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IRRATIONAL ROTATION NUMBERS AND UNBOUNDEDNESS OF SOLUTIONS OF THE SECOND ORDER DIFFERENTIAL EQUATIONS WITH ASYMMETRIC NONLINEARITIES

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ABSTRACT. In this paper, we study the dynamics of the mappings

$$\begin{cases}\theta_1 = \theta + 2\alpha\pi + \frac{1}{r}\mu_1(\theta) + o(r^{-1}),\\ r_1 = r + \mu_2(\theta) + o(1), \qquad r \to +\infty,\end{cases}$$

where α is a irrational rotation number. We prove the existence of orbits that go to infinity in the future or in the past by using the well-known Birkhoff Ergodic Theorem. Applying this conclusion, we deal with the unboundedness of solutions of Liénard equations with asymmetric nonlinearities.

1. Introduction

We are concerned with the unboundedness of solutions of the second order differential equations

$$x'' + f(x)x' + ax^+ - bx^- = p(t), \tag{1.1}$$

where a and b are positive constants, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, $f(x)$ is a continuous function and $p(t)$ is a continuous 2π-periodic function. Throughout this paper, we define $F(x) = \int_0^x f(x)dx$ and so $F(x) \in C^1(\mathbb{R})$.

When $f(x) \equiv 0$, Eq. (1.1) becomes

$$x'' + ax^+ - bx^- = p(t), \tag{1.2}$$

which was first studied by Dancer in [1], [2] and Fucik in [3]. Up to now, there have appeared many results about the existence of periodic solutions and boundedness (or unboundedness) of solutions of Eq. (1.2) [4], [12]. When a and b are different and satisfy

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q},$$

J. M. Alonso and R. Ortega [12] proved the existence of periodic functions $p(t)$ such that all the solutions of (1.2) with large initial conditions are unbounded. In order

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to prove the unboundedness of solutions of Eq. (1.2), they studied the dynamics of a class of mappings defined on the plane, which have an asymptotic expression

$$\begin{cases} \theta_1 = \theta + 2\pi\frac{p}{q} + \frac{1}{r}\mu_1(\theta) + o(r^{-1}), \\ r_1 = r + \mu_2(\theta) + o(1), \qquad r \to +\infty, \end{cases}$$

where p/q is a rational number and μ_1, μ_2 are continuous and 2π-periodic functions. They proved the existence of orbits that go to infinity in the future provided that there exists $\omega \in R$ such that

$$\mu_2(\omega) > 0, \mu_1(\omega) = 0, \mu_1(\theta)(\theta - \omega) < 0 \text{ for } \theta \neq \omega \text{ and } |\theta - \omega| \text{ is small}$$

or in the past provided that there exists $\omega \in R$ such that

$$\mu_2(\omega) < 0, \mu_1(\omega) = 0, \mu_1(\theta)(\theta - \omega) > 0 \text{ for } \theta \neq \omega \text{ and } |\theta - \omega| \text{ is small.}$$

In the present paper, we will study the unboundedness of solutions of Eq. (1.1) when a and b satisfy

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{R}\backslash\mathbb{Q}.$$

Similarly, we will study the dynamics of mappings

$$\begin{cases} \theta_1 = \theta + 2\alpha\pi + \frac{1}{r}\mu_1(\theta) + o(r^{-1}), \\ r_1 = r + \mu_2(\theta) + o(1), \qquad r \to +\infty, \end{cases}$$

where α is an irrational number. Under certain conditions, we prove the existence of orbits that go to infinity in the future or in the past by using the well-known Birkhoff Ergodic Theorem. On the basis of this conclusion, we obtain the following theorems.

