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NONEXISTENCE RESULTS FOR HIGHER–ORDER EVOLUTION PARTIAL DIFFERENTIAL INEQUALITIES

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Abstract. Nonexistence of global solutions to semilinear higher-order (with respect to t) evolution partial differential inequalities $u_t^{(k)} - \Delta u \geq |x|^\sigma |u|^q$ with $k = 1, 2, \dots$ in the complement of a ball is studied. The critical exponents q^* are found and the nonexistence results are proved for $1 < q \leq q^*$. The corresponding results for $k = 1$ (parabolic problem) are sharp.

Introduction

In this paper we study the nonexistence of global solutions for higher-order (with respect to t) semilinear partial differential inequalities

$$\frac{\partial^k u}{\partial t^k} - \Delta u \geq |x|^\sigma |u|^q, \qquad (x, t) \in \Omega \times (0, \infty).$$

Let us introduce the "critical singularity" $\sigma^* = -2$. We consider the problem in the complement of a ball $\Omega = \mathbf{R}^N \setminus B_R$, $R > 0$, and in the case $+\infty > \sigma > \sigma^*$ we obtain the critical exponent q^*. In another case, $-\infty < \sigma \leq \sigma^*$, for the problem in the ball $\Omega = B_R$ we formulate some results from the author's paper [17] (see also [6, 29]) on nonexistence for all $q > 1$.

Corresponding investigations for parabolic equations were initiated by Bandle and Levine [2] (see also the references therein).

Nonexistence theory for evolution equations is well developed. We do not cite all the results, but let us mention the book [30] and the surveys [20, 8] (parabolic problems), and the books [1, 12, 24] and the papers [7, 10] (hyperbolic problems).

In this paper we obtain some generalization for higher-order evolution equations and inequalities, which includes, among others, parabolic and hyperbolic problems. We use the nonlinear test-function method, developed by Pohozaev [27], Mitidieri and Pohozaev [22, 23, 24], Pohozaev and Tesei [28], Veron and Pohozaev [32], and Zhang [33, 34, 35] (see also the papers by Bandle, Levine and Zhang [3], Kurta [14, 15], Levine and Zhang [21] and author's papers [16, 17, 18, 19]).

Let $N \geq 3$, Q be an (unbounded) domain in $\mathbf{R}^{N+1}$ with piecewise smooth boundary. We will use the well-known Sobolev spaces $W_q^2(Q)$ and the local space $L_{q,\mathrm{loc}}(Q)$, the elements of which belong to $L_q(Q')$ for any compact subset Q': $\overline{Q'} \subset$

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Q. Denote the space of continuous functions by $C(\overline{Q})$ and the space of smooth functions by $C^m(\overline{Q})$. Similar anisotropic spaces $W_q^{2,k}(Q)$ and $C^{2,k}(\overline{Q})$ are also introduced.

The symbol Δ stands for the Laplace operator. The expression $\frac{\partial u}{\partial n}$ denotes the derivative of u in the direction of the outward normal n to the boundary of the domain under consideration. Indexed c's and C's are used to designate constants.

1. Main results

Let $R > 0$, $k \in \mathbf{N}$ and $-2 < \sigma < +\infty$. Let us introduce the domain $\Omega = \mathbf{R}^N \setminus B_R$ and consider the problem

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \geq |x|^\sigma |u|^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) \geq 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0, & x \in \Omega. \end{cases} \tag{1.1}$$

Definition 1.1. Let $u(x,t) \in C(\overline{\Omega} \times [0,\infty))$ and the locally integrable traces $\frac{\partial^i u}{\partial t^i}(x,0)$, $i = 1, \ldots, k-1$, are well defined. The function $u(x,t)$ is called a weak solution to problem (1.1) if, for any nonnegative test-function $\varphi(x,t) \in W_\infty^{2,k}(\Omega \times (0,\infty))$ with compact support, such that $\varphi|_{\partial\Omega \times (0,\infty)} = 0$, the integral inequality

$$\begin{aligned} \int_0^\infty \int_{\partial\Omega} u \frac{\partial\varphi}{\partial n}\, dxdt + \int_0^\infty \int_\Omega u \left((-1)^k \frac{\partial^k \varphi}{\partial t^k} - \Delta\varphi \right) dxdt \\ \geq \int_0^\infty \int_\Omega |x|^\sigma |u|^q \varphi\, dxdt + \sum_{i=1}^{k-1} (-1)^i \int_\Omega \frac{\partial^{k-1-i} u}{\partial t^{k-1-i}}(x,0) \frac{\partial^i \varphi}{\partial t^i}(x,0)\, dx \\ + \int_\Omega \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \varphi(x,0)\, dx \end{aligned} \tag{1.2}$$

holds.

