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ON PRO-UNIPOTENT GROUPS SATISFYING THE GOLOD–SHAFAREVICH CONDITION

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Abstract. We prove that a pro-unipotent group satisfying the Golod–Shafarevich condition contains a free non-abelian pro-unipotent group. Together with the result of A. Magid this implies that such a group is not linear.

1. Introduction

In [4] Golod and Shafarevich found a sufficient condition for a (pro-p) group represented by generators and relations to be infinite. Recently, Zelmanov [12] confirmed an earlier conjecture by Wilson [11] by showing that a pro-p group satisfying the Golod–Shafarevich condition contains a free non-abelian pro-p group.

In the early 80's Lubotzky and Magid (see [6, 5, 7]) developed a theory of pro-unipotent groups presented by generators and relators. They also extended the Golod–Shafarevich result to pro-unipotent groups.

In this article we prove that if G is a pro-unipotent group satisfying the Golod–Shafarevich condition, then it contains a free pro-unipotent group (Theorem 5.3).

The proof in the pro-p case cannot be translated directly to the pro-unipotent case because it relies on the fact that a free pro-p group contains a countable dense subset, which is not true in the pro-unipotent case. To avoid this problem we need some facts about ideals in free Lie algebras.

This result can be combined with the result of Magid [7], who showed that a free pro-unipotent group is not linear. Together, these results imply that a pro-unipotent group satisfying the Golod–Shafarevich condition is not linear.

A large class of pro-unipotent groups satisfying the Golod–Shafarevich condition can be constructed from fundamental groups of 3-manifolds (see [2]).

The outline of this paper is as follows: §2 gives the definition and basic properties of pro-unipotent and free pro-unipotent groups; §3 presents the Golod–Shafarevich condition for pro-unipotent groups; §4 is supplementary and states some facts about free Lie algebras which are used in the proofs that follow; §5 gives the proof of the main result.

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2. Pro-unipotent groups

Let G be a pro-unipotent group, which is an inverse limit of a series G_i of unipotent groups. A Lie algebra $\mathrm{Lie}(G)$ of G is defined as a (projective) limit of Lie algebras corresponding to the groups G_i. There exists a map $\log : G \to \mathrm{Lie}(G)$ and its inverse $\exp : \mathrm{Lie}(G) \to G$, which satisfies the usual properties of the logarithm map in the case of unipotent Lie groups; for example, it is an isomorphism between pro-affine varieties. As in the case of nilpotent groups, it is convenient instead of working with the group to pass to (the completion of) the universal enveloping algebra of $\mathrm{Lie}(G)$. This algebra is denoted by $k\langle\langle G\rangle\rangle$, and is called a group algebra of G. In [6] it is proved that there exists a group embedding $\rho : G \hookrightarrow k\langle\langle G\rangle\rangle^*$, such that the image of G generates $k\langle\langle G\rangle\rangle$ as a pro-nilpotent algebra. Moreover, if the group G is generated by the elements g_i, then its group algebra $k\langle\langle G\rangle\rangle$ is generated by $s_i = \rho(g_i) - 1$.

Let $G = U(X)$ be the free pro-unipotent group (the free object in the category of the pro-unipotent groups whose existence is proved in [5, 6]) on the finite set $X = \{g_1, \dots, g_n\}$. Its group algebra, $k\langle\langle U(X)\rangle\rangle$, is isomorphic to $k\langle\langle s_1, \dots, s_n\rangle\rangle$, the pro-nilpotent completion of the free associative algebra generated by s_i. Also, the group $U(X)$ can be considered as a subgroup of the multiplicative group of $k\langle\langle s_1, \dots, s_n\rangle\rangle$.

For a more detailed introduction to pro-unipotent groups the reader is referred to [5, 6, 7].

3. The Golod–Shafarevich condition

Let $\mathcal{G}$ be the category of pro-unipotent groups over a field k of characteristic zero. Let $F_{\mathcal{G}}(n)$ denote the free (in $\mathcal{G}$) group on n generators. Let G be a finitely generated pro-unipotent group.

The commutator subgroups of G form a filtration

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$$

defined by $G_1 = G$ and $G_{i+1} = \overline{[G, G_i]}$. Here $\overline{[A, B]}$ is the closure of the subgroup generated by the commutators $[a, b] = aba^{-1}b{-}1$ for $a \in A$ and $b \in B$. These subgroups form a basis of the pro-unipotent topology on G and the group G is complete in this topology.

