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ADJOINTS OF A CLASS OF COMPOSITION OPERATORS

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ABSTRACT. Adjoints of certain operators of composition type are calculated. Specifically, on the classical Hardy space $H_2(D)$ of the open unit disk D operators of the form $C_B(f) = f \circ B$ are considered, where B is a finite Blaschke product. C_B^* is obtained as a finite linear combination of operators of the form $T_g A_B T_h$, where g and h are rational functions, T_g, T_h are associated Toeplitz operators and A_B is defined by

$$A_B(f)(z) = \frac{1}{n} \sum_{B(\xi)=z} f(\xi).$$

1. Introduction

In the theory of composition operators determination of adjoints is a problem of some interest. (See [2].) In this paper we calculate the adjoints of certain operators of composition type. Specifically, on the classical Hardy space $H_2(D)$ of the open unit disk D, we consider operators of the form $C_B(f) = f \circ B$, where B is a finite Blaschke product

$$B(z) = e^{i\alpha} \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a}_j z}, |a_j| < 1.$$

In the case $n = 1$ it follows from a result of Cowen and MacCluer [1, Th.9.2] that

$$C_B^* = T_g C_{B^{-1}} T_h,$$

where T_g and T_h are certain Toeplitz operators. We will extend this result to the case $n > 1$ by obtaining C_B^* as a finite linear combination of operators of the form $T_g A_B T_h$, where g and h are rational functions and the operator A_B is defined by

$$A_B(f)(z) = \frac{1}{n} \sum_{B(\xi)=z} f(\xi).$$

For the remainder of the paper we assume that $n > 1$ and B has the form

$$B(z) = z \prod_{j=2}^{n} \frac{z - a_j}{1 - \overline{a}_j z}, |a_j| < 1. \tag{1}$$

That this restriction does not effect the generality of our results follows from [1, Th.9.2] and the fact that any finite Blaschke product may be written in the form

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$\sigma \circ B \circ \phi$, where ϕ and σ are linear fractional transformations of the disk onto itself and B satisfies (1).

2. Extending $A_B(f)$ to the unit circle

We recall that the space $H_2(D)$ consists of all analytic functions on D that satisfy

$$\|f\|^2 = \sup_{0<r<1} \|f_r\|_2^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f_r(e^{it})|^2 dt,$$

where $f_r(z) = f(rz)$. It is well known that the radial limits

$$\tilde{f}(e^{it}) = \lim_{r \to 1^-} f_r(e^{it})$$

exist for almost all $t \in [0, 2\pi]$, and that the correspondence $f \to \tilde{f}$ is an isometric linear map from $H_2(D)$ onto the closed subspace H_2 of $L_2([0, 2\pi])$ spanned by $1, e^{it}, e^{2it}, \ldots$, where the norm on $L_2([0, 2\pi])$ is defined by $\|g\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{it})|^2 dt$. It has been shown by Ryff [6] that $\widetilde{f \circ \phi} = \tilde{f} \circ \tilde{\phi}$ when $\phi : D \to D$ is analytic. It follows that the operator C_ϕ is unitarily equivalent to the operator $C_{\tilde{\phi}}$ defined on H_2. In this section we obtain an analogous result for the operator A_B, namely, A_B is unitarily equivalent via $f \to \tilde{f}$ to the operator $\tilde{A}_B(g)(e^{it}) = \frac{1}{n} \sum_{B(e^{is})=e^{it}} g(e^{is})$.

Lemma 1. *There is a strictly increasing continuously differentiable function $\beta : [0, 2\pi) \to [0, 2\pi n)$ with $\beta(0) = 0, \beta(2\pi) = 2\pi n$ and a constant c such that $B(e^{it}) = ce^{i\beta(t)}$. Furthermore, there is a constant b such that $\beta'(t) \le b$.*

Proof. For $t \in [0, 2\pi)$ let $\beta_1(t) = t$ and let $\beta_j(t)$ be a single valued branch of $\frac{1}{i} \log(\frac{e^{it} - a_j}{1 - \overline{a}_j e^{it}})$ for $j = 2, \ldots, n$. We note that $\beta_j(2\pi-) = \beta_j(0) + 2\pi$ and

$$\beta'_j(t) = \frac{1 - |a_j|^2}{|1 - \overline{a}_j e^{it}|^2} \le \frac{1 + |a_j|}{1 - |a_j|}$$

for $j = 2, \ldots, n$. Next, let $\beta(t) = d + t + \sum_{k=2}^n \beta_k(t)$, where the constant d is chosen so that $\beta(0) = 0$. Then β satisfies the asserted conditions.

