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LOCALLY PRE-C^*-EQUIVALENT ALGEBRAS

WEI WU

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Abstract. We prove a weaker form of Cuntz's theorem: every locally pre-C^*-equivalent Banach *-algebra is C^*-equivalent. Using this result, we obtain local conditions for the existence of an equivalent C^*-norm.

1. Introduction

A Banach *-algebra $\mathcal{A}$ is called C^*-*equivalent* if there is a norm equivalent to the given norm on $\mathcal{A}$ which makes $\mathcal{A}$ a C^*-algebra. In [2] and [3], Barnes attempted to characterize C^*-algebras in terms of their commutative closed *-subalgebras. He specifically asked the following question: if $\mathcal{A}$ is a Banach *-algebra such that each of its commutative closed *-subalgebras is C^*-equivalent, is $\mathcal{A}$ C^*-equivalent? In several important special cases, Barnes answered the question affirmatively.

A Banach *-algebra $\mathcal{A}$ is said to be *locally* C^*-*equivalent* if for every self-adjoint element h in $\mathcal{A}$, the closed *-subalgebra $C(h)$ generated by h is C^*-equivalent. Cuntz, using a restricted form of uniform C^*-equivalence, answered Barnes' question affirmatively: every locally C^*-equivalent Banach *-algebra is C^*-equivalent [5].

In Banach algebra theory, it is very useful to characterize global properties of a Banach algebra by local properties, since the commutative subalgebras of a Banach algebra are usually better known than the algebra itself [14]. On the other hand, the investigation of locally C^*-equivalent algebras is also motivated by quantum physics: the observables of quantum theory correspond to the self-adjoint elements of a C^*-algebra [16, 1].

The problem discussed here is also a version of the Kaplansky's problem in [8]. Kaplansky conjectured in [8] that a Banach *-algebra satisfying $\|x^*x\| \geq \alpha\|x^*\|\|x\|$ for some constant $\alpha > 0$ and all $x \in \mathcal{A}$ admits an equivalent C^*-norm, and a symmetric Banach *-algebra satisfying $r(h) \geq \beta\|h\|$ for some constant $\beta > 0$ and every self-adjoint element $h \in \mathcal{A}$ admits an equivalent C^*-norm. These were demonstrated by Yood (see [17, 18]). Further weakenings of the axioms of C^*-equivalent algebras appeared in [13], [11], [12], [10] and [6].

In this paper, we weaken Cuntz's theorem first, and then, inspired by the ideas of Barnes and Magyar, use the result to prove several local conditions for the existence of an equivalent C^*-norm.

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2. A weakening of Cuntz's theorem

In this section we show a weaker form of Cuntz's theorem. To begin, we give the following definitions.

Definition 2.1. A *pre-C^*-algebra* is a normed *-algebra $(\mathcal{A}, \|\cdot\|)$ which satisfies $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$ and the norm $\|\cdot\|$ is called a C^*-norm on $\mathcal{A}$. A normed *-algebra $\mathcal{A}$ is called *pre-C^*-equivalent* if there is a norm on $\mathcal{A}$, equivalent to the given norm, which makes $\mathcal{A}$ into a pre-C^*-algebra. In a *-algebra $\mathcal{A}$, we denote by $\langle h\rangle$ the commutative *-subalgebra generated by a self-adjoint element $h \in \mathcal{A}$. A Banach *-algebra $\mathcal{A}$ is said to be *locally pre-C^*-equivalent*, if for every self-adjoint element h in $\mathcal{A}$, $\langle h\rangle$ is pre-C^*-equivalent.

The following lemma is important for the proof of the main result in this section.