This filtration satisfies the condition $\bigcap G_i = (1)$, since G is a pro-unipotent group. Hence, every non-trivial element $g \in G$ falls into some gap $g \in G_i \backslash G_{i+1}$ for some unique i. We will call i the degree of the element g and denote it by $\deg g$. The Hilbert series of G is defined using this filtration by $H_G(t) = \sum \dim(G_i/G_{i+1}) \cdot t^i$.

Consider a group $G = \langle x_1, \dots, x_n | r = 1,\ r \in R_0\rangle$ in the category $\mathcal{G}$ presented by generators x_i and relators $R_0 \subset F_{\mathcal{G}}$, i.e., $G = F_{\mathcal{G}}/N$, where N is the minimal closed normal subgroup of $F_{\mathcal{G}}$ which contains R_0.

Without loss of generality, we may assume that $R_0 \subset (F_{\mathcal{G}})_2$, because relations in $(F_{\mathcal{G}})_1 \backslash (F_{\mathcal{G}})_2$ are equivalent to an elimination of a generator. Define the Hilbert series of R_0 by $H_{R_0}(t) = \sum r_i t^i$, where r_i is the number of elements in R_0 which have degree i.

Lubotzky and Magid [6] extended the Golod and Shafarevich result [4] (see also [9, 11]) from the case of discrete groups and associative algebras to unipotent groups. They proved that

$$\frac{H_G(t)}{1-t}(1-nt+H_{R_0}(t)) \geq \frac{1}{1-t},$$

where the inequality of formal series is termwise.

If there exists a real number $0 < t_0 < 1$ such that $H_{R_0}(t_0)$ converges and

$$1 - nt_0 + H_{R_0}(t_0) < 0,$$

we will say that the presentation for the group G satisfies the Golod–Shafarevich condition. In this case $H_G(t_0)$ cannot converge; in particular, the group G is infinite dimensional.

Remark 3.1. If the number of elements in R_0 is less than $n^2/4$, the presentation satisfies the Golod–Shafarevich condition because $1 - nt_0 + H_{R_0}(t_0) \leq 1 - nt_0 + n^2t_0/4$ for all $0 < t_0 < 1$, and the last expression is zero for some t_0.

For example, the pro-unipotent completion of the fundamental group of a 3-dimensional manifold satisfies the Golod–Shafarevich condition (except in few cases) because they are balanced, i.e., they have presentation with n generators and n relations (see [2]).

Remark 3.2. The Golod–Shafarevich condition was originally defined for associative algebras. A presentation of a pro-nilpotent algebra $A = \langle s_i | r = 0,\ r \in R\rangle$ is said to satisfy the Golod–Shafarevich condition if there exists $0 < t_0 < 1$ such that $H_R(t_0)$ converges and $1 - nt_0 + H_R(t_0) < 0$.

Remark 3.3. Having the presentation of a pro-unipotent group G, we can obtain a presentation of its (pro-nilpotent) group algebra $k\langle\langle G\rangle\rangle$ in the following way: for each $r_0 \in R_0 \subset F_{\mathcal{G}}(X)$ we construct $r \in k\langle\langle T\rangle\rangle$ by $r = r_{0|g_i \to 1+s_i} - 1$, i.e., we substitute every generator g_i with $1+s_i$ and expand the result as a series in the non-commutative variables s_i. Thus, we have that $\deg r = \deg r_0$. Let R be the set of all r for $r_0 \in R_0$. The group algebra $k\langle\langle G\rangle\rangle$ has a presentation $k\langle\langle s_i | r = 0,\ r \in R\rangle\rangle$ (see [6]). Therefore, the group algebra (with that presentation) satisfies the Golod–Shafarevich condition if and only if the group satisfies it. This result allows us to work mainly with the group algebra $k\langle\langle G\rangle\rangle$ for the remainder of the article.

The following lemma is a generalization of Golod's result (see [3]). In the case of a countable field, it was proven by Zelmanov in [12].

