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A TRACE FORMULA FOR ISOMETRIC PAIRS

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ABSTRACT. It is well known that for every isometry V, $tr[V^*,\ V] = -ind(V)$. This fact for the shift operator is a basis for many important developments in operator theory and topology. In this paper we prove an analogous formula for a pair of isometries $(V_1,\ V_2)$, namely

$$tr[V_1^*, V_1, V_2^*, V_2] = -2ind(V_1, V_2),$$

where $[V_1^*, V_1, V_2^*, V_2]$ is the complete anti-symmetric sum and $ind(V_1, V_2)$ is the Fredholm index of the pair $(V_1,\ V_2)$. The major tool is what we call the *fringe* operator. Two examples are considered.

0. INTRODUCTION

In recent decades, studies of operator tuples have been very active. Because of their connections to complex function theory, canonical model theory, control theory, signal processing, etc., commuting isometric tuples have received special attention. The work in [ACD], [BCL], [DF], [GS], [Su] and many other articles gave us a much better understanding of the structure of isometric tuples. However, quantitative properties of isometric tuples remain mystic, and so far little has been done to reveal them. The difficulty lies in the fact that one does not know what to expect. One useful idea to overcome this difficulty is looking for analogues of quantitative facts about a single isometry in a multi-variable setting.

If V is an isometry acting on a separable Hilbert space $\mathcal{H}$, its self-commutator $[V^*,\ V]$ is the orthogonal projection from $\mathcal{H}$ onto its range $R([V^*,\ V])= Coker(V) := \mathcal{H} \ominus V\mathcal{H}$. Therefore,

$$(*) \qquad tr[V^*,\ V] = -ind(V).$$

Although it looks simple, (*) is a very useful quantative property of V. In fact, in the case when $\mathcal{H}$ is $H^2(D)$, the Hardy space over the unit disc, and V is the Toeplitz operator T_z, this fact becomes the basis for many important developments in Toeplitz operator theory (cf. [Do]), C^*-algebras (cf. [BDF]), topology (cf. [At]) and in many other places. In this paper, we show an analogous formula of (*) for a pair of isometries. Most ideas and techniques in this paper are from [Ya3] where a pair of shift operators on $H^2(D^2)$ were studied in detail.

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In (*), trace of the self-commutator for V is related to the Fredholm index of V. But for a pair (V_1, V_2), the notion of "self-commutator" is not that clear. Here we give a brief introduction. Details can be found in the references herein.

Joint self-commutators of operator tuples. The *joint self-commutator* of operator tuples are defined through *complete anti-symmetric form* for operator tuples. For an m tuple $(T_1,\ ...,\ T_m)$, the complete anti-symmetric form $[T_1,\ ...,\ T_m]$ is defined to be

$$[T_1,\ ...,\ T_m] = \sum_{\sigma\in S_m} \epsilon(\sigma) T_{\sigma(1)}\cdots T_{\sigma(m)},$$

where S_m is the symmetric group on $(1,\ ...,\ m)$ and $\epsilon(\sigma)$ is the signum of the permutation σ. The *joint self-commutator* of $(T_1,\ ...,\ T_m)$ is $[T_1^*,\ T_1, ...,\ T_m^*,\ T_m]$.

The complete anti-symmetric form for operator tuples was studied in the 70's and 80's, in which period many efforts were made to detect topological ingredients in multi-variable operator theory (cf. [CP], [DV], [HH]). Results in [CP] and [DV], among other things, revealed connections between the trace of the joint self-commutator and the Fredholm index of the tuple. Their results were based on slightly different conditions, but one thing they have in common is the requirement that the tuple $(T_1,\ ...,\ T_m)$ satisfies some essential commutativity conditions, namely $[T_i,\ T_j]$ and $[T_i,\ T_j^*]$ are in Schatten p-class for some fixed p and all i and j. Their idea of proof, roughly speaking, is to assign a Bott operator $\hat{T}$, which is a $2^{m-1}\times 2^{m-1}$ matrix with operator entries, to $(T_1,\ ...,\ T_m)$. For example, the Bott operator for $(T_1,\ T_2)$ is