Lemma 2.2. *Let K be a compact subset of $\mathbb{C}$ with $K = \overline{K}$, where $\overline{K} = \{\overline{k} : k \in K\}$. Denote $\langle z\rangle_K = \{p(z) : p$ is a complex polynomial of complex variable z and without constant term, $z \in K\}$. For any $p \in \langle z\rangle_K$, we define*

$$p^*(z) = \overline{p(z)},\ \textit{for } z \in K$$

and

$$p^+(z) = \overline{p(\overline{z})},\ \textit{for } z \in K.$$

Let $\|\cdot\|_\infty$ be the sup-norm on $(\langle z\rangle_K, *)$ and $\|\cdot\|$ a C^*-norm on $(\langle z\rangle_K, +)$ which is equivalent to $\|\cdot\|_\infty$. Then:
1. *$\|p\| = \|p\|_\infty$ for all $p \in \langle z\rangle_K$.*
2. *$K \subseteq \mathbb{R}$.*
3. *$p^+ = p^*$ for all $p \in \langle z\rangle_K$.*

Proof. Let $\overline{\langle z\rangle}_K^{\|\cdot\|}$ be the completion of $\langle z\rangle_K$ in the norm $\|\cdot\|$. Then $\overline{\langle z\rangle}_K^{\|\cdot\|}$ is a commutative C^*-algebra. Because $\|\cdot\|_\infty$ is equivalent to $\|\cdot\|$, $\|\cdot\|_\infty$ has a unique continuous extension to $\overline{\langle z\rangle}_K^{\|\cdot\|}$ such that $(\overline{\langle z\rangle}_K^{\|\cdot\|}, \|\cdot\|_\infty)$ is a Banach algebra. So $\|f\| \le \|f\|_\infty$ for all $f \in \overline{\langle z\rangle}_K^{\|\cdot\|}$ (see [15, Theorem 1.2.4]). Similarly, we can deduce that $\|f\| \ge \|f\|_\infty$ for all $f \in \overline{\langle z\rangle}_K^{\|\cdot\|_\infty}$, the completion of $\langle z\rangle_K$ in the norm $\|\cdot\|_\infty$. Therefore, $\|f\| = \|f\|_\infty$ for all $f \in \langle z\rangle_K$. So (1) holds.

Now, for $f \in \langle z\rangle_K$, we have

$$\|f^+f\|_\infty = \|f^+f\| = \|f\|^2 = \|f\|_\infty^2.$$

Without loss of generality, we may assume that $K \neq \{0\}$. Suppose there exists a $z_0 \in K$ with $\Im z_0 \neq 0$. Since $K = \overline{K}$, we can choose $z_0 \in K$ such that $\Im z_0 < 0$. Let

$$D = \max_{z\in K}\{|z|\}, \quad t_0 = \frac{1}{-2\Im z_0} \ln\left(\frac{6D^2}{|z_0|^2}\right) + 1.$$

Setting

$$f_k(z) = z\sum_{n=0}^{k} \frac{(it_0 z)^n}{n!}, \quad k \in \mathbb{N},$$

we have that $f_k(z) \in \langle z\rangle_K$ for all $k \in \mathbb{N}$. In the norm $\|\cdot\|_\infty$,

$$\lim_{k\to\infty} f_k(z) = z\exp(it_0 z), \quad \lim_{k\to\infty} \overline{f_k(\overline{z})} = z\exp(-it_0 z),$$

and hence, in the norm $\|\cdot\|_\infty$,

$$\lim_{k\to\infty}\overline{f_k(\overline{z})}f_k(z)=z^2.$$

So there is an $N\in\mathbb{N}$ such that

$$\frac{1}{2}|z_0\exp(it_0z_0)|\le|f_k(z_0)|,\quad \|f_k^+f_k\|_\infty<\frac{3}{2}\|z^2\|_\infty,$$

for $k\ge N$. Hence we have

$$\|f_N^+f_N\|_\infty<\frac{3}{2}D^2,$$

and

$$\begin{aligned}\|f_N\|_\infty^2 &= \|f_N^*f_N\|_\infty\\ &\ge |f_N^*(z_0)f_N(z_0)|\\ &= |f_N(z_0)|^2\\ &\ge \tfrac{1}{4}|z_0\exp(it_0z_0)|^2\\ &= \tfrac{1}{4}|z_0|^2\exp(-2t_0\Im z_0)\\ &> \tfrac{3}{2}D^2.\end{aligned}$$

Now we get that $\frac{3}{2}D^2>\frac{3}{2}D^2$. This is a contradiction. Therefore, $K\subseteq\mathbb{R}$, and hence $p^+=p^*$ for all $p\in\langle z\rangle_K$. □

Now, we are in a position to state and prove the weaker form of Cuntz's theorem.