Lemma 3.4. *For any $\epsilon > 0$ and any $0 < t_0 < 1$, there exists a subset J of $K\langle s_1, \dots, s_m\rangle$ and an increasing function $i \to h(i)$ such that:*

1. *$H_J(t_0) \leq \epsilon$,*
2. *every i elements from the algebra $k\langle s_1, \dots, s_m \mid v = 0,\ v \in J\rangle$ of degree higher than $h(i)$ generate a nilpotent subalgebra.*

Proof. A proof in the case of a countable field can be found in [12]. That proof relies on the fact that the free algebra $k\langle X\rangle$ is a countable set. The proof that follows uses only that this algebra has a countable dimension, i.e., any element can be expressed as a finite linear combination of countably many basis elements.

Let $h : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $rt_0^{h(r)} < 1/4$ for any r. Let us fix r. Then, for any l, the expression $(N + rl)^{rl}(rt_0^{h(r)})^N$ converges to 0 as N tends to ∞. Therefore, for any integer $s \geq 1$, there exists a number $N_{r,l,s}$ such that

$$(N_{r,l,s} + rl)^{rl}(rt_0^{h(r)})^{N_{r,l,s}} \leq 2^{-(r+l+s)}\epsilon.$$

Let W_n be the set of all words on letters $s_1, \ldots, s_m$ of length greater than n. Also, denote by $(W_{h(r)})^{(r,l)}$ the set of all rl-tuples of elements in W_n of the form $\{w_{i,j}\}_{i=1,r}^{j=1,l}$. The set $(W_{h(r)})^{(r,l)}$ is countable for any r and l. Let $\gamma_{r,l}$ be a bijection from this set to N. Given an element $\alpha = \{w_{i,j}\}$, we can define the formal sum $b(\alpha)_p$ as $b(\alpha)_p = \sum_j t_{p,j} w_{p,j}$, where $\{t_{p,j}\}$ are formal commuting variables.

Let $\alpha = \gamma_{r,l}^{-1}(s) \in (W_{h(r)})^{(r,l)}$ be an element such that $\gamma_{r,l}(\alpha) = s$. Let u be a word of length $N_{r,l,s}$ composed of letters $y_1, \ldots, y_r$. Then, for any such word, denote by $J_{(\alpha,u)}$ the set of the coefficients of the monomial in $t_{i,j}$ in the expansion of

$$u_{|y_i \to b(\alpha)_i},$$

which is just a product of length $N_{r,l,s}$ on $b(\alpha)_i$-es. The set $J_{\alpha,u}$ contains less than $(N_{r,l,s} + rl)^{rl}$ elements (since this is greater than the number of monomials of degree $N_{r,l,s}$ of rl commuting variables), and each element is of degree greater than $N_{r,l,s}h(r)$. Let

$$J_\alpha = \bigcup_u J_{(\alpha,u)}, \quad J_{r,l} = \bigcup_{\alpha \in (W_{h(r)})^{(l,r)}} J_\alpha, \quad J_r = \bigcup_l J_{r,l}, \quad J = \bigcup_r J_r.$$

Then,

$$\begin{aligned} H_J(t_0) &\leq \sum_{r,l,\alpha,u} H_{J_{(\alpha,u)}}(t_0) < \sum_{r,l,s} r^{N_{r,l,s}} (N_{r,l,s} + rl)^{rl} t_0^{N_{r,l,s}h(r)} \\ &\leq \sum_{r,l,s} 2^{-r-l-s}\epsilon = (\sum_r 2^{-r})^3 \epsilon = \epsilon. \end{aligned}$$

It remains to be shown that the set J satisfies the conditions of the lemma. Let $y_1, \ldots, y_r$ be r elements in $k\langle s_1, \ldots, s_m\rangle$, each of degree greater than $h(r)$. Then there exists l such that each element can be expressed as a linear combination of at most l monomials, i.e.,

$$y_i = \sum_{j=1}^{l} t_{i,j} w_{i,j},$$

where $w_{i,j}$ are monomials and $t_{i,j}$ are elements of the field k. Let $\alpha = \{w_{i,j}\}$ be the element in $(W_{h(r)})^{l,r}$ corresponding to the monomials $w_{i,j}$. Then y_i can be obtained from $b(\alpha)_i$ by substituting $t_{i,j}$-es with suitable elements from the field k. By construction, any product of length $N_{r,l,\gamma_{k,l}(\alpha)}$ on $b(\alpha)_i$-es is in the ideal generated by the set J, i.e., any product of this length on y_i-es lies in $\langle J\rangle$. Therefore, every r elements of degree greater than $h(r)$ generate a nilpotent subalgebra of $k\langle s_1, \ldots, s_m \mid v = 0,\ v \in J\rangle$. □

