D$ maps $B^1$ boundedly into $B^{n+1}$. Since $D$ maps $B^1$ onto $B^2$, the weighted composition operator $\varphi^{(n)}C_{\varphi}$ maps $B^2$ boundedly onto $B^{n+1}$. By condition (b) of Theorem 2 this implies that

$$|\varphi^{(n+1)}(z)| \frac{(1 - |z|^2)^{n+1}}{1 - |\varphi(z)|^2} = |(\varphi^{(n)})'(z)| \frac{(1 - |z|^2)^n}{1 - |\varphi(z)|^2}$$

is bounded. This completes the induction, and the proof. \qed

Theorem 3 can easily be generalized as follows.

**Theorem 3.** Let $\varphi : D \to D$ be an analytic map such that

$$\sup_{z \in D} \frac{(1 - |z|^2)^{\beta} |\varphi(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} < \infty$$

for some $\alpha, \beta > 0$. Then for each integer $n \geq 2$,

$$\sup_{z \in D} \frac{(1 - |z|^2)^{\beta + n - 1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} < \infty.$$

**Proof.** The hypothesis insures that $C_{\varphi}$ is bounded from $B^\alpha$ to $B^{\beta}$ (\cite{3}, Corollary 2.4) so $DC_{\varphi}$ is bounded from $B^\alpha$ to $B^{\beta+1}$. Since $DC_{\varphi} = \varphi' C_{\varphi} D$ it follows that $\varphi' C_{\varphi}$ must be bounded from $B^{n+1}$ to $B^{\beta+1}$. Part (b) of Theorem 2 gives the desired conclusion for $n = 2$. We proceed by induction in much the same way as was done in the proof of Theorem 1. Assume the result holds for all positive integers less than or equal to $n$. To obtain the result for $n + 1$ we show that $\varphi^{(n)} C_{\varphi}$ is bounded from $B^{n+1}$ to $B^{\beta+n}$, and then appeal to Theorem 2. As in the proof of Theorem 1 boundedness of $\varphi^{(n)} C_{\varphi}$ will follow from the boundedness of $D^n C_{\varphi}$ from $B^\alpha$ to $B^{\beta+n}$ and \cite{1} if we can show that $u C_{\varphi}$ is bounded from $B^{n+k}$ to $B^{\beta+n}$, $1 \leq k < n$, when $u$ is given by \cite{2}. Condition (a) of Theorem 2 follows from the observation that

$$|u(z)| \frac{(1 - |z|^2)^{\beta+n}}{(1 - |\varphi(z)|^2)^{\alpha+k} |\varphi'(z)|} = \frac{(1 - |z|^2)^{\beta} |\varphi(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} \prod_{j=1}^{n-1} \left( \frac{(1 - |z|^2)^{\beta+j} |\varphi^{(j)}(z)|}{1 - |\varphi(z)|^2} \right)^{k_j}.$$

The first factor is bounded on $D$ by hypothesis, and the other factors are bounded on $D$ by Theorem 1.

Similarly, to check condition (b) we must show that

$$\frac{|u'(z)|(1 - |z|^2)^{\beta+n}}{(1 - |\varphi(z)|^2)^{\alpha+k-1}}$$
is bounded on \( D \). Using the expression for \( u'(z) \) given in (4) this follows by observing that for \( k_i \geq 1 \) the expression

\[
\left| \varphi^{(i)}(z) \right|^{k_i - 1} \left| \varphi^{(i+1)}(z) \right| \prod_{j=1, j \neq i}^{n-1} \left| \varphi^{(j)}(z) \right|^{k_j}
\]

is bounded above by a constant multiple of

\[
\frac{(1 - |\varphi(z)|^2)^\alpha}{(1 - |z|^2)^{\beta+1}} \frac{(1 - |\varphi(z)|^2)^{k_i - 1}}{(1 - |z|^2)^{(k_i - 1)}} \prod_{j=1, j \neq i}^{n-1} \left( \frac{1 - |\varphi(z)|^2}{(1 - |z|^2)} \right)^{k_j} = \frac{(1 - |\varphi(z)|^2)^{\alpha + k_i - 1}}{(1 - |z|^2)^{\beta+n}},
\]

which gives the desired result. This completes the verification of the boundedness of \( uC_{\varphi} \) from \( B_\alpha \) to \( B_\beta \), and the theorem follows exactly as in Theorem 1.