Theorem 2.3. *Every locally pre-C^*-equivalent Banach *-algebra is C^*- equivalent.*

Proof. Assume that $(\mathcal{A},\|\cdot\|)$ is a locally pre-C^*-equivalent Banach *-algebra. For any self-adjoint element h in $\mathcal{A}$, let $|\cdot|_h$ be the C^*-norm on $\langle h\rangle$ which is equivalent to $\|\cdot\|$ on $\langle h\rangle$. For $x\in\mathcal{A}$, the spectral radius of x is denoted by $r(x)$, that is, $r(x)=\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}}$. We break the proof into four steps.

Step 1. Let h be a self-adjoint element in $\mathcal{A}$. For $x\in\langle h\rangle$,

$$|x|_h=r(x^*x)^{\frac{1}{2}}.$$

In particular, for self-adjoint $y\in\langle h\rangle$,

$$|y|_h=r(y).$$

In fact, for any self-adjoint element $x\in\langle h\rangle$, we have

$$|x^{2^n}|_h=|x|_h^{2^n}$$

since $|\cdot|_h$ is a C^*-norm. By the equivalence of $|\cdot|_h$ and $\|\cdot\|$ on $\langle h\rangle$, we get

$$|x|_h=\lim_{n\to\infty}|x^{2^n}|_h^{2^{-n}}=\lim_{n\to\infty}\|x^{2^n}\|^{2^{-n}}=r(x).$$

So, for $x\in\langle h\rangle$, we obtain

$$|x|_h=|x^*x|_h^{\frac{1}{2}}=r(x^*x)^{\frac{1}{2}}.$$

Step 2. $\mathcal{A}$ is hermitian.

Let h be a self-adjoint element in $\mathcal{A}$. Then for self-adjoint element $y \in \langle h \rangle$, $|y|_h = r(y)$ by Step 1. If $x \in \langle h \rangle$, we have that

$$\begin{array}{rcl} r(x^*) & = & \lim_{n\to\infty} \|(x^*)^n\|^{\frac{1}{n}} \\ & = & \lim_{n\to\infty} |(x^*)^n|_h^{\frac{1}{n}} \\ & = & \lim_{n\to\infty} |x^n|_h^{\frac{1}{n}} \\ & = & \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \\ & = & r(x). \end{array}$$

Note that spectral radius is a seminorm on a commutative normed algebra (see [4, p. 19, Corollary 3]). Thus

$$\begin{array}{rcl} |x|_h & \leq & |\frac{x+x^*}{2}|_h + |\frac{x-x^*}{2i}|_h \\ & = & \frac{1}{2}(r(x+x^*) + r(x-x^*)) \\ & \leq & 2r(x) \leq 2\|x\| \end{array}$$

for any $x \in \langle h \rangle$. So $|\cdot|_h$, $\|\cdot\|$ and $r(\cdot)$ are equivalent norms on $\langle h \rangle$.

For $p \in \langle z \rangle_{\sigma(h)}$, define

$$\|p\|_1 = |p(h)|_h.$$

If $p(z) = 0$ for $z \in \sigma(h)$, then $\sigma(p(h)) = p(\sigma(h)) = \{0\}$. Thus $r(p(h)) = 0$, and so $|p(h)|_h = 0$. Therefore, $\|\cdot\|_1$ is well defined and it is a norm on $\langle z \rangle_{\sigma(h)}$. Since $\mathcal{A}$ is a *-algebra, $\overline{\sigma(h)} = \sigma(h)$. For $p, q \in \langle z \rangle_{\sigma(h)}$, we have