Theorem 1. *Assume that* $1/\sqrt{a} + 1/\sqrt{b} \in \mathbb{R}\backslash\mathbb{Q}$ *and the limits* $\lim_{x\to+\infty} F(x) = F(+\infty)$, $\lim_{x\to-\infty} F(x) = F(-\infty)$ *exist and are finite. Moreover,* $F(+\infty) < 0 < F(-\infty)$. *Then there exists* $R_0 > 0$ *such that every solution* $x(t)$ *of* (1.1) *with*

$$x(t_0)^2 + x'(t_0)^2 \geq R_0^2$$

with some $t_0 \in \mathbb{R}$ *goes to infinity in the future.*

Theorem 2. *Assume that* $1/\sqrt{a} + 1/\sqrt{b} \in \mathbb{R}\backslash\mathbb{Q}$ *and the limits* $\lim_{x\to+\infty} F(x) = F(+\infty)$, $\lim_{x\to-\infty} F(x) = F(-\infty)$ *exist and are finite. Moreover,* $F(-\infty) < 0 < F(+\infty)$. *Then there exists* $R_0 > 0$ *such that every solution* $x(t)$ *of* (1.1) *with*

$$x(t_0)^2 + x'(t_0)^2 \geq R_0^2$$

with some $t_0 \in \mathbb{R}$ *goes to infinity in the past.*

2. Unbounded orbits of planar mappings

Let $\sigma > 0$ be a sufficiently large constant. Set

$$E_\sigma = \{(x, y) : x^2 + y^2 \geq \sigma^2\}.$$

Assume that $\mathcal{P} : E_\sigma \to R^2$ is a one-to-one and continuous mapping, whose lift can be expressed in the form

$$\begin{cases} \theta_1 = \theta + 2\alpha\pi + \frac{1}{r}\mu_1(\theta) + H(\theta, r), \\ r_1 = r + \mu_2(\theta) + G(\theta, r), \end{cases} \tag{2.1}$$

where

(2.2) $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\mu_1, \mu_2 : S^1 \to S^1$ are Lipschitz continuous , $S^1 = \mathbb{R}/2\pi Z$

and H, G are 2π-periodic in θ and satisfy

$$r|H(\theta, r)| + |G(\theta, r)| \to 0 \text{ as } r \to +\infty, \tag{2.3}$$

uniformly with respect to $\theta \in R$.

Given a point (θ_0, r_0), denote by $\{(\theta_n, r_n)\}$ the orbit of the mapping $\mathcal{P}$ through the point (θ_0, r_0). That is to say

$$(\theta_{n+1}, r_{n+1}) = \mathcal{P}(\theta_n, r_n).$$

Proposition 2.1. *Assume that conditions* (2.2), (2.3) *hold and*

$$\int_0^{2\pi} \mu_2(\theta) d\theta > 0.$$

Then there exists $R_0 > \sigma$ such that if $r_0 \geq R_0$, the orbit $\{(\theta_n, r_n)\}$ satisfies

$$\lim_{n \to +\infty} r_n = +\infty.$$

Proof. From the expression of the mapping $\mathcal{P}$ we have that

$$\begin{cases} \theta_2 = \theta_1 + 2\alpha\pi + \frac{1}{r_1}\mu_1(\theta_1) + H(\theta_1, r_1), \\ r_2 = r_1 + \mu_2(\theta_1) + G(\theta_1, r_1). \end{cases}$$

Therefore,

$$\begin{cases} \theta_2 = \theta_0 + 4\alpha\pi + \frac{1}{r_0}\mu_1(\theta_0) + H(\theta_0, r_0) + \frac{1}{r_1}\mu_1(\theta_1) + H(\theta_1, r_1), \\ r_2 = r_0 + \mu_2(\theta_0) + G(\theta_0, r_0) + \mu_2(\theta_1) + G(\theta_1, r_1). \end{cases}$$

Since

$$\frac{1}{r_1} = \frac{1}{r_0 + \mu_2(\theta_0) + G(\theta_0, r_0)} = \frac{1}{r_0} + O(\frac{1}{r_0^2})$$

and

$$\mu_1(\theta_1) = \mu_1(\theta_0 + 2\alpha\pi + \frac{1}{r_0}\mu_1(\theta_0) + H(\theta_0, r_0)) = \mu_1(\theta_0 + 2\alpha\pi) + O(\frac{1}{r_0}),$$

we know that

$$\frac{1}{r_1}\mu_1(\theta_1) = \frac{1}{r_0}\mu_1(\theta_0 + 2\alpha\pi) + O(\frac{1}{r_0^2}).$$