Now we define functions $\theta_k(t) = \beta^{-1}(t + 2\pi k)$ for $t \in [0, 2\pi)$ and $k = 0, 1, \ldots, n-1$. Then $e^{it} = ce^{i\beta(\theta_k(t))} = B(e^{i\theta_k(t)})$. Thus, the terms $e^{i\theta_k(t)}, k = 0, \ldots, n-1$, are the n distinct roots of the equation $e^{it} = B(z)$. We now consider the operator $\tilde{A}_B$ defined on H_2 by

$$\tilde{A}_B(f)(e^{it}) = \frac{1}{n} \sum_{k=0}^{n-1} f(e^{i\theta_k(t)}).$$

Lemma 2. *$\tilde{A}_B$ is a bounded linear operator mapping H_2 into itself.*

Proof. By Cauchy's inequality

$$\begin{aligned} \int_0^{2\pi} |\tilde{A}_B(f)(e^{it})|^2 dt &\le \frac{1}{n} \sum_{k=0}^{n-1} \int_0^{2\pi} |f(e^{i\beta^{-1}(t+2\pi k)})|^2 dt \\ &= \frac{1}{n} \int_0^{2\pi n} |f(e^{i\beta^{-1}(t)})|^2 dt \\ &= \frac{1}{n} \int_0^{2\pi} |f(e^{is})|^2 \beta'(s) ds. \end{aligned}$$

Thus, by Lemma 1 we get $\int_0^{2\pi}|\tilde{A}_B(f)(e^{it})|^2dt \le \frac{b}{n}\|f\|_2^2$. It follows that $\tilde{A}$ is a bounded linear operator from H_2 into $L_2([0,2\pi])$. To show that $\tilde{A}_B(H_2) \subseteq H_2$ it suffices to prove that $\tilde{A}_B(z^m) \in H_2$ for $m = 0, 1, 2, \ldots$. For suitably chosen $r > 1$

$$\begin{aligned}\tilde{A}_B(z^m)(e^{it}) &= \frac{1}{n}\sum_{k=0}^{n-1} e^{im\theta_k(t)} \\ &= \frac{1}{2\pi n}\int_{|\xi|=r}\frac{\xi^m B'(\xi)}{B(\xi)-e^{it}}d\xi.\end{aligned}$$

Since $|B(\xi)| > 1$ on $|\xi| = r$, it follows that

$$\int_0^{2\pi} e^{ikt}\tilde{A}_B(z^m)(e^{it})dt = \frac{1}{2\pi in}\int_{|\xi|=r}\xi^m B'(\xi)\int_0^{2\pi}\frac{e^{ikt}}{B(\xi)-e^{it}}dtd\xi = 0$$

for $k = 1, 2, \ldots$.

The next result shows that the operator $\tilde{A}_B$ is the H_2-version of the operator A_B.

Theorem 1. (i) $A_B(f) \in H_2(D)\ \ \forall f \in H_2(D)$,
(ii) $\widetilde{A_B(f)} = \tilde{A}_B(\tilde{f})\ \ \forall f \in H_2(D)$,
(iii) *A_B is a bounded linear operator on $H_2(D)$.*

Proof. Suppose that g is analytic in a neighborhood of $\overline{D}$ so that g agrees with $\tilde{g}$ on the unit circle. Since

$$A_B(g)(z) = \frac{1}{2\pi in}\int_{|\xi|=1+\epsilon}\frac{g(\xi)B'(\xi)}{B(\xi)-z}d\xi$$

for some sufficiently small positive ϵ, it follows that $A_B(g)$ is also analytic in a neighborhood of $\overline{D}$ and $A_B(g)(e^{it}) = \tilde{A}_B(\tilde{g})(e^{it})$. Thus, from Lemma 2,

$$\|A_B(g)\| = \lim_{r\to 1^-}\|(A_B(g))_r\|_2 = \|\tilde{A}_B(\tilde{g})\|_2 \le c\|\tilde{g}\|_2 = c\|g\| \tag{2}$$

for some constant c.