$$\|pq\|_1 = |p(h)q(h)|_h \leq |p(h)|_h |q(h)|_h = \|p\|_1 \|q\|_1,$$

$$\|p^+ p\|_1 = |p^+(h)p(h)|_h = |p(h)^* p(h)|_h = |p(h)|_h^2 = \|p\|_1^2$$

and

$$\begin{array}{rcl} \|p\|_\infty & = & \max_{z\in\sigma(h)}\{|p(z)|\} \\ & = & \max_{w\in p(\sigma(h))}\{|w|\} \\ & = & \max_{w\in\sigma(p(h))}\{|w|\} \\ & = & r(p(h)). \end{array}$$

Since $r(\cdot)$ and $|\cdot|_h$ are equivalent on $\langle h \rangle$, there are positive constants M and N such that $M|s(h)|_h \leq r(s(h)) \leq N|s(h)|_h$ for every $s \in \langle z \rangle_{\sigma(h)}$. So $M\|p\|_1 = M|p(h)|_h \leq \|p\|_\infty \leq N|p(h)|_h = N\|p\|_1$ for all $p \in \langle z \rangle_{\sigma(h)}$. By Lemma 2.2, we obtain that $\sigma(h) \subseteq \mathbb{R}$. Therefore, $\mathcal{A}$ is hermitian.

Step 3. $\mathcal{A}$ is semisimple and the involution on $\mathcal{A}$ is continuous.

Let $R(\mathcal{A})$ be the Jacobson radical of $\mathcal{A}$. Then $R(\mathcal{A}) = \{x \in \mathcal{A} : 1 + xy$ is invertible in $\mathcal{A}$ for every $y \in \mathcal{A}\}$ if $\mathcal{A}$ has identity 1, or $R(\mathcal{A}) = \{x \in \mathcal{A} : 1 + xy$ is invertible in $\mathcal{A}\dot{+}\mathbb{C}$ for every $y \in \mathcal{A}\}$ if $\mathcal{A}$ has no identity [7]. So $R(\mathcal{A})$ is self-adjoint. Suppose h is a self-adjoint element in $R(\mathcal{A})$. Then $r(h) = 0$ (see [4, p. 126]). By Step 1, $|h|_h = r(h) = 0$. Since $|\cdot|_h$ and $\|\cdot\|$ are equivalent on $\langle h \rangle$, $\|h\| = 0$, and hence $h = 0$. $\mathcal{A}$ is hermitian implies that $R(\mathcal{A}) = \{a \in \mathcal{A} : r(a^*a) = 0\}$ (see [4, p. 227, Theorem 9]). So $R(\mathcal{A}) = \{0\}$, that is, $\mathcal{A}$ is semisimple. Therefore, the involution on $\mathcal{A}$ is continuous (see [4, p. 191, Theorem 2]).

Step 4. For every self-adjoint element h in $\mathcal{A}$, $C(h)$ is C^*-equivalent.

By Step 3, the involution on $\mathcal{A}$ is continuous. So we get that

$$C(h) = \overline{\langle h \rangle}^{\|\cdot\|},$$

where $\overline{\langle h \rangle}^{\|\cdot\|}$ is the closure of $\langle h \rangle$ in the norm $\|\cdot\|$. Since $|\cdot|_h$ and $\|\cdot\|$ are equivalent on $\langle h \rangle$, we can extend $|\cdot|_h$ to $C(h)$. Denote this extension by $|\cdot|_h$, too. Clearly, $|\cdot|_h$ and $\|\cdot\|$ are still equivalent on $C(h)$ and $(C(h), |\cdot|_h)$ is a C^*-algebra. So $C(h)$ is C^*-equivalent. Therefore, $\mathcal{A}$ is locally C^*-equivalent, and hence $\mathcal{A}$ is C^*-equivalent by Cuntz's theorem. □

3. Conditions for existence of an equivalent C^*-norm

In this section, we discuss the existence of an equivalent C^*-norm on a Banach *-algebra. As a motivation for the discussion at hand, consider the following theorem which was proved by Magyar in [11].