$$\hat{T} = \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix},$$

and then we use a lemma of Calderon to express the trace of $[\hat{T}^*,\ \hat{T}]$ by $ind(\hat{T})$. Since every T_i has a position in $\hat{T}$, one needs the conditions on $[T_i,\ T_j^*]$ for every i and j to make $[\hat{T}^*,\ \hat{T}]$ trace class. But $[T_i,\ T_i^*]$ being compact is a very strong condition on T_i. As we will show in this paper, in the case of isometric pairs, this condition is not really necessary. But we need different techniques to sort things out.

Let us first take a closer look at $[V_1^*, V_1, V_2^*, V_2]$. Using the fact that $V_1V_2 = V_2V_1$, one has the expression

$$\begin{aligned}[V_1^*, V_1, V_2^*, V_2] = {} & [V_1^*, V_1][V_2^*, V_2] + [V_2^*, V_2][V_1^*, V_1] - [V_1^*, V_2][V_2^*, V_1] \\ & - [V_2^*, V_1][V_1^*, V_2].\end{aligned}$$

So what really matters are the product $[V_1^*, V_1][V_2^*, V_2]$ and the cross commutator $[V_2^*, V_1]$ (note that $[V_2^*, V_2][V_1^*, V_1] = ([V_1^*, V_1][V_2^*, V_2])^*$ and $[V_1^*, V_2] = [V_2^*, V_1]^*$), so $[V_1^*, V_1, V_2^*, V_2]$ will be trace class as long as $[V_1^*, V_1][V_2^*, V_2]$ is trace class and $[V_2^*, V_1]$ is Hilbert-Schmidt.

In fact, $[V_1^*, V_1][V_2^*, V_2]$ and $[V_2^*, V_1]$ are both natural objects associated with the pair $(V_1,\ V_2)$. In vague terms, $tr[V_1^*, V_1][V_2^*, V_2]$ measures the orthogonality of range $R([V_1^*, V_1])$ to range $R([V_2^*, V_2])$, and $tr[V_2^*, V_1][V_1^*, V_2]$ measures the commutativity of V_1^* with V_2. $[V_1^*, V_1][V_2^*, V_2]$ and $[V_1^*, V_2]$ are better understood through the Berger-Coburn-Lebow model for isometric pairs. In [BCL], the authors showed that every completely non-unitary isometric pair $(V_1,\ V_2)$, up to unitary equivalence, is of the form

$$(V_1 f)(z) = (P + zP^{\perp})V^* f(z), \quad (V_2 f)(z) = V(zP + P^{\perp}) f(z) \quad (z\in D, f\in H^2(\mathcal{E})),$$

where $\mathcal{E}$ is a Hilbert space, V is a unitary and P is a projection acting on $\mathcal{E}$, and $H^2(\mathcal{E})$ is the $\mathcal{E}$-valued Hardy space. Using this representation, one computes that

$$[V_1^*, V_2]f = VPVP^{\perp}f(0),$$
$$[V_1^*, V_1][V_2^*, V_2]f = P^{\perp}VPV^*f(0).$$

Since V is unitary, properties of $[V_1^*, V_2]$ and $[V_1^*, V_1][V_2^*, V_2]$ are determined by $PVP^{\perp}$ and $P^{\perp}VP$, respectively. From this viewpoint, we see that $[V_2^*, V_1]$ and $[V_1^*, V_1][V_2^*, V_2]$ play symmetric roles. The following proposition is a direct consequence of the computation. Here $\|A\|_{HS}$ denotes the Hilbert-Schmidt norm of A.