Theorem 1.2. *For* $\sigma > -2$ *and*

$$1 < q \leq q_k^* = \frac{N + 2/k + \sigma}{N - 2 + 2/k} = 1 + \frac{2 + \sigma}{N - 2 + 2/k} \tag{1.3}$$

problem (1.1) *has no nontrivial global weak solution.*

This theorem includes, among others, the sharp results for parabolic problems ($k = 1$, Fujita–Hayakawa's critical exponent $q_1^* = 1 + \frac{2+\sigma}{N}$; see also [25, 4, 26, 20, 25, 8, 3, 21]) and hyperbolic problems ($k = 2$, Kato's critical exponent $q_2^* = 1 + \frac{2+\sigma}{N-1}$; see also [32, 34, 9]). It is interesting that formally passing to the limit as $k \to \infty$ in Theorem 1.2 we arrive at the sharp elliptic critical exponent $q_\infty^* = \frac{N+\sigma}{N-2}$ (see, for example, [22, 23, 24, 13, 16, 5] and the references therein).

Now let $-\infty < \sigma \leq -2$. Let us consider the problem (1.1) in the ball B_R, i.e. $\Omega = B_R$.

Definition 1.3. Let $u(x,t) \in C(\overline{\Omega} \setminus \{0\} \times [0,\infty))$ and let the traces $\frac{\partial^i u}{\partial t^i}(x,0) \in L_{\mathrm{loc}}(\Omega \setminus \{0\})$, $i = 1, \ldots, k-1$, be well defined. The function $u(x,t)$ is called a weak solution to problem (1.1) if, for any nonnegative test-function $\varphi(x,t) \in W_\infty^{2,k}(\Omega \times (0,\infty))$, such that $\varphi(x,t) \equiv 0$ in a neighborhood of $x = 0$ for all $t \geq 0$ and $\varphi|_{\partial\Omega \times (0,\infty)} = 0$, the integral inequality (1.2) holds.

Theorem 1.4 ([17]). *For $-\infty < \sigma \le -2$ and*

$$1 < q < +\infty$$

problem (1.1) *in the ball $\Omega = B_R$ has no global (with respect to t) nontrivial solution.*

Let us explain the meaning of the above theorem. To do this, it is convenient to use an analogy with Theorem 1.2. In the case $\sigma \le -2$, the solution has a singularity at $x = 0$ rather than at infinity. Accordingly, the test function vanishes in a neighborhood of $x = 0$, and no assumption is made about the integrability of $u(x,t)$ at zero. This argument can be formalized by applying the Kelvin transform in order to proceed to an exterior problem in $\mathbf{R}^N \setminus B_R$.

2. Auxiliary estimates

In this section we obtain some estimates depending on the parameter ρ, $\rho \to \infty$. These estimates play a fundamental role in the test function method.

Let us consider the "standard cut-off function" $\zeta(y) \in C^\infty(\mathbf{R}_+)$ with the following properties:

$$0 \le \zeta(y) \le 1, \qquad \zeta(y) = \begin{cases} 1, & \text{if } 0 \le y \le 1, \\ 0, & \text{if } y \ge 2. \end{cases}$$

For the function

$$\eta(y) = (\zeta(y))^{kp_0}$$

with some positive p_0 and $k \in \mathbf{N}$, by direct calculation one can obtain the estimates (for $1 < p \le p_0$)

$$\begin{aligned}
&|\eta'(y)|^p = (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-p} |\zeta'|^p \le c_\eta \eta^{p-1}(y), \\
&|\eta''(y)|^p \le (kp_0)^p \zeta^{kp_0(p-1)} \zeta^{kp_0-2p}((kp_0-1)|\zeta'|^2 + \zeta|\zeta''|)^p \le c_\eta \eta^{p-1}(y), \\
&\dots \\
&|\eta^{(k)}(y)|^p \le c_\eta \eta^{p-1}(y)
\end{aligned}$$

with a positive constant c_η.