Next, let f be an arbitrary member of $H_2(D)$ and $0 < s, r < 1$. It will be shown that $A_B(f_s)(re^{it})$ is bounded in s and t for fixed r and tends to $A_B(f)(re^{it})$ as $s \uparrow 1$. This follows from the compactness of the set $K = B^{-1}(\{re^{it} : 0 \le t < 2\pi\})$. Thus, by standard properties of H_2-functions, if $B(\xi) = re^{it}$, then $|f_s(\xi)| \le \frac{1+\rho}{1-\rho}\|f_s\|_2$, where $\rho = \sup\{|w| : w \in K\}$.

Hence,

$$|A_B(f_s)(re^{it})| = \frac{1}{n}|\sum_{B(\xi)=re^{it}} f_s(\xi)| \le \frac{1+\rho}{1-\rho}\|f_s\|_2 \le \frac{1+\rho}{1-\rho}\|f\|.$$

Since $A_B(f_s)(re^{it}) \to A_B(f)(re^{it})$ as $s \uparrow 1$, it follows that

$$\lim_{s\uparrow 1}\int_0^{2\pi}|A_B(f_s)(re^{it})|^2dt = \int_0^{2\pi}|A_B(f)(re^{it})|^2dt.$$

Thus, by (2) it follows that

$$\int_0^{2\pi}|A_B(f)(re^{it})|^2dt \le c\lim_{s\uparrow 1}\|f_s\| \le c\|f\|.$$

Both (i) and (iii) now follow easily.

Since both sides are bounded linear operators from $H_2(D)$ into H_2, it suffices to prove (ii) in the case where f is analytic in a neighborhood of $\overline{D}$. It was pointed out above that $A_B(f)$ is analytic in a neighborhood of $\overline{D}$ when f is. On the unit circle, $\widetilde{A_B(f)}(z)$ agrees with $A_B(f)(z)$ and the latter agrees with $\frac{1}{n}\sum_{B(\xi)=z} f(z)$. Since $|B(\xi)| = 1$ if and only if $|\xi| = 1$, it follows that

$$\widetilde{A_B(f)}(z) = \frac{1}{n}\sum_{B(\xi)=z} f(z) = \frac{1}{n}\sum_{B(\xi)=z} \tilde{f}(z) = \tilde{A}_B(\tilde{f})(z).$$

Hence, (ii) is also shown.

Theorem 1 shows that the formula

$$A_B(f)(z) = \frac{1}{n}\sum_{B(\xi)=z} f(\xi)$$

extends to the case $z = e^{it}$ for almost all $t \in [0, 2\pi)$. For this reason we will abuse notation only slightly, when, for the rest of the paper, we drop the tilde from $\tilde{A}_B$ and identify $H_2(D)$ and H_2 via $f \to \tilde{f}$.

3. Main results

Let

$$B_0(z) = 1, B_1(z) = z, B_k(z) = z\prod_{j=1}^{k} \frac{z - a_j}{1 - \hat{a}_j z} \quad \text{and} \quad B_n(z) = B(z).$$

It was shown by Rochberg [5] that every $f \in H_2$ has the unique decomposition

$$f = \sum_{k=0}^{n-1} B_k f_k \circ B, \tag{3}$$

where each f_k belongs to H_2. It follows from Rochberg's arguments that the operators $P_k : H_2 \to H_2$ defined by $P_k(f) = f_k$ are bounded and furthermore that $P_k(f)$ is continuous on $\overline{D}$ when f is. In [3] we gave some rather complicated formulas for the P_k's. Here we modify our previous approach to obtain expressions for the P_k's in terms of the operator A_B. We will then use the P_k's to compute C_B^*.