Proposition 0.1.

$$[V_1^*, V_1][V_2^*, V_2] = \|P^{\perp}VP\|_{HS}, \quad tr[V_2^*, V_1][V_1^*, V_2] = \|PVP^{\perp}\|_{HS}.$$

We observe further that when $[V_2^*, V_1]$ is Hilbert-Schmidt, $tr[V_1^*, V_2][V_2^*, V_1] = tr[V_2^*, V_1][V_1^*, V_2]$, and moreover since $[V_1^*, V_1]$ and $[V_2^*, V_2]$ are projections, then $[V_1^*, V_1][V_2^*, V_2]$ is trace class if and only if it is Hilbert-Schmidt and

$$tr[V_1^*, V_1][V_2^*, V_2] = \|[V_2^*, V_2][V_1^*, V_1]\|_{HS}^2 = tr[V_2^*, V_2][V_1^*, V_1].$$

In conclusion, when $[V_1^*, V_1][V_2^*, V_2]$ and $[V_2^*, V_1]$ are Hilbert-Schmidt, then $[V_1^*, V_1, V_2^*, V_2]$ is trace class with

$$tr[V_1^*, V_1, V_2^*, V_2] = 2(tr[V_1^*, V_1][V_2^*, V_2] - tr[V_2^*, V_1][V_1^*, V_2]).$$

The goal of this paper is to show that under this conditon, there is a simple trace formula for $[V_1^*, V_1, V_2^*, V_2]$ in terms of $ind(V_1, V_2)$. This formula is different from the ones in [CP] and [DV].

Koszul complex. For every pair of commutating operators $(T_1,\ T_2)$ on Hilbert space H, there is a short sequence

$$0 \to H \xrightarrow{d_1} H \oplus H \xrightarrow{d_2} H \to 0,$$

where

$$d_1 x = (-T_2 x,\ T_1 x), \quad d_2(x,\ y) = T_1 x + T_2 y, \quad x,\ y \in H,$$

and it is easy to check that

$$d_2 d_1 = 0.$$

The tuple $(T_1,\ T_2)$ is said to be Fredholm if d_1 and d_2 both have closed range and

$$dim(Ker(d_1)) + dim(Ker(d_2) \ominus d_1(H)) + dim(H \ominus d_2(H \oplus H)) < +\infty,$$

and its index

$$ind(T_1, T_2) := -dim(Ker(d_1)) + dim(Ker(d_2) \ominus d_1(H)) - dim(H \ominus d_2(H \oplus H)).$$

Fredholm complexes for n-tuples can also be defined in this combinatoric way. We refer the readers to [Cu] for detailed discussions. Here we look at the Fredhom complex for $(V_1,\ V_2)$. It is easy to see that in this case $Ker(d_1) = 0$ and $d_2(\mathcal{H} \oplus \mathcal{H}) = V_1\mathcal{H} + V_2\mathcal{H}$, and the following lemma is from [Ya1].

Lemma 0.2. $Ker(d_2) \ominus d_1(\mathcal{H}) = \{(x, -V_2^* V_1 x) : x \in \mathcal{H} \ominus V_2\mathcal{H}$ *and* $V_1 x \in V_2\mathcal{H}\}$.

1. Fringe operator

Our idea in proving the trace formula is also to assign a single operator to operator tuples and then study the trace formula of the single operator. But to relax the strong conditions required in [CP], [DV], we use the *fringe* operator instead of the Bott operator for $(V_1,\ V_2)$. By some standard, the Bott operator is not really a single operator, but the fringe operator is.

If $(V_1,\ V_2)$ is a pair of commuting isometries, then $p_\perp := 1-V_1V_1^*$ is a projection from $\mathcal{H}$ onto $\mathcal{H}\ominus V_1\mathcal{H}$. The fringe operator F acting on $\mathcal{H}\ominus V_1\mathcal{H}$ is defined by

$$Fx = p_\perp V_2 x$$

for $x\in\mathcal{H}\ominus V_1\mathcal{H}$. It is believable that properties of $(V_1,\ V_2)$ are encoded in this single operator F. More precisely, properties of V_1 are encoded in the space $\mathcal{H}\ominus V_1\mathcal{H}$ on which F is defined, while properties of V_2 are encoded in the composition $p_\perp V_2$. In the following, we will make this intuition concrete and see that the fringe operator fits pieces together perfectly.