Then θ_2 can be expressed in the form

$$\theta_2 = \theta_0 + 4\alpha\pi + \frac{1}{r_0}[\mu_1(\theta_0) + \mu_1(\theta_0 + 2\alpha\pi)] + H_2(\theta_0, r_0),$$

where $H_2(\theta_0, r_0) = H(\theta_0, r_0) + H(\theta_1, r_1) + \frac{1}{r_1}\mu_1(\theta_1) - \frac{1}{r_0}\mu_1(\theta_0 + 2\alpha\pi)$. Obviously, we have that

$$\lim_{r_0 \to +\infty} r_0 |H_2(\theta_0, r_0)| = 0.$$

On the other hand, since

$$\mu_2(\theta_1) = \mu_2(\theta_0 + 2\alpha\pi + \frac{1}{r_0}\mu_1(\theta_0) + H(\theta_0, r_0)) = \mu_2(\theta_0 + 2\alpha\pi) + O(\frac{1}{r_0}),$$

we get that

$$r_2 = r_0 + \mu_2(\theta_0) + \mu_2(\theta_0 + 2\alpha\pi) + G_2(\theta_0, r_0),$$

where $G_2(\theta_0, r_0) = G(\theta_0, r_0) + G(\theta_1, r_1) + \mu_2(\theta_1) - \mu_2(\theta_0 + 2\alpha\pi)$. It is easy to check that

$$\lim_{r_0 \to +\infty} |G_2(\theta_0, r_0)| = 0.$$

Inductively, we have that

$$\begin{cases} \theta_n = \theta_0 + 2n\alpha\pi + \frac{1}{r_0} \sum_{i=0}^{i=n-1} \mu_1(\theta_0 + 2i\alpha\pi) + H_n(\theta_0, r_0), \\ r_n = r_0 + \sum_{i=0}^{i=n-1} \mu_2(\theta_0 + 2i\alpha\pi) + G_n(\theta_0, r_0), \end{cases}$$

where $H_n(\theta_0, r_0)$ and $G_n(\theta_0, r_0)$ satisfy

$$r_0 |H_n(\theta_0, r_0)| + |G_n(\theta_0, r_0)| \to 0 \text{ as } r_0 \to +\infty.$$

Next, we define a transformation $T : S^1 \to S^1$, $T(\theta) = \theta + 2\alpha\pi$. Since α is an irrational number, T is ergodic. By the Birkhoff Ergodic Theorem [13] we get that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \mu_2(\theta + 2i\alpha\pi) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \mu_2(T^i\theta) = \frac{1}{2\pi} \int_0^{2\pi} \mu_2(\theta) d\theta > 0$$

for almost every $\theta \in S^1$. Since μ_2 is continuous and S^1 is compact, we can further obtain that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \mu_2(\theta + 2i\alpha\pi) = \frac{1}{2\pi} \int_0^{2\pi} \mu_2(\theta) d\theta > 0$$

uniformly for every $\theta \in S^1$. Therefore, there exist a positive integer $m >> 1$ and a constant $c > 0$ such that

$$\frac{1}{m} \sum_{i=0}^{i=m-1} \mu_2(\theta_0 + 2i\alpha\pi) \geq c > 0$$

for all $\theta_0 \in S^1$. Recalling that $\lim_{r_0 \to +\infty} G_m(\theta_0, r_0) = 0$, we have that there exists a constant $R_0 > \sigma$ such that for $r_0 \geq R_0$, $|G_m(\theta_0, r_0)| \leq c$. Then for $r_0 \geq R_0$, we get that

$$\begin{aligned} r_m = r_0 + m \cdot \frac{1}{m} \sum_{i=0}^{i=m-1} \mu_2(\theta_0 + 2i\alpha\pi) + G_m(\theta_0, r_0) &\geq r_0 + mc + G_m(\theta_0, r_0) \\ &\geq r_0 + (m-1)c. \end{aligned}$$