4. Some facts about free Lie algebras

Let $\mathcal{L}[X]$ be the free Lie algebra generated by the set X. Define the ideals I_n by $I_0 = \mathcal{L}$ and $I_{n+1} = [\mathcal{L}, I_n]$. These ideals form a basis of the pro-nilpotent topology on $\mathcal{L}[X]$ and its completion in this topology is denoted by $\mathcal{L}[[X]]$. Each nonzero element $f \in \mathcal{L}[[X]]$ lies in $I_n \backslash I_{n+1}$ for some n, which is called the degree of f and denoted by $\deg f$. The homogeneous component of f of degree n is called the leading term.

The algebra $\mathcal{L}[[X]]$ is a non-compact topological algebra and its topology is equivalent to the one defined by the norm $||f|| = \exp(-\deg f)$.

Let $d(n)$ be the dimension of the space of homogeneous elements of degree n in the free Lie algebra $\mathcal{L}[x, y]$. There is an explicit formula for $d(n)$ which can be found in [8].

Lemma 4.1. *There exists a nonzero element l_n in the free Lie algebra*

$$\mathcal{L}[x_1, \dots, x_n, y]$$

which is poly-linear and skew symmetric in the x_i-es.

Proof. Let l_n be the element

$$l_n(x_1, \dots, x_n, y) = \sum_{\sigma \in S_n} (-1)^\sigma [y, x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}],$$

where $[z_1, z_2, \dots, z_k]$ is the left normed commutator of the elements $z_1, \dots, z_k$ defined inductively by $[z_1, z_2, \dots, z_k] = [[z_1, z_2, \dots, z_{k-1}], z_k]$.

By construction, the element l_n is poly-linear and skew symmetric in x_i. This element is not zero because in the lexicographic order such that $y > x_1 > x_2 > \dots > x_n$, its leading term (in the free associative algebra) is $yx_1x_2 \dots x_n$. □

Remark 4.2. In the decompositon of $\mathcal{L}[x_i, y]$ under the natural action of the group GL_{n+1}, there is a unique submodule which corresponds to the partition $(2, 1^{n-1})$. This shows that the element l_n is unique up to a scalar multiplication.

Lemma 4.3. *For an arbitrary integer n, there exists an element $f_n \in \mathcal{L}[x, y]$ with the following property: Let $x, y \in L$ be elements of a Lie algebra L such that there exists a homogeneous element $g \in \mathcal{L}[x, y]$ of degree n for which the condition $g(x, y) = 0$ in L holds. Then $f_n(x, y) = 0$ in L.*

Proof. If $n = 1$ or $n = 2$ we may take $f_1 = f_2 = [x, y]$, which is the only homogeneous Lie polynomial of degree 2 and, therefore, has the property in the lemma.

Let $n > 2$. Let the elements $g_1, \dots, g_{d(n)}$ form a basis of the space of homogeneous elements of degree n in the free Lie algebra $\mathcal{L}[x, y]$.

Using degree arguments one can see that none of the elements x, g_i can be expressed as a Lie polynomial of the others. This implies that elements x, g_i freely generate a Lie subalgebra. Hence, there is no non-trivial Lie polynomial of x, g_i which is zero in $\mathcal{L}[x, y]$ (see [8]). Therefore, the element

$$f_n = l_{d(n)}(g_1, \dots, g_{d(n)}, x)$$

is nonzero. (Here, $l_{d(n)}$ is the element from Lemma 4.1.)

If elements x and y in the Lie algebra L satisfy the condition of this lemma, then the elements $g_i(x, y)$ are linearly dependent in L. Therefore, any skew symmetric expression of them vanishes, which implies that $f_n(x, y) = 0$ in L. □

Remark 4.4. The statement of the lemma is equivalent to the following: Let $h \in \mathcal{L}[x, y]$ be a homogeneous element of degree n. Then f_n lies in the ideal generated by h.