Lemma 1.1. $R(F) = (V_1\mathcal{H}+V_2\mathcal{H})\ominus V_1\mathcal{H}$.

Proof. First of all, for every $x\in\mathcal{H}\ominus V_1\mathcal{H}$,

$$Fx = p_\perp V_2x = (1-V_1V_1^*)V_2x = V_2x - V_1V_1^*V_2x$$

which is in $(V_1\mathcal{H}+V_2\mathcal{H})\ominus V_1\mathcal{H}$, so $R(F)\subset(V_1\mathcal{H}+V_2\mathcal{H})\ominus V_1\mathcal{H}$.

On the other hand, for every $y\in(V_1\mathcal{H}+V_2\mathcal{H})\ominus V_1\mathcal{H}$, $y=V_1x_1+V_2x_2$ for some x_1 and x_2 in $\mathcal{H}$. Since $y\perp V_1\mathcal{H}$, for all $h\in\mathcal{H}$,

$$\begin{aligned}0 &= \langle V_1x_1+V_2x_2,\ V_1h\rangle\\ &= \langle x_1,\ h\rangle + \langle V_1^*V_2x_2,\ h\rangle\\ &= \langle x_1+V_1^*V_2x_2,\ h\rangle.\end{aligned}$$

This implies that $x_1+V_1^*V_2x_2=0$ and hence

$$y = V_1x_1+V_2x_2 = -V_1V_1^*V_2x_2+V_2x_2 = (1-V_1V_1^*)V_2x_2.$$

If we write $x_2=(1-V_1V_1^*)x_2+V_1V_1^*x_2$, by the commutativity of V_1 and V_2 we easily obtain

$$y=(1-V_1V_1^*)V_2(1-V_1V_1^*)x_2 = Fp_\perp x_2.$$

This establishes the inclusion $(V_1\mathcal{H}+V_2\mathcal{H})\ominus V_1\mathcal{H}\subset R(F)$, and the lemma is proved. $\square$

It follows easily from Lemma 1.1 that $R(F)$ is closed if and only $V_1\mathcal{H}+V_2\mathcal{H}$ is closed and moreover,

$$Ker(F^*) = \mathcal{H}\ominus(V_1\mathcal{H}+V_2\mathcal{H}).$$

The following lemma regarding $Ker(F)$ follows directly from the definition of F.

Lemma 1.2. $Ker(F) = \{x\in\mathcal{H}\ominus V_1\mathcal{H}:\ V_2x\in V_1\mathcal{H}\}$.

By this lemma and Lemma 0.2, the map which sends x to $(x,\ -V_2^*V_1x)$ is a bijection between $Ker(F)$ and $Ker(d_2)\ominus d_1(\mathcal{H})$ and hence we have the following:

Theorem 1.3. *F is Fredholm if and only if $(V_1,\ V_2)$ is Fredholm, and moreover*

$$ind(F) = ind(V_1,\ V_2).$$

We now establish a relationship between commutators $[V_1^*,\ V_2]$, $[V_1^*,\ V_1][V_2^*,\ V_2]$ and the fringe operator F.

Proposition 1.4. *For every $x \in \mathcal{H} \ominus V_1\mathcal{H}$,*
(a) $x - F^*Fx = [V_2^*,\ V_1][V_1^*,\ V_2]x$;
(b) $x - FF^*x = [V_1^*,\ V_1][V_2^*,\ V_2]x$.