Meanwhile, we have that

$$\begin{aligned} r_{2m} &= r_m + m \cdot \frac{1}{m} \sum_{i=0}^{i=m-1} \mu_2(\theta_m + 2i\alpha\pi) + G_m(\theta_m, r_m) \\ &\geq r_m + mc + G_m(\theta_m, r_m) \geq r_m + (m-1)c \geq r_0 + 2(m-1)c. \end{aligned}$$

Inductively, we have that

$$\begin{aligned} r_{km} &= r_{(k-1)m} + m \cdot \frac{1}{m} \sum_{i=0}^{i=m-1} \mu_2(\theta_{(k-1)m} + 2i\alpha\pi) + G_m(\theta_{(k-1)m}, r_{(k-1)m}) \\ &\geq r_{(k-1)m} + mc + G_m(\theta_{(k-1)m}, r_{(k-1)m}) \geq r_0 + k(m-1)c. \end{aligned}$$

Therefore, we get that

$$\lim_{k \to +\infty} r_{km} = +\infty. \tag{2.4}$$

Because $\mu_2(\theta)$ is continuous and $\lim_{r_0\to+\infty} G(\theta_0, r_0) = 0$, there exists a constant $d > 0$ such that

$$|\mu_2(\theta_0) + G(\theta_0, r_0)| \le d,$$

for $\theta_0 \in S^1$ and $r_0 > \sigma$. From

$$r_{(km+i)} = r_{(km+i-1)} + \mu_2(\theta_{(km+i-1)}) + G(\theta_{(km+i-1)}, r_{(km+i-1)}),\ i = 1, \cdots, m-1,$$

we get that

$$|r_{(km+i)} - r_{(km+i-1)}| \le d, \quad i = 1, \cdots, m-1.$$

Consequently, we have that

$$|r_{(km+i)} - r_{km}| \le id, \quad i = 1, \cdots, m-1. \tag{2.5}$$

From (2.4) and (2.5) we know that

$$\lim_{n\to+\infty} r_n = +\infty.$$

□

Proposition 2.2. *Assume that conditions* (2.2), (2.3) *hold and*

$$\int_0^{2\pi} \mu_2(\theta) d\theta < 0.$$

Then there exists $R_0 > \sigma$ *such that if* $r_0 \ge R_0$, *the orbit* $\{(\theta_n, r_n)\}$ *satisfies*

$$\lim_{n\to-\infty} r_n = +\infty.$$

The proof of Proposition 2.2 is identical to the proof of Proposition 2.1. We only give some explanations. At first, from (2.2), (2.3) we know that $\mathcal{P}(E_\sigma)$ contains a neighborhood of infinity. Next, by using the inductive method, we can also obtain that

$$\begin{cases} \theta_{-n} = \theta_0 - 2n\alpha\pi - \frac{1}{r_0}\sum_{i=0}^{i=n-1} \mu_1(\theta_0 - 2i\alpha\pi) - H_{-n}(\theta_0, r_0), \\ r_{-n} = r_0 - \sum_{i=0}^{i=n-1} \mu_2(\theta_0 - 2i\alpha\pi) - G_{-n}(\theta_0, r_0), \end{cases}$$

where $H_{-n}(\theta_0, r_0)$ and $G_{-n}(\theta_0, r_0)$ satisfy

$$r_0|H_{-n}(\theta_0, r_0)| + |G_{-n}(\theta_0, r_0)| \to 0 \text{ as } r_0 \to +\infty.$$

Thus, by applying the same ideas in proving Proposition 2.1, we can prove that the conclusion of Proposition 2.2 holds.