Theorem 3.1 (Magyar's theorem)**.** *Let $\mathcal{A}$ be a *-algebra. Let $\|\cdot\|$ be a norm on it, and assume that the following hold with suitable positive constants α and β:*

(1) $\|x^*x\| \leq \alpha\|x^*\|\|x\|$ *for all* $x \in \mathcal{A}$,
(2) $\|x^*x\| \geq \beta\|x^*\|\|x\|$ *if* $x \in \langle h \rangle$, $h = h^* \in \mathcal{A}$.

Then $(\mathcal{A}, \|\cdot\|)$ is pre-C^*-equivalent.

This is a very weak condition which can hardly be weakened. In the theorem above, the usual condition of the submultiplicativity on the norm $\|\cdot\|$ is replaced by the weaker assumption: $\|x^*x\| \leq \alpha\|x^*\|\|x\|$ for all $x \in \mathcal{A}$. The following theorem, on the other hand, indicates that the uniform condition: $\|x^*x\| \geq \beta\|x^*\|\|x\|$, on all commutative *-subalgebras $\langle h \rangle$, $h = h^* \in \mathcal{A}$, can also be localized further if the norm on $\mathcal{A}$ is submultiplicative.

Theorem 3.2. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach *-algebra. If, for every self-adjoint element h in $\mathcal{A}$, there are a norm $|\cdot|_h$ on $\langle h \rangle$, which is equivalent to $\|\cdot\|$ on $\langle h \rangle$, and a positive constant α_h such that $\alpha_h|x^*|_h|x|_h \leq |x^*x|_h$ for all $x \in \langle h \rangle$, then $\mathcal{A}$ is C^*-equivalent.*

Proof. Since $|\cdot|_h$ and $\|\cdot\|$ are equivalent on $\langle h \rangle$, there exist positive constants μ_h and λ_h such that

$$\mu_h\|x\| \leq |x|_h \leq \lambda_h\|x\|,\ x \in \langle h \rangle.$$

Then, for any $x \in \langle h \rangle$, we have

$$\alpha_h|x^*|_h|x|_h \leq |x^*x|_h \leq \frac{\lambda_h}{\mu_h^2}|x^*|_h|x|_h.$$

On the *-algebra $\langle h \rangle$, applying Magyar's theorem, we obtain that $(\langle h \rangle, |\cdot|_h)$ is pre-C^*-equivalent. So $(\langle h \rangle, \|\cdot\|)$ is pre-C^*-equivalent. By Theorem 2.3, $\mathcal{A}$ is C^*-equivalent. □

Corollary 3.3. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach *-algebra. If, for every self-adjoint element h in $\mathcal{A}$, there are a norm $|\cdot|_h$ on $\langle h \rangle$, which is equivalent to $\|\cdot\|$ on $\langle h \rangle$, and a positive constant α_h such that $\alpha_h|x|_h^2 \leq |x^*x|_h$ for all $x \in \langle h \rangle$, then $\mathcal{A}$ is C^*-equivalent.*

Proof. Suppose h is a self-adjoint element in $\mathcal{A}$. Since $|\cdot|_h$ and $\|\cdot\|$ are equivalent on $\langle h \rangle$, there exist positive constants μ_h and λ_h such that

$$\mu_h\|x\| \leq |x|_h \leq \lambda_h\|x\|,\ x \in \langle h \rangle.$$

Then, for any $x \in \langle h\rangle$, we have

$$\alpha_h |x|_h^2 \leq |x^*x|_h \leq \frac{\lambda_h^2}{\alpha_h \mu_h^4}|x|_h^2.$$

So

$$\alpha_h |x|_h^2 \leq \lambda_h \|x^*x\| \leq \lambda_h \|x^*\|\|x\| \leq \frac{\lambda_h}{\mu_h^2}|x^*|_h |x|_h,$$

that is, $\alpha_h |x|_h \leq \frac{\lambda_h}{\mu_h^2}|x^*|_h$. Thus

$$\frac{\alpha_h^2 \mu_h^2}{\lambda_h}|x^*|_h |x|_h \leq \alpha_h |x|_h^2 \leq |x^*x|_h.$$

By Theorem 3.2, $\mathcal{A}$ is C*-equivalent. □

The next theorem gives the local spectral radius characterization of C^*-equivalent algebras.