Proof. (a)

$$\begin{aligned} F^*Fx &= [V_1^*,\ V_1]V_2^*[V_1^*,\ V_1]V_2x \\ &= (1 - V_1^*V_1)(1 - V_2^*V_1V_1^*V_2)x \\ &= x - (1 - V_1V_1^*)V_2^*V_1V_1^*V_2x \\ &= x - (1 - V_1V_1^*)(V_2^*V_1[V_1^*,\ V_2]x + V_2^*V_1V_2V_1^*x) \\ &= x - (1 - V_1V_1^*)(V_2^*V_1[V_1^*,\ V_2]x + V_1V_1^*x) \\ &= x - (1 - V_1V_1^*)V_2^*V_1[V_1^*,\ V_2]x. \end{aligned}$$

Since $(1 - V_1V_1^*)V_2^*V_1 = V_2^*V_1 - V_1V_2^*$, it follows that

$$x - F^*Fx = [V_2^*,\ V_1][V_1^*,\ V_2]x.$$

(b) Since $V_1^*x = 0$, $F^*x = (1 - V_1V_1^*)V_2^*x = V_2^*x$. Therefore,

$$\begin{aligned} FF^*x &= (1 - V_1V_1^*)V_2V_2^*x \\ &= x - (1 - V_1V_1^*)(1 - V_2V_2^*)x \end{aligned}$$

and hence

$$x - FF^*x = [V_1^*,\ V_1][V_2^*,\ V_2]x.$$

□

As an immediate consequence, F is Fredholm when $[V_1^*,\ V_2]$ and $[V_1^*,\ V_1][V_2^*,\ V_2]$ are compact. In particular, the compactness of $[V_1^*,\ V_1][V_2^*,\ V_2]$ will imply the closedness of $V_1M + V_2M$. We do not know how to prove this fact without using the fringe operator.

Note also that for every $x \in V_1M$, we can write $x = V_1y$. Therefore

$$[V_1^*,\ V_2]x = V_1^*V_1V_2y - V_2V_1^*V_1y = V_2y - V_2y = 0$$

and

$$\langle [V_1^*,\ V_1][V_2^*,\ V_2]x,\ x\rangle = \langle [V_2^*,\ V_2]x,\ [V_1^*,\ V_1]V_1y\rangle = 0.$$

So it follows from Proposition 1.4 that

$$tr(1 - F^*F) = tr[V_2^*,\ V_1][V_1^*,\ V_2];\ tr(1 - FF^*) = tr[V_1^*,\ V_1][V_2^*,\ V_2],$$

and in particular,

$$2tr[F^*,\ F] = tr[V_1^*, V_1, V_2^*, V_2].$$

Since F is much easier to study, the identities in Proposition 1.4 sometimes provide an estimate for $tr[V_2^*,\ V_1][V_1^*,\ V_2]$ and $tr[V_1^*,\ V_1][V_2^*,\ V_2]$. In fact, for every $x \in \mathcal{H} \ominus V_1\mathcal{H}$ with $\|x\| = 1$,

$$\langle (1 - F^*F)x,\ x\rangle = 1 - \|Fx\|^2 \le 1 - |\langle Fx,\ x\rangle|^2$$

and

$$\langle (1 - FF^*)x,\ x\rangle = 1 - \|F^*x\|^2 \le 1 - |\langle Fx,\ x\rangle|^2.$$

So if $\{x_n\ :\ n > 0\}$ is an orthonormal basis for $\mathcal{H} \ominus V_1\mathcal{H}$ and $\sum_{n=1}^{\infty} 1 - |\langle Fx_n,\ x_n\rangle|^2 < \infty$, then both $1 - F^*F$ and $1 - FF^*$ are trace class, and their traces are dominated by $\sum_{n=1}^{\infty} 1 - |\langle Fx_n,\ x_n\rangle|^2$. These observations will be used in Example 1.