Now let us introduce the change of variables $y = t/\rho^\theta$, with $\theta > 0$, $\rho > 2R$. For the function $\eta(t/\rho^\theta)$ we have

$$\operatorname{supp} \left| \eta\left(\frac{t}{\rho^\theta}\right) \right| = \{t < 2\rho^\theta\}, \qquad \operatorname{supp} \left| \frac{d^k \eta(t/\rho^\theta)}{dt^k} \right| = \{\rho^\theta < t < 2\rho^\theta\},$$

and

$$\int_{\operatorname{supp} |\frac{d^k \eta(t/\rho^\theta)}{dt^k}|} \frac{|\frac{d^k \eta(t/\rho^\theta)}{dt^k}|^p}{\eta^{p-1}(t/\rho^\theta)}\, dt \le c_\eta \rho^{-\theta(kp-1)}. \tag{2.1}$$

The parameter θ will be chosen later.

For the variable x, $|x| = r$, we introduce the functions $\eta(r/\rho)$,

$$\xi(x) \equiv \xi(r) = \frac{1}{R^s} - \frac{1}{r^s}, \tag{2.2}$$

and

$$\psi_\rho(x) \equiv \psi_\rho(r) = \left(\frac{1}{R^s} - \frac{1}{r^s}\right) \eta\left(\frac{r}{\rho}\right). \tag{2.3}$$

Here we will put $s = N - 2$. It is evident that $\psi_\rho = 0$ and $\frac{\partial \psi_\rho}{\partial r} \ge 0$ as $r = R$.

For the derivatives of the function $\psi_\rho(r)$ (as $r > 2R$) we have

$$\left|\frac{\partial\psi_\rho}{\partial r}\right|^p \le \left|\frac{s}{r^{s+1}}\eta\left(\frac{r}{\rho}\right) + \left(\frac{1}{R^s} - \frac{1}{r^s}\right)\eta'\left(\frac{r}{\rho}\right)\frac{1}{\rho}\right|^p \le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{ps}r^p}\left(1 + \frac{r^p}{\rho^p}\right),$$

$$\begin{aligned}\left|\frac{\partial^2\psi_\rho}{\partial r^2}\right|^p &\le \left|-\frac{s(s+1)}{r^{s+2}}\eta\left(\frac{r}{\rho}\right) + \frac{2s}{r^{s+1}\rho}\eta'\left(\frac{r}{\rho}\right) + \left(\frac{1}{R^s} - \frac{1}{r^s}\right)\frac{1}{\rho^2}\eta''\left(\frac{r}{\rho}\right)\right|^p \\ &\le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{sp}r^{2p}}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right);\end{aligned}$$

here c does not depend on r and ρ. Using these estimates we arrive at the inequality for the Laplace operator:

$$\begin{aligned}|\Delta\psi_\rho(x)|^p &= \left|\frac{\partial^2\psi_\rho}{\partial r^2} + \frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r}\right|^p \le c\left|\frac{\partial^2\psi_\rho}{\partial r^2}\right|^p + c\frac{1}{r^p}\left|\frac{\partial\psi_\rho}{\partial r}\right|^p \\ &\le c\eta^{p-1}\left(\frac{r}{\rho}\right)\frac{1}{R^{sp}r^{2p}}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right) \le c\psi_\rho^{p-1}(x)\frac{1}{r^{2p}}\left(1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}}\right).\end{aligned} \tag{2.4}$$