Theorem 3.4. *Let $(\mathcal{A}, \|\cdot\|)$ be a hermitian Banach *-algebra. If, for every self-adjoint element h in $\mathcal{A}$, there is a positive constant M_h such that $\|x\| \leq M_h r(x)$ holds for every self-adjoint element $x \in \langle h\rangle$, where $r(x)$ is the spectral radius of x in $\mathcal{A}$, then $\mathcal{A}$ is C^*-equivalent.*

Proof. Let $R(\mathcal{A})$ be the Jacobson radical of $\mathcal{A}$. Then $R(\mathcal{A}) = \{x \in \mathcal{A} : 1 + xy$ is invertible in $\mathcal{A}$ for every $y \in \mathcal{A}\}$ if $\mathcal{A}$ has identity 1, or $R(\mathcal{A}) = \{x \in \mathcal{A} : 1 + xy$ is invertible in $\mathcal{A}\dot{+}\mathbb{C}$ for every $y \in \mathcal{A}\}$ if $\mathcal{A}$ has no identity, and hence $R(\mathcal{A})$ is self-adjoint. Suppose h is a self-adjoint element in $R(\mathcal{A})$. Then $r(h) = 0$. So $\|h\| \leq M_h r(h) = 0$, that is, $h = 0$. Thus $R(\mathcal{A}) = \{0\}$ since $\mathcal{A}$ is hermitian. We obtain that $\mathcal{A}$ is semisimple, and so the involution on $\mathcal{A}$ is continuous.

Suppose α is a positive constant such that $\|x^*\| \leq \alpha\|x\|$ for all $x \in \mathcal{A}$. By Pták's fundamental inequality [13]:

$$r(x) \leq \rho(x),\ x \in \mathcal{A},$$

where $\rho(x) = r(x^*x)^{\frac{1}{2}}$, we have

$$\begin{array}{rcl} M_h^{-1}\|x\| & \leq & M_h^{-1}(\|k_1\| + \|k_2\|) \\ & \leq & r(k_1) + r(k_2) \\ & \leq & r(x) + r(x^*) \\ & \leq & 2r(x) \\ & \leq & 2\rho(x) \\ & \leq & 2\alpha^{\frac{1}{2}}\|x\| \end{array}$$

for $x = k_1 + ik_2 \in \langle h\rangle$, where $k_1^* = k_1$, $k_2^* = k_2$. Since $\rho(\cdot)$ is a C^*-seminorm on a hermitian Banach *-algebra [13], $(\langle h\rangle, \|\cdot\|)$ is pre-C^*-equivalent. By Theorem 2.3, $\mathcal{A}$ is C^*-equivalent. □

Our last result is the local analytic characterization which is the generalization and the localization of the exponential characterization (see [9], [12]).

Theorem 3.5. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach *-algebra, and let g be an entire function on $\mathbb{C}$ satisfying $g'(0) \neq 0$, and $\sup_{t\in\mathbb{R}}\{|g(tz)|\} = +\infty$ for all $z \in \mathbb{C}$ with $\Im z \neq 0$. If, for every self-adjoint element h in $\mathcal{A}$, there is a positive constant M_h such that $\|g(y) - g(0)\| \leq M_h$ for all $y = y^* \in \langle h\rangle$, then $\mathcal{A}$ is C^*-equivalent.*