A result of Calderon (cf. Lemma 7.1 in [Ho]) says that if $A,\ B$ are bounded linear operators such that $(1-AB)^N$ and $(1-BA)^N$ are of trace class for some natural number N, then B is Fredholm and $ind(B) = tr(1-AB)^N - tr(1-BA)^N$. Using this fact in the case $A = F^*,\ B = F$ and $N = 1$, we establish the following:

Corollary 1.5. *If $[V_1^*,\ V_2]$ is Hilbert-Schmidt and $[V_1^*,\ V_1][V_2^*,\ V_2]$ is trace class, then F is Fredholm and*

$$tr[F^*,\ F] = -ind(F).$$

Theorem 1.3, the remarks following Proposition 1.4 and Corollary 1.5 are combined to produce our main result.

Theorem 1.6. *If $(V_1,\ V_2)$ is a commuting isometric pair with $[V_1^*,\ V_2]$ and $[V_1^*,\ V_1][V_2^*,\ V_2]$ both Hilbert-Schmidt, then*

$$tr[V_1^*, V_1, V_2^*, V_2] = -2ind(V_1,\ V_2). \tag{**}$$

As seen in Section 0, the anti-symmetric sum and the Koszul complex are both classical objects associated with operator tuples. So Theorem 1.6, which manifests a quantitative connection between the two objects, is a nice and meaningful two variable analogue of (*). The conditions in Theorem 1.6 are much weaker than requiring that V_1 and V_2 be essentially normal. Many isometric pairs satisfy these conditions. In Section 2, we will see two well-known isometric pairs which have this property.

One observes that (**) makes sense as long as $[V_1^*, V_1, V_2^*, V_2]$ is trace class (which is weaker than the conditions in Theorem 1.6!) and the Koszul complex for $(V_1,\ V_2)$ is Fredholm, so a positive answer to the following question will establish (**) in its best generality.

Question 1. Is the Fredholmness of $(V_1,\ V_2)$ equivalent to $[V_1^*, V_1, V_2^*, V_2]$ being trace class?

As we have seen, the fringe operator works very efficiently for isometric pairs. Similar ideas can be used to reduce a commuting n-tuple of isometries to an $n-1$-tuple of commuting contractions. But so far there is no good method to reduce an $(n-1)$-tuple of commuting contractions further. So we probably need a different technique to settle the following:

Question 2. Is there an analogous trace formula for n-tuples of commuting isometries?

2. Examples

Compared to the pair $(V_1,\ V_2)$, the fringe operator F is much easier to deal with. In this section we give two examples where this advantage will be exhibited.

Example 1. The paper [BCL] by Berger, Coburn and Lebow studied the C^*-algebra generated by a pair of commuting isometries (denoted by $C^*(V_1, V_2)$). A key element the authors used to classify $C^*(V_1, V_2)$ is an invariant κ. If ϕ and ψ are two inner functions in $H^2(D)$, then the pair of Toeplitz operators $(T_\phi,\ T_\psi)$ is a pair of commuting isometries, and [BCL] showed that if $[T_\phi^*,\ T_\psi]$ is compact, then $\kappa = 0$. It was discovered in [Ya1] that this κ for $C^*(V_1, V_2))$ is actually the Fredholm index of $(V_1,\ V_2)$. So if $[T_\phi^*,\ T_\psi]$ is compact, then $(T_\phi,\ T_\psi)$ is Fredholm with $ind(T_\phi, T_\psi) = 0$. By Theorem 1.6, this implies that

$$tr[T_\psi^*,\ T_\phi][T_\phi^*,\ T_\psi] = tr[T_\phi^*,\ T_\phi][T_\psi^*,\ T_\psi]$$

when both are finite. This fact was not known in [BCL].