Now we take $s = N-2$. Due to

$$\Delta\left(\frac{1}{r^{N-2}}\right) = 0, \quad r \ne 0,$$

we have $\Delta\psi_\rho = 0$ for $r < \rho$ and

$$\operatorname{supp}|\Delta\psi_\rho| = \{\rho < r < 2\rho\}.$$

On the set $\operatorname{supp}|\Delta\psi_\rho|$ the estimate

$$1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}} \le c$$

holds, where c does not depend on r and ρ. Therefore, it follows from (2.4) (for $\rho < r < 2\rho$) that

$$|\Delta\psi_\rho(x)|^p \le c\psi_\rho^{p-1}(x)\frac{1}{\rho^{2p}};$$

hence for $\sigma \in \mathbf{R}$ we get

$$\begin{aligned}&\int_{\operatorname{supp}|\Delta\psi_\rho|}\frac{|\Delta\psi_\rho(x)|^p}{\psi_\rho^{p-1}(x)|x|^{\sigma(p-1)}}\,dx \\ &\qquad\le c\int_\rho^{2\rho}\frac{\psi_\rho^{p-1}(x)}{\psi_\rho^{p-1}(x)}\frac{r^{N-1}}{\rho^{2p+\sigma(p-1)}}\,dr \le c_\psi\rho^{-p(\sigma+2)+N+\sigma}.\end{aligned} \tag{2.5}$$

Finally, for the general test-function

$$\varphi_\rho(x,t) = \eta\left(\frac{t}{\rho^\theta}\right)\psi_\rho(x) \tag{2.6}$$

we obtain the inequality

$$\text{(2.7)}\quad \iint_{\operatorname{supp}|\Delta\varphi_\rho|} \frac{|\Delta\varphi_\rho(x,t)|^p}{\varphi_\rho^{p-1}(x,t)|x|^{\sigma(p-1)}}\,dx\,dt \le \int_0^{2\rho^\theta} \eta(t/\rho^\theta)\,dt \int_{\operatorname{supp}|\Delta\psi_\rho|} \frac{|\Delta\psi_\rho|^p}{\psi_\rho^{p-1}|x|^{\sigma(p-1)}}\,dx \le c_\varphi \rho^{\theta-p(\sigma+2)+N+\sigma}.$$

Analogously, using (2.1), we obtain

$$\text{(2.8)}\quad \iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|} \left|\frac{\partial^k\varphi_\rho(x,t)}{\partial t^k}\right|^p \frac{1}{\varphi_\rho^{p-1}(x,t)|x|^{\sigma(p-1)}}\,dx\,dt$$

$$\le \int_{\operatorname{supp}|\frac{d^k\eta(t/\rho^\theta)}{dt^k}|} \left|\frac{d^k\eta(t/\rho^\theta)}{dt^k}\right|^p \frac{1}{\eta^{p-1}(t/\rho^\theta)}\,dt \int_{R<|x|<2\rho} \frac{\psi_\rho(x)}{|x|^{\sigma(p-1)}}\,dx$$

$$\le c_\eta \rho^{-\theta(kp-1)} c \int_R^{2\rho} \frac{r^{N-1}}{r^{\sigma(p-1)}}\,dr \le c_\varphi \begin{cases} \rho^{N-\sigma(p-1)-\theta(kp-1)}, & N-\sigma(p-1)>0,\\ \rho^{-\theta(kp-1)}\ln\rho, & N-\sigma(p-1)=0,\\ \rho^{-\theta(kp-1)}R^{N-\sigma(p-1)}, & N-\sigma(p-1)<0. \end{cases}$$

For $\theta = 2/k$ the powers in these estimates (under condition $N-\sigma(p-1)>0$) are equal:

$$N-\sigma(p-1)-\theta(kp-1) = \theta - p(\sigma+2)+N+\sigma \equiv -p(\sigma+2)+N+\sigma+2/k.$$

3. Proof of Theorem 1.2

Let $u(x,t)$ be a global nontrivial solution of problem (1.1). From Definition 1.1 with the test function $\varphi(x,t)=\varphi_\rho(x,t)$, defined by (2.6) with $p=q'>1$ and $\theta = 2/k$, using the equalities

$$\frac{\partial^i\varphi_\rho}{\partial t^i}(x,0)\equiv 0, \qquad i=1,\dots,k-1,$$

we obtain

$$\text{(3.1)}\quad \int_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx + \int_0^\infty\int_\Omega |u|^q|x|^\sigma\varphi_\rho\,dxdt$$