Theorem 2. *There is a matrix $Q(z) = (q_{\ell k}(z))_{\ell,k=0}^{n-1}$ of rational functions uniquely determined by B such that*

$$P_\ell(f) = \sum_{k=0}^{n-1} q_{\ell k} A_B(b_k f), \ell = 0, \ldots, n-1.$$

Proof. It follows from [5, Prop.1] and [3, Lemma 3.2] that, when $|z| = 1$, the roots $\xi_0, \ldots, \xi_{n-1}$ of the equation $B(\xi) = z$ are distinct and that the matrix $C(z) = (B_k(\xi_j))_{j,k=0}^{n-1}$ is non-singular. Suppose that f is continuous and in H_2. Then from (3)

$$\begin{pmatrix} P_0(f)(z) \\ P_1(f)(z) \\ \vdots \\ P_{n-1}(f)(z) \end{pmatrix} = C(z)^{-1} \begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_{n-1}) \end{pmatrix} = (C(z)^* C(z))^{-1} C(z)^* \begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_{n-1}) \end{pmatrix}. \tag{4}$$

We observe that

$$(5)\qquad C(z)^* \begin{pmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_{n-1}) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{n-1} \overline{B_0(\xi_j)} f(\xi_j) \\ \sum_{j=0}^{n-1} \overline{B_1(\xi_j)} f(\xi_j) \\ \vdots \\ \sum_{j=0}^{n-1} \overline{B_{n-1}(\xi_j)} f(\xi_j) \end{pmatrix} = n\overline{z} \begin{pmatrix} A_B(b_0 f)(z) \\ A_B(b_1 f)(z) \\ \vdots \\ A_B(b_{n-1} f)(z) \end{pmatrix}$$

where $b_k = B/B_k$.

Let $c_{\ell k}(z)$ denote the ℓ, k-entry of $C(z)^*C(z)$. Then

$$c_{\ell k}(z) = \sum_{j=0}^{n-1} B_\ell(\xi_j)\overline{B_k(\xi_j)}.$$

If $k < \ell$, then $c_{\ell k}(z) = nA_B(\overline{B_\ell/B_k})$. In particular $c_{\ell k}(z) \in H_2$. On the other hand, since $|z| = 1$, it follows that $z\overline{c_{\ell k}(z)} = nA_B(B_\ell b_k) \in H_2$. Hence, in the case $k < \ell$, the entry $c_{\ell k}(z)$ is of the form $\alpha_{\ell k} + \beta_{\ell k} z$ where the constants are determined by Blaschke product B. In the case $\ell < k$, $c_{\ell k}(z) = \overline{c_{k\ell(z)}}$. Finally, $c_{\ell k}(z) = n$, when $\ell = k$.

It has been shown that all entries of $C(z)^*C(z)$ are in one of the forms $\alpha + \beta z$ or $\alpha + \beta\frac{1}{z}$. By Cramer's rule $(C(z)^*C(z))^{-1}$ has rational entries. Furthermore, since $\det(C(z)^*C(z))$ does not vanish, it follows that the entries of $(C(z)^*C(z))^{-1}$ are continuous for $|z| = 1$. The assertions of the theorem now follow from (4), (5) and the fact that the continuous functions in H_2 are dense.

Corollary 1. *Suppose that B^* is a finite Blaschke product that is a factor of B. Then $A_B(B^*) = a + zb$, for some constants a and b.*

Proof. This follows by the same argument used to obtain the formula $c_{\ell k}(z) = \alpha_{\ell k} + \beta_{\ell k} z$ in the proof of the previous theorem.

Corollary 2. *The matrix function $Q(z)$ has the form $Q(z) = (\alpha + z\beta + z^2\gamma)^{-1}$, where α, β and γ are constant matrices depending only on $B(z)$.*

Proof. Since $P_\ell(B_j) = \delta_{j\ell}$, it follows from Theorem 2 that for $|z| = 1$, $Q(z)M(z)$ is the identity matrix, where $M(z) = (A_B(b_k B_j)(z))_{j,k=0}^{n-1}$. If $j \le k$, then $b_k B_j$ is a factor of B, while, if $k < j$, then B_j/B_k is a factor of B and $b_k B_j = B(B_j/B_k)$. The assertion of the corollary is now immediate from Corollary 1.