As an example, [BCL] showed that if ϕ and ψ are Blaschke products with distinct zeros $\{a_n\}$ and $\{b_m\}$ respectively, then the condition $\sum_{n=1}^\infty (1 - |\psi(a_n)|^2) + \sum_{m=1}^\infty (1 - |\phi(b_{nm})|^2) < \infty$ implies that $[T_\psi^*,\ T_\phi][T_\phi^*,\ T_\psi]$ and $[T_\phi^*,\ T_\phi][T_\psi^*,\ T_\psi]$ are trace class with

$$tr[T_\psi^*,\ T_\phi][T_\phi^*,\ T_\psi] + tr[T_\phi^*,\ T_\phi][T_\psi^*,\ T_\psi] \leq \sum_{n=1}^\infty (1 - |\psi(a_n)|^2) + \sum_{m=1}^\infty (1 - |\phi(b_m)|^2).$$

In this first example, we use the fringe operator and the remarks following the proof of Proposition 1.4 to show that the condition ψ being a Blaschke product can be dropped.

We still assume that ϕ is a Blaschke product with distinct zeros $\{a_n\}$ and write $b_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$, $k_\alpha(z) = \frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}z}$. If we let $f_1(z) = k_{a_1}(z)$ and $f_{n+1}(z) = \prod_{i=1}^n \frac{-\overline{a_i}}{|a_i|} b_{\alpha_i}(z) k_{a_{n+1}}(z)$, for $n = 1,\ 2,\ 3,\ ...$, then one checks that $\{f_n :\ n > 0\}$ is an orthonormal basis for $H^2(D) \ominus \phi H^2(D)$ (cf. [BCL]). Also, one computes that $\langle Ff_1,\ f_1\rangle = \psi(a_1)$, and for $n > 0$,

$$\begin{aligned}
\langle Ff_{n+1},\ f_{n+1}\rangle &= \langle \psi f_{n+1},\ f_{n+1}\rangle \\
&= \langle \psi \prod_{i=1}^n \frac{-\overline{a_i}}{|a_i|} b_{\alpha_i} k_{a_{n+1}},\ \prod_{i=1}^n \frac{-\overline{a_i}}{|a_i|} b_{\alpha_i} k_{a_{n+1}}\rangle \\
&= \langle \psi k_{a_{n+1}},\ k_{a_{n+1}}\rangle = \psi(a_{n+1}).
\end{aligned}$$

These computations together with Theorem 1.6 produce the following:

Theorem 2.1. *Suppose ϕ and ψ are two inner functions in $H^2(D)$ and ϕ is a Blaschke product with distinct zeros $\{a_n \in D :\ n > 0\}$. If $\sum_{n=1}^\infty 1 - |\psi(a_n)|^2 < \infty$, then $[T_\phi^*,\ T_\psi]$ is Hilbert-Schmidt and $[T_\phi^*,\ T_\phi][T_\psi^*,\ T_\psi]$ is trace class with*

$$tr[T_\psi^*,\ T_\phi][T_\phi^*,\ T_\psi] = tr[T_\phi^*,\ T_\phi][T_\psi^*,\ T_\psi] \leq \sum_{n=1}^\infty 1 - |\psi(a_n)|^2.$$

Note that $\sum_{n=1}^\infty 1 - |\psi(a_n)|^2 < \infty$ is equivalent to saying that $\{\psi(a_n)\}$ is a zero set for $H^2(D)$.

Example 2. We now consider a two variable example. On the Hardy space over the bidisk $H^2(D^2)$, multiplications by z and w (denoted by T_z and T_w, respectively) are a pair of commuting isometries. It is an easy computation to check that $[T_z^*,\ T_w] = 0$ and $[T_z^*, T_z][T_w^*, T_w]$ is the rank 1 projection onto the constants, from which it follows that $tr[T_z^*, T_z, T_w^*, T_w] = 2$. On the other hand, $H^2(D^2) \ominus zH^2(D^2)$ is the Hardy space $H^2(D)$ in variable w and the fringe operator F is just multiplication by w. Therefore, $ind(T_z, T_w) = ind(F) = -1$ by Theorem 1.3, and hence