$$\le -\int_0^\infty\int_{\partial B_R} u\frac{\partial\varphi_\rho}{\partial r}\,dxdt + (-1)^k \iint_{\frac{\partial^k\varphi_\rho}{\partial t^k}=0} u\frac{\partial^k\varphi_\rho}{\partial t^k}\,dxdt - \iint_{\Delta\varphi_\rho=0} u\Delta\varphi_\rho\,dxdt$$

$$+ (-1)^k \iint_{\frac{\partial^k\varphi_\rho}{\partial t^k}\neq 0} u\frac{\partial^k\varphi_\rho}{\partial t^k}\,dxdt - \iint_{\Delta\varphi_\rho\neq 0} u\Delta\varphi_\rho\,dxdt.$$

As it was mentioned above $\frac{\partial\psi_\rho}{\partial r}|_{r=R}\ge 0$, so that $\frac{\partial\varphi_\rho}{\partial r}|_{r=R}\ge 0$ and the first integral on the right-hand side is nonpositive due to our assumption $u|_{\partial\Omega\times(0,\infty)}\ge 0$. The second and the third integrals equal zero.

Using the Hölder inequality we obtain

$$
\begin{aligned}
(3.2)\quad & \int_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx + \int_0^\infty \int_\Omega |u|^q|x|^\sigma \varphi_\rho\,dxdt \\
&= \int_\Omega \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\,dx + \iint_{\varphi_\rho(x,t)\neq\xi(x)} |u|^q|x|^\sigma\varphi_\rho\,dxdt \\
&\qquad + \iint_{\varphi_\rho(x,t)=\xi(x)} |u|^q|x|^\sigma\xi(x)\,dxdt \\
&\le \iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|} |u|\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right| dxdt + \iint_{\operatorname{supp}|\Delta\varphi_\rho|} |u||\Delta\varphi_\rho|\,dxdt \\
&\le \left(\iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|} |u|^q|x|^\sigma\varphi_\rho\,dxdt\right)^{1/q}\left(\iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|}\frac{\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|^{q'}}{\varphi_\rho^{q'-1}|x|^{\sigma(q'-1)}}\,dxdt\right)^{1/q'} \\
&\quad + \left(\iint_{\operatorname{supp}|\Delta\varphi_\rho|} |u|^q|x|^\sigma\varphi_\rho\,dxdt\right)^{1/q}\left(\iint_{\operatorname{supp}|\Delta\varphi_\rho|}\frac{|\Delta\varphi_\rho|^{q'}}{\varphi_\rho^{q'-1}|x|^{\sigma(q'-1)}}\,dxdt\right)^{1/q'}.
\end{aligned}
$$

Finally, using the Young inequality, we get

$$
\begin{aligned}
(3.3)\quad \iint_{\varphi_\rho(x,t)=\xi(x)} |u|^q|x|^\sigma\xi(x)\,dxdt &\le c\iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|}\frac{\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|^{q'}}{\varphi_\rho^{q'-1}|x|^{\sigma(q'-1)}}\,dxdt \\
&\quad + c\iint_{\operatorname{supp}|\Delta\varphi_\rho|}\frac{|\Delta\varphi_\rho|^{q'}}{\varphi_\rho^{q'-1}|x|^{\sigma(q'-1)}}\,dxdt.
\end{aligned}
$$

The last term due to (2.7) ($p = q'$, $\theta = 2/k$) is less than

$$c\rho^{-q'(\sigma+2)+N+\sigma+2/k}.$$

If

$$(3.4)\qquad -q'(\sigma+2)+N+\sigma+2/k \le 0,$$

then the last integral in (3.3) is bounded as $\rho\to\infty$. As for the first integral in the right-hand side of (3.3), it is easy to see that under condition (3.4) this integral is bounded. Passing to the limit as $\rho\to\infty$ in the case (3.4) we obtain

$$\int_0^\infty\int_\Omega |u|^q|x|^\sigma\xi\,dxdt \le c_0.$$

Then by the inequality $\varphi_\rho\le\xi$ and taking into account that the Lebesgue integral is absolutely continuous with respect to the Lebesgue measure, we have

$$
\begin{aligned}
\iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|} |u|^q|x|^\sigma\varphi_\rho\,dxdt &\le \iint_{\operatorname{supp}|\frac{\partial^k\varphi_\rho}{\partial t^k}|} |u|^q|x|^\sigma\xi\,dxdt < \varepsilon(\rho)\to 0, \\
\iint_{\operatorname{supp}|\Delta\varphi_\rho|} |u|^q|x|^\sigma\varphi_\rho\,dxdt &\le \iint_{\operatorname{supp}|\Delta\varphi_\rho|} |u|^q|x|^\sigma\xi\,dxdt < \varepsilon(\rho)\to 0
\end{aligned}
$$

as $\rho\to\infty$.