3. Action and angle variables

At first, we consider the piecewise linear equation

$$x'' + ax^+ - bx^- = 0 \tag{3.1}$$

and denote by $C(t)$ the solution of (3.1) satisfying the initial condition $x(0) = 1, x'(0) = 0$. It is a periodic function with period

$$\tau = \frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}$$

and can be expressed by

$$C(t) = \begin{cases} \cos\sqrt{a}t, \quad 0 \le |t| \le \frac{\pi}{2\sqrt{a}}, \\ -\sqrt{\frac{a}{b}}\sin\sqrt{b}(t - \frac{\pi}{2\sqrt{a}}), \quad \frac{\pi}{2\sqrt{a}} \le |t| \le \frac{\tau}{2}. \end{cases}$$

The derivative of $C(t)$ will be denoted by $S(t) = C'(t)$. Obviously, $C(t)$ and $S(t)$ satisfy the following properties,

(i) $C(t+\tau) = C(t)$, $S(t+\tau) = S(t)$ and $C(0) = 1$, $S(0) = 0$.

(ii) $C(t) \in C^2(\mathbb{R})$, $S(t) \in C^1(\mathbb{R})$.

(iii) $C'(t) = S(t)$, $S'(t) = -(aC^+(t) - bC^-(t))$.

(iv) $S(t)^2 + aC^+(t)^2 + bC^-(t)^2 = a, \quad \forall t \in \mathbb{R}$.

Define the mapping

$$\Phi : \quad (\theta, I) \in S^1 \times (0, +\infty) \to (x, y) \in \mathbb{R}^2 \backslash \{0\}$$

with

$$x = \gamma I^{\frac{1}{2}} C(\frac{\theta}{\omega}), \quad y = \gamma I^{\frac{1}{2}} S(\frac{\theta}{\omega}),$$

with $\omega = \frac{2\pi}{\tau}$, $\gamma = \sqrt{\frac{2\omega}{a}}$. It is easy to check that Φ is an area-preserving C^1-diffeomorphism.

Now, we deal with Eq. (1.1). Consider the equivalent system of Eq. (1.1)

$$x' = y - F(x), \quad y' = -(ax^+ - bx^-) + p(t). \tag{3.2}$$

Under the transformation Φ, Eq. (3.2) becomes

$$\begin{cases} \dfrac{d\theta}{dt} = \omega + \dfrac{\gamma}{2} I^{-\frac{1}{2}} F(\gamma I^{\frac{1}{2}} C(\dfrac{\theta}{\omega})) S(\dfrac{\theta}{\omega}) - \dfrac{\gamma}{2} I^{-\frac{1}{2}} p(t) C(\dfrac{\theta}{\omega}), \\ \dfrac{dI}{dt} = \dfrac{2}{a\gamma} I^{\frac{1}{2}} [-aC^+(\frac{\theta}{\omega}) + bC^-(\frac{\theta}{\omega})] F(\gamma I^{\frac{1}{2}} C(\dfrac{\theta}{\omega})) + \dfrac{2}{a\gamma} I^{\frac{1}{2}} p(t) S(\dfrac{\theta}{\omega}). \end{cases} \tag{3.3}$$

Denote by $(\theta(t; \theta_0, I_0), I(t; \theta_0, I_0))$ the solution of (3.3) satisfying an initial condition $\theta(0) = \theta_0$, $I(0) = I_0$. If $F(x)$ is bounded, then for large values of I_0, this solution is defined for all $t \in [0, 2\pi]$. Thus we can define the Poincaré mapping

$$\theta_1 = \theta(2\pi; \theta_0, I_0), \quad I_1 = I(2\pi; \theta_0, I_0).$$

From the second equality of (3.3) we get that

$$\frac{dI^{\frac{1}{2}}}{dt} = \frac{4}{a\gamma} [(-aC^+(\frac{\theta}{\omega}) + bC^-(\frac{\theta}{\omega})) F(\gamma I^{\frac{1}{2}} C(\frac{\theta}{\omega})) + p(t) S(\frac{\theta}{\omega})]. \tag{3.4}$$