Proof. Let $f(z) = g(z) - g(0)$. Then f is an entire function on $\mathbb{C}$ satisfying $f'(0) \neq 0$, $g(0) = 0$ and $\sup_{t\in\mathbb{R}}\{|f(tz)|\} = +\infty$ for all $z \in \mathbb{C}$ with $\Im z \neq 0$. Let h be a self-adjoint element in $\mathcal{A}$. If $z \in \mathbb{C}$ and $\Im z \neq 0$, then the set $\{f(tz) : t \in \mathbb{R}\}$ is not bounded. So $\{r(f(th)) : t \in \mathbb{R}\}$ is not bounded if $\sigma(h) \subset \mathbb{C} \setminus \mathbb{R}$. Since $r(a) \leq \|a\|$ for $a \in \mathcal{A}$, $\{\|f(th)\| : t \in \mathbb{R}\}$ is not bounded if $\sigma(h) \subset \mathbb{C} \setminus \mathbb{R}$. Therefore the set $\{\|g(th)\| : t \in \mathbb{R}\}$ is not bounded if $\sigma(h) \subset \mathbb{C} \setminus \mathbb{R}$. So $\sigma(h) \subset \mathbb{R}$, that is, $\mathcal{A}$ is hermitian.

Assume $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Let h be a self-adjoint element in $\mathcal{A}$. For any nonzero self-adjoint element $y \in \langle h\rangle$ and $t \in \mathbb{R}^+$, we have

$$\begin{aligned} t|a_1| &= \left\|a_1 t \tfrac{y}{\|y\|}\right\| \\ &= \left\|f(t\tfrac{y}{\|y\|}) - \textstyle\sum_{n=2}^{\infty} a_n t^n \tfrac{y^n}{\|y\|^n}\right\| \\ &\leq M_h + \textstyle\sum_{n=2}^{\infty}|a_n| t^n \left(\left\|\tfrac{y^2}{\|y\|^2}\right\|^{\frac{1}{3}}\right)^n. \end{aligned}$$

Letting $t = \left\|\frac{y^2}{\|y\|^2}\right\|^{-\frac{1}{3}}$, we get

$$\left\|\frac{y^2}{\|y\|^2}\right\|^{-\frac{1}{3}} \leq |a_1|^{-1}\left(M_h + \sum_{n=2}^{\infty}|a_n|\right).$$

Therefore

$$\|y\|^2 \leq \left[|a_1|^{-3}\left(M_h + \sum_{n=2}^{\infty}|a_n|\right)^3\right]\|y^2\|.$$

Now we have

$$\begin{aligned} \|y\|^{2^n} &= (\|y\|^2)^{2^{n-1}} \\ &\leq \left[|a_1|^{-3}\left(M_h + \textstyle\sum_{n=2}^{\infty}|a_n|\right)^3\right]^{2^{n-1}} \|y^2\|^{2^{n-1}} \\ &\leq \cdots \\ &\leq \left[|a_1|^{-3}\left(M_h + \textstyle\sum_{n=2}^{\infty}|a_n|\right)^3\right]^{2^n-1} \left\|y^{2^n}\right\|, \end{aligned}$$

and hence we get

$$\|y\| \leq \left[|a_1|^{-3}\left(M_h + \sum_{n=2}^{\infty}|a_n|\right)^3\right] r(y).$$

When $y = 0$, it is clear that the inequality above holds. By Theorem 3.4, $\mathcal{A}$ is C^*-equivalent. □

Corollary 3.6. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach *-algebra. If, for every self-adjoint element h in $\mathcal{A}$, there is a positive constant M_h such that $\|\exp(iy) - 1\| \leq M_h$ for all $y = y^* \in \langle h\rangle$, then $\mathcal{A}$ is C^*-equivalent.*

Corollary 3.7. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach *-algebra. If, for every self-adjoint element h in $\mathcal{A}$, there is a positive constant M_h such that $\|\sin y\| \leq M_h$ for all $y = y^* \in \langle h\rangle$, then $\mathcal{A}$ is C^*-equivalent.*

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Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China

E-mail address: wwu@math.ecnu.edu.cn