Then from inequality (3.2) we finally get

$$\iint_{\varphi_\rho(x,t)=\xi(x)} |u|^q|x|^\sigma \xi(x)\,dxdt \le 2\varepsilon^{1/q}(\rho)c_0^{1/q'} \to 0$$

as $\rho \to \infty$, and

$$\int_0^\infty \int_\Omega |u|^q|x|^\sigma \xi\,dxdt = 0,$$

that is, the solution $u(x,t)$ must be trivial under condition (3.4), which is equivalent to the condition of Theorem 1.2. $\square$

4. Possible generalizations

Using the described technique one can obtain nonexistence results for more general quasilinear equations and systems. Let us mention some examples in this direction.

For the inequality with damping

$$\begin{cases} \frac{\partial^{k_1} u}{\partial t^{k_1}} + \frac{\partial^{k_2} u}{\partial t^{k_2}} - \Delta u \ge |x|^\sigma|u|^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) \ge 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \frac{\partial^{k_1-1} u}{\partial t^{k_1-1}}(x,0) \ge 0,\ \ \frac{\partial^{k_2-1} u}{\partial t^{k_2-1}}(x,0) \ge 0, & x \in \Omega, \end{cases} \tag{4.1}$$

where $k_1 \ge 1$, $k_2 \ge 1$, we have

Theorem 4.1. *Let $\Omega = \mathbf{R}^N \setminus B_R$, $-2 < \sigma < +\infty$, $k = \min\{k_1, k_2\}$ and $1 < q \le q_k^*$, where q_k^* is defined in* (1.3). *Then problem* (4.1) *has no nontrivial global solution.*

The special case of this problem ($k_1 = 1$, $k_2 = 2$, $\sigma = 0$) was investigated in [31] (on $\mathbf{R}^N$) and in [35] (on some noncompact manifolds) with additional assumption that initial data are compactly supported.

Lemma 4.2. *Let $\Omega = \mathbf{R}^N \setminus B_R$ and*

$$1 \le m < q \le q^* = m + 2\frac{m-(m-1)/k}{N-2+2/k}.$$

Then the problem

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u^m \ge |u|^q, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) \ge 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \ge 0, & x \in \Omega, \end{cases} \tag{4.2}$$

has no nontrivial global solution.

Lemma 4.3. *Let $\Omega = \mathbf{R}^N \setminus B_R$, $q_1 > 1$, $q_2 > 1$, and*

$$\max\{\gamma_1,\gamma_2\} \ge \frac{N-2+2/k}{2}, \qquad \gamma_1 = \frac{q_1+1}{q_1q_2-1}, \quad \gamma_2 = \frac{q_2+1}{q_1q_2-1}.$$

Then the problem

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u \ge |v|^{q_1}, & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial^k v}{\partial t^k} - \Delta v \ge |u|^{q_2}, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) \ge 0, \quad v(x,t) \ge 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \ge 0, \quad \frac{\partial^{k-1} v}{\partial t^{k-1}}(x,0) \ge 0, & x \in \Omega, \end{cases} \tag{4.3}$$

has no nontrivial global solution.

Remark 4.4. The referee's report on the earlier version of this paper contained a very useful remark that the method works if the complement of a circular cylinder (with respect to t) is replaced by the complement of other cylinder. In this case $R = R(t)$ and we have to assume that $R(t) \geq R_0 > 0$ (so the domain under consideration is infinite with respect to time t).

Finally it should be mentioned that the nonlinear test function method can be applied also to semilinear inequalities in cone-like domains (see, for example, the author's papers [16, 18, 19]).

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