3. THE HYPERBOLIC LITTLE BLOCH CLASS

Recall that an analytic self-map of the disk \( \varphi \) is said to be in the hyperbolic little Bloch class \( B_h^0 \) if

\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| = 0.
\]

Note this implies in particular that \( \varphi \) is in the little Bloch space \( B_0 \), the subspace of \( B \) consisting of Bloch functions \( f \) satisfying \( \lim_{|w| \to 1} \frac{1 - |w|^2}{1 - |f'(w)|} (1 - |w|^2) = 0 \). The hyperbolic little Bloch class appears in the characterization of those composition operators which are compact on the little Bloch space: \( C_{\varphi} \) is compact from \( B_0 \) to itself if and only if \( \varphi \in B_h^0 \) (2, Theorem 1).

A particular case of the next result shows that functions in the hyperbolic little Bloch class satisfy a little-oh version of our generalized Schwarz-Pick estimates.

**Theorem 4.** Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be an analytic map such that

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0
\]

for some \( \alpha, \beta > 0 \). Then for each integer \( n \geq 2 \),

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0.
\]

In particular, if \( \varphi \in B_h^0 \), then

\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0
\]

for every positive integer \( n \).

Theorem 4 can be proved by similar techniques to those employed in Theorem 3 using Theorem 3.1 of [3] which characterizes compactness of weighted composition operators from \( B_\alpha^0 \) to \( B_\beta^0 \) by little oh analogues of (a) and (b) of Theorem 2. We omit the details.
4. Converse results

For certain positive $\alpha$ and $\beta$ the implications in Theorem 3 and Theorem 4 are actually logical equivalences.

**Theorem 5.** Let $\varphi$ be an analytic self-map of the unit disk and $\beta > \alpha > 0$. Then

$$\sup_{z \in D} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty$$

if and only if

$$\sup_{z \in D} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty$$

for each positive integer $n$.

Furthermore,

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0$$

if and only if

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0$$

for each positive integer $n$.

We do not give the proof of this result here, but note that the interest in the first part of Theorem 5 in the “if” direction is when $0 < \alpha < \beta < 1$, as the condition

$$\sup_{z \in D} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty$$

holds automatically for all self-maps when $\alpha \leq \beta$ and $\beta \geq 1$.

The “if” directions of the two statements in Theorem 5 need not hold if $\beta < \alpha$. For example, if $n \geq 2$ and $\varphi(z) = \frac{1}{2}z^n - \frac{1}{2}$, then $\varphi^{(n)}(z) = 0$ so that

$$\sup_{z \in D} \frac{|\varphi^{(n)}(z)|(1 - |z|^2)^{\beta+n-1}}{(1 - |\varphi(z)|^2)^\alpha} = \lim_{|z| \to 1^-} \frac{|\varphi^{(n)}(z)|(1 - |z|^2)^{\beta+n-1}}{(1 - |\varphi(z)|^2)^\alpha} = 0$$

for any $\alpha, \beta > 0$. However if we consider $z = r \in (0, 1)$ we have that

$$\frac{(1 - r^2)^\beta |\varphi'(r)|}{(1 - |\varphi(r)|^2)^\alpha}$$

is unbounded as $r \to 1^-$ if $\beta < \alpha$, and tends to a finite positive constant as $r \to 1^-$ if $\beta = \alpha$. Thus the hypothesis $\alpha < \beta$ is sharp for the second statement in Theorem 5 and close to sharp for the first statement.
REFERENCES


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