$$tr[T_z^*, T_z, T_w^*, T_w] = -2ind(T_z, T_w) = 2.$$

Restrictions of T_z and T_w to their common invariant subspaces (which we denote by R_z and R_w, respectively) are much more complex isometric pairs, for example $[R_z^*,\ R_w] = 0$ occurs only for common invariant subspaces of the form $\psi H^2(D^2)$ where $\psi(z, w)$ is an inner function (cf. [GM]). It is very easy to give a common invariant subspace which is not of this type, for example the closure of $\{(z+\phi(w))f : f \in H^2(D^2)\}$ in $H^2(D^2)$ (which we denote by $[z+\phi]$), where $\phi(w)$ is a non-constant inner function. On $[z+\phi]$, depending on what ϕ is, the trace $tr[R_w^*,\ R_z][R_z^*,\ R_w]$ or $tr[R_z^*,\ R_z][R_w^*,\ R_w]$ can be very difficult to compute. However, $ind(R_z,\ R_w)$ can still be efficiently computed through the fringe operator.

For simplicity we let $M = [z+\phi]$. The associated fringe operator $F :\ M \ominus zM \longrightarrow M \ominus zM$ is expressed by

$$Ff = p_\perp wf,$$

where $p_\perp$ is the projection from M onto $M \ominus zM$. A very useful tool in this setting is the evaluation operator $L(0)$ defined by

$$L(0)f := f(0,w), \quad f \in H^2(D^2).$$

One observes that for every $h \in M$, $h - p_\perp h$ is in zM and therefore $L(0)(h - p_\perp h) = 0$, so it follows that for every $g \in M \ominus zM$

$$L(0)Fg = L(0)wg - L(0)(wg - p_\perp wg) = wL(0)g,$$

or in other words, $L(0)|_{M\ominus zM}$ intertwines F with the multiplication by w on $L(0)(M \ominus zM)$. Since $L(0)(z+\phi(w)) = \phi(w)$,

$$L(0)(M \ominus zM) = L(0)(M) = \phi H^2(D),$$

which implies that $L(0)|_{M\ominus zM}$ has closed range, and more importantly, multiplication by w on $L(0)(M \ominus zM)$ is equivalent to the unilateral shift of multiplicity 1. So if $L(0)|_{M\ominus zM}$ has trivial kernel, it will furnish a similarity between F and the unilateral shift, from which it will follow that $ind(R_z, R_w) = ind(F) = -1$.

We now show that $L(0)|_{M\ominus zM}$ does have trivial kernel. In fact, since every $h \in M \ominus zM$ is of the form

$$h(z,w) = (z+\phi(w))f(z,w)$$

for some f holomorphic on D^2 (note here f may not be in $H^2(D^2)$!),

$$L(0)h = \phi(w)f(0,w).$$

If $L(0)h = 0$, then $f(0,w) = 0$, and we can write $f(z,w) = z\frac{f}{z}$. Since $\|(z+\phi)\frac{f}{z}\| = \|(z+\phi)f\| < \infty$, $(z+\phi)\frac{f}{z} \in M$ and hence $h \in zM$ which concludes that $h = 0$.

The fact that $L(0)|_{M\ominus zM}$ has trivial kernel and closed range also implies that $[R_z^*,\ R_z,\ R_w^*,\ R_w]$ is trace class. But we refer the reader to [Ya3] for details. Summing up all these arguments, we see that for $M = [z+\phi]$, $tr[R_z^*,\ R_z,\ R_w^*,\ R_w] = -2ind(R_z,\ R_w) = 2$.

Since what really matters is the condition that $L(0)|_{M\ominus zM}$ has trivial kernel and closed range, the arguments in this example actually produce the following:

Corollary 2.2. *If $M \in H^2(D^2)$ is invariant for multiplications by z and w, and $L(0)|_{M\ominus zM}$ has trivial kernel and closed range, then $tr[R_z^*,\ R_z,\ R_w^*,\ R_w] = 2$.*

The fact that $tr[R_z^*,\ R_z,\ R_w^*,\ R_w]$ is equal to 2 is not accidental. In [Ya4] we will see a deep connection between this fact and boundary behaviors of the reproducing kernels for common invariant subspaces.

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