It follows from (3.4) that

$$I(t)^{\frac{1}{2}} = I_0^{\frac{1}{2}} + O(1), t \in [0, 2\pi], I_0 \to +\infty.$$

Furthermore, we have that

$$I(t)^{-\frac{1}{2}} = I_0^{-\frac{1}{2}} + O(I_0^{-1}), t \in [0, 2\pi], I_0 \to +\infty. \tag{3.5}$$

From (3.5) and the first equality of (3.3) we know that

$$\frac{d\theta}{dt} = \omega + O(I_0^{-\frac{1}{2}}).$$

Consequently,

$$\theta(t) = \theta_0 + \omega t + O(I_0^{-\frac{1}{2}}), t \in [0, 2\pi], \tag{3.6}$$

which, together with (3.4), yields

$$\begin{aligned} \frac{dI^{\frac{1}{2}}}{dt} =& \frac{4}{a\gamma} [(-aC^+(t + \frac{\theta_0}{\omega}) + bC^-(t + \frac{\theta_0}{\omega})) F(\gamma I_0^{\frac{1}{2}} C(t + \frac{\theta_0}{\omega}) + O(1)) + p(t) S(t + \frac{\theta_0}{\omega})] \\ &+ O(I_0^{-\frac{1}{2}}). \end{aligned}$$

An integration shows that

$$I_1^{\frac{1}{2}} = I_0^{\frac{1}{2}} + \frac{4}{a\gamma}\int_0^{2\pi}(-aC^+(t+\frac{\theta_0}{\omega}) + bC^-(t+\frac{\theta_0}{\omega}))F(\gamma I_0^{\frac{1}{2}}C(t+\frac{\theta_0}{\omega}) + O(1))dt$$
$$+ \frac{4}{a\gamma}\int_0^{2\pi} p(t)S(t+\frac{\theta_0}{\omega})dt + O(I_0^{-\frac{1}{2}}).$$

Similarly, substituting (3.6) in the first equality of (3.3), we obtain that for $t \in [0, 2\pi]$

$$\frac{d\theta}{dt} = \omega + \frac{\gamma}{2}I_0^{-\frac{1}{2}}F(\gamma I_0^{\frac{1}{2}}C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega}) - \frac{\gamma}{2}I_0^{-\frac{1}{2}}p(t)C(t+\frac{\theta_0}{\omega}) + O(I_0^{-1}).$$

Therefore, we have that

$$\begin{aligned}\theta_1 = \quad & \theta_0 + 2\pi\omega + \frac{\gamma}{2}I_0^{-\frac{1}{2}}\int_0^{2\pi} F(\gamma I_0^{\frac{1}{2}}C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt \\ & -\frac{a\gamma}{2}I_0^{-\frac{1}{2}}\int_0^{2\pi} p(t)C(t+\frac{\theta_0}{\omega})dt + O(I_0^{-1}).\end{aligned}$$

Set $r = I^{1/2}$. Then we get

$$\begin{cases}\theta_1 = & \theta_0 + 2\pi\omega + \frac{\gamma}{2}r_0^{-1}\int_0^{2\pi} F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt \\ & -\frac{\gamma}{2}r_0^{-1}\int_0^{2\pi} p(t)C(t+\frac{\theta_0}{\omega})dt + O(r_0^{-2}), \\ r_1 = & r_0 + \frac{4}{a\gamma}\int_0^{2\pi}(-aC^+(t+\frac{\theta_0}{\omega}) + bC^-(t+\frac{\theta_0}{\omega}))F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))dt \\ & +\frac{4}{a\gamma}\int_0^{2\pi} p(t)S(t+\frac{\theta_0}{\omega})dt + O(r_0^{-1}).\end{cases}$$