Lemma 3. (a) $C_B^*(f \circ B)(z) = f(z) \forall f \in H_2$.
(b) $C_B^*(B_k f \circ B)(z) = \overline{b_k(0)} z f(z) \forall f \in H_2$ for $k = 1, 2, \ldots, n-1$.

Proof. Nordgren [4] has shown that C_B is an isometry when, as in this case, $B(0) = 0$. It follows that $C_B^* C_B$ is the identity on H_2. Thus, (a) is shown.

Since the polynomials in z are dense in H_2, it suffices to prove (b) in the case $f(z) = z^m$ where m is a non-negative integer. Recall the definitions $\langle f, g\rangle = \int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dt$ of the inner product on H_2 and $\hat{f}(k) = (1/2\pi)\langle f, z^k\rangle$ of Fourier coefficient. Let $c(m, j)$ denote the j-th Fourier coefficient of the left-hand side of (b) in the case of $f(z) = z^m$. Then

$$2\pi c(m, j) = \langle C_B^*(B_k B^m), z^j\rangle = \langle B_k B^m, B^j\rangle = \int_0^{2\pi} B_k(e^{it})B^{m-j}(e^{it})dt.$$

It follows that $c(m, m+1) = \overline{b_k(0)}$ and $c(m, k) = 0$ when $j \ne m+1$. This shows that both sides of (ii) have the same Fourier coefficients. Thus, (b) is shown.

Theorem 3. *There exist rational functions $r_k, k = 0, 1, \ldots, n-1$, with poles off the unit circle and depending only on B such that*

$$C_B^*(f) = \sum_{k=0}^{n-1} r_k A_B(b_k f).$$

Proof. It follows from (3) and Lemma 3 that

$$C_B^*(f) = P_0(f) + \sum_{\ell=1}^{n-1} \overline{b_\ell(0)} z P_\ell(f).$$

Applying Theorem 2 yields

$$C_B^*(f) = \sum_{k=0}^{n-1} q_{0k} A_B(b_k f) + \sum_{\ell=1}^{n-1} \overline{b_\ell(0)} z \sum_{k=0}^{n-1} q_{\ell k} A_B(b_k f) \tag{6}$$

$$= \sum_{k=0}^{n-1} r_k A_B(b_k f), \tag{7}$$

where $r_k = q_{0k} + \sum_{\ell=1}^{n-1} \overline{b_\ell(0)} z q_{\ell k}$.

4. Examples

Example 1. Suppose that $B(z) = z^n$. It is not hard to see that the matrix Q in Theorem 2 reduces to $1/z$ times the identity matrix. It follows that in Theorem 3 $r_k = 0$ for $k \neq 0$ and $r_0 = 1/z$. Thus, for $|z| = 1$,

$$C_{z^n}^*(f) = \overline{z} A_{z^n}(z^n f) = A_{z^n}(f).$$

It follows that $C_{z^n}^* = A_{z^n}$.

Example 2. Suppose that $B(z) = z\frac{z-b}{1-\overline{b}z}$. In this case explicit calculation of the matrix Q in Theorem 2 can be carried out. From it the rational functions r_k in Theorem 3 can be obtained. The result is the following formula:

$$C_B^*(f)(z) = z\frac{p_1(z)}{q_1(z)} A_B(f)(z) + \frac{1}{z}\frac{p_2(z)}{q_2(z)} A_B(b_1 f)(z),$$

where $p_1(z), q_1(z), p_2(z), q_2(z)$ are explicitly determined polynomials, where $p_1(z)$ has degree 1, where $q_1(z), p_2(z), q_2(z)$ each have degree 2, and where the zeros of $q_1(z)$ and $q_2(z)$ are outside $\overline{D}$.

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