Write

$$\psi_1(\theta_0, r_0) = \int_0^{2\pi} F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt,$$
$$\psi_2(\theta_0, r_0) = \int_0^{2\pi}(-aC^+(t+\frac{\theta_0}{\omega}) + bC^-(t+\frac{\theta_0}{\omega}))F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))dt,$$
$$\psi_3(\theta_0) = \int_0^{2\pi} p(t)C(t+\frac{\theta_0}{\omega})dt,$$
$$\psi_4(\theta_0) = \int_0^{2\pi} p(t)S(t+\frac{\theta_0}{\omega})dt.$$

Lemma 1. *Assume that the limits* $\lim_{x\to+\infty} F(x) = F(+\infty)$, $\lim_{x\to-\infty} F(x) = F(-\infty)$ *exist and are finite. Then, for* $r_0 \to +\infty$,

$$\psi_1(\theta_0, r_0) = F(+\infty)\int_{J_1} S(t+\frac{\theta_0}{\omega})dt + F(-\infty)\int_{J_2} S(t+\frac{\theta_0}{\omega})dt + o(1),$$
$$\psi_2(\theta_0, r_0) = -aF(+\infty)\int_{J_1} C^+(t+\frac{\theta_0}{\omega})dt + bF(-\infty)\int_{J_2} C^-(t+\frac{\theta_0}{\omega})dt + o(1),$$

where $J_1 = \{t : t \in (0, 2\pi), C(t+\frac{\theta_0}{\omega}) \geq 0\}$, $J_2 = \{t : t \in (0, 2\pi), C(t+\frac{\theta_0}{\omega}) \leq 0\}$.

Proof. We only check that

$$\lim_{r_0\to+\infty}\int_{J_1} F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt = F(+\infty)\int_{J_1} S(t+\frac{\theta_0}{\omega})dt.$$

From $\lim_{x\to+\infty} F(x) = F(+\infty)$ we have that, for any sufficiently small $\eta > 0$,

$$\lim_{r_0\to+\infty} \int_{J_{11}} F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt = F(+\infty)\int_{J_{11}} S(t+\frac{\theta_0}{\omega})dt,$$

with $J_{11} = \{t : t \in (0, 2\pi), C(t+\frac{\theta_0}{\omega}) \geq \eta\}$. On the other hand, it is easy to see that

$$\lim_{\eta\to 0^+} \int_{J_{12}} F(\gamma r_0 C(t+\frac{\theta_0}{\omega}) + O(1))S(t+\frac{\theta_0}{\omega})dt = 0, \quad \lim_{\eta\to 0^+} \int_{J_{12}} S(t+\frac{\theta_0}{\omega})dt = 0,$$

where $J_{12} = \{t : t \in (0, 2\pi), 0 \leq C(t+\frac{\theta_0}{\omega}) \leq \eta\}$. Thus we get the conclusion. □

Lemma 2. $\int_0^{2\pi} \psi_4(\theta_0)d\theta_0 = 0$.

Proof. Since $\psi_4(\theta_0) = \psi_3'(\theta_0)$ and $\psi_4(\theta_0)$, $\psi_3(\theta_0)$ are 2π-periodic functions, we have $\int_0^{2\pi} \psi_4(\theta_0)d\theta_0 = 0$. □

Now, we prove Theorem 1. The proof of Theorem 2 can be treated similarly.

Proof of Theorem 1. Consider the Poincaré mapping $P : (\theta_0, r_0) \to (\theta_1, r_1)$. From Lemma 1 we know that P can be expressed in the form:

$$\begin{cases} \theta_1 = & \theta_0 + 2\pi\omega + r_0^{-1}\mu_1(\theta_0) + H(\theta_0, r_0), \\ r_1 = & r_0 + \mu_2(\theta_0) + G(\theta_0, r_0), \end{cases}$$

where H, G are continuous functions and satisfy

$$H(\theta_0, r_0) = o(\frac{1}{r_0}), \quad G(\theta_0, r_0) = o(1) \text{ as } r_0 \to +\infty$$

and $\mu_1(\theta_0) = \frac{\gamma}{2}[\phi_1(\theta_0) - \psi_3(\theta_0)]$, $\mu_2(\theta_0) = \frac{4}{a\gamma}[\phi_2(\theta_0) + \psi_4(\theta_0)]$ with

$$\phi_1(\theta_0) = F(+\infty)\int_{J_1} S(t+\frac{\theta_0}{\omega})dt + F(-\infty)\int_{J_2} S(t+\frac{\theta_0}{\omega})dt,$$

$$\phi_2(\theta_0) = -aF(+\infty)\int_{J_1} C^+(t+\frac{\theta_0}{\omega})dt + bF(-\infty)\int_{J_2} C^-(t+\frac{\theta_0}{\omega})dt,$$

where J_1 and J_2 are defined in Lemma 1. Clearly, $\mu_1, \mu_2 : S^1 \to S^1$ are Lipschitz continuous. Since $1/\sqrt{a} + 1/\sqrt{b} \in \mathbb{R}\backslash\mathbb{Q}$ and $\omega = 2\pi/\tau$, $\tau = \pi/\sqrt{a} + \pi/\sqrt{b}$, we have that ω is an irrational number. On the other hand, it follows from $F(+\infty) < 0 < F(-\infty)$ that $\phi_2(\theta_0) > 0$ for $\theta_0 \in S^1$. Therefore, from Lemma 2 we get that

$$\int_0^{2\pi} \mu_2(\theta_0)d\theta_0 = \frac{4}{a\gamma}\int_0^{2\pi} \phi_2(\theta_0)d\theta_0 > 0.$$

Applying the result of Proposition 2.1, we obtain the conclusion of Theorem 1. □

References

[1] N. Dancer, On the Dirichlet problem for weakly nonlinear elliptic partial differential equations, Proc. Roy. Soc. Edinburg Scct, A(76)1977, 283-300. MR **82i:**35063, MR **58:**17506

[2] N. Dancer, Boundary-value problems for weakly nonlinear ordinary differential equations, Bull. Austral. Math. Soc., 15(1976), 321-328. MR **55:**3389

[3] S. Fučik, Solvability of nonlinear equations and boundary value problems, Reidel Dordrecht, 1980. MR **83c:**47079

[4] P. Drabek and S. Invernizzi, On the periodic boundary value theorem for forced Duffing equation with jumping nonlinearity, Nonlinear Anal, 10(1986), 643-650. MR **87j:**34077

[5] C. Fabry, Landesman-lazer conditions for periodic boundary value problems wih asymmetric nonlinearities, J. Differential Equations, 116(1995), 405-418. MR **96c:**34033
[6] C. Fabry and A. Fonda, Nonlinear resonance in asymmetric oscillators, J. Differential Equations, 147(1998), 58-78. MR **99d:**34070
[7] A. C. Lazer and D. E. Leach, Bounded perturbations of forced harmonic oscillations at resonance, Ann. Mat. Pura. Appl., (82)1969, 49-68. MR **40:**2972
[8] T. Ding, Nonlinear oscillations at a point of resonance, Scientia Sinica(series A), (8)1982, 918-931. MR **84c:**34058
[9] B. Liu, Boundedness in nonlinear oscillations at resonance, J. Differential Equations, 153(1999), 142-172. MR **2000d:**34075
[10] R. Ortega, Asymmetric oscillators and twist mappings, J. London Math. Soc., 53(1996), 325-342. MR **96k:**34093
[11] J. M. Alonso and R. Ortega, Unbounded solutions of semilinear equations at resonance, Nonlinearity, 9(1996), 1099-1111. MR **97m:**35155
[12] J. M. Alonso and R. Ortega, Roots of unity and unbounded motions of an asymmetric oscillator, J. Differential Equations, 143(1998), 201-220. MR **99a:**34102
[13] P. Walters, An introduction to ergodic theory, Springer-Verlag, 1982. MR **84e:**28017

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