Lemma 4.5. *Let J be a homogeneous ideal of $\mathcal{L}[x, y]$. Then J contains the ideal $\langle f_n \rangle$ for some n.*

5. On pro-unipotent groups satisfying the Golod–Shafarevich condition

Let $G = \langle g_1, \ldots, g_m | R_0 = 0\rangle$ be a pro-unipotent group satisfying the Golod–Shafarevich condition. Therefore, there exists a real number t_0 such that $H_{R_0}(t_0) < 0$. Let p be an integer which satisfies $H_{R_0}(t_0) + 2t_0^p < 0$ (such an integer exists since $t_0 < 1$).

Let L be the Lie algebra of G and let A be the group algebra

$$k\langle\langle G\rangle\rangle = k\langle\langle s_1, \ldots, s_m | r = 0, r \in R\rangle\rangle.$$

Thus, $L \hookrightarrow A^-$ and L is the Lie subalgebra in A generated by the generators s_i of A (see [6]).

Assume that G does not contain a free pro-unipotent group. Then L does not contain a free (pro-nilpotent) Lie algebra $\mathcal{L}[[x,y]]$, i.e., for any $l_1, l_2 \in L$, there exists an element $m_{(l_1,l_2)} \in \mathcal{L}[[x,y]]$ such that $m_{(l_1,l_2)}(l_1, l_2) = 0$.

For any homogeneous Lie polynomial h, let $(L \times L)_h$ denote the set of all pairs $(l_1, l_2) \in L \times L$ such that there exists an $h' \in \mathcal{L}[[x,y]]$ with $\deg h' > \deg h$ and $h(l_1, l_2) = h'(l_1, l_2)$ in L. This set is closed in the pro-nilpotent topology on L. Since every element in $\mathcal{L}[[x,y]]$ has a leading term, we have that

$$(L \times L) = \bigcup_h (L \times L)_h.$$

Let $\bar{I}_i$ be the ideal in L which is the image of the ideal $I_i \subset \mathcal{L}[[s_1, \ldots, s_m]]$ for any integer i.

The first step in the proof is to show that the set $(L \times L)_h$ cannot contain an open subset of $(\bar{I}_p \times \bar{I}_p) \subset (L \times L)$. The proof is by contradiction.

Lemma 5.1. *For any homogeneous element h of degree N in $\mathcal{L}[x,y]$, the set $(L \times L)_h$ does not contain an open subset of $(\bar{I}_p \times \bar{I}_p)$.*

Proof. Assume that for some h we have the inclusion

$$(x_0 + \bar{I}_q) \times (y_0 + \bar{I}_q) \subset (L \times L)_h,$$

where $x_0, y_0 \in \bar{I}_p$, and $q \geq p$ (since every open subset of $(\bar{I}_p \times \bar{I}_p)$ contains a subset of this type). Then the factor algebra $A' = A/\langle x_0, y_0\rangle$ also satisfies the Golod–Shafarevich condition because the inequality $H_{R_0 \cap \{x_0, y_0\}}(t) \leq H_{R_0} + 2t^p$ holds for all $t < 1$, and the last expression is less than 0 for $t = t_0$. Let L' denote the Lie subalgebra of A' generated by the images of s_i (the generators of A). This subalgebra is the image of L under projection from A to A'.

The assumption on $(L \times L)_h$ implies that if x, y are elements of $L' \cap A'^{(q)}$, then there exists some $h' \in \mathcal{L}$ such that $\deg h' > N$ and such that $h(x,y) = h'(x,y)$.

Let f be the full linearization of some (non-trivial) poly-homogeneous component of h. It is known (see [1]) that $f(z_1, \ldots, z_N)$ can be expressed as a linear combination of values of h in which x and y have been substituted by some linear combinations of the z_i-es. Therefore, for any $z_i \in L' \cap A'^{(q)}$, there exists a $f' \in \mathcal{L}[[z_i]]$ of degree greater than N such that $f(z_1, \ldots, z_n) = f'(z_1, \ldots, z_n)$.

Let A'' be the factor algebra of A' obtained by adding relations obtained from Lemma 3.4. Then the algebra A'' satisfies the Golod–Shafarevich condition together with the following conditions:

(1) The algebra $A'' = \langle s_1, \ldots, s_m\rangle$ is a finitely generated associative algebra.

(2) For all i, there exists an $h(i)$ such that every i elements from ${A''}^{h(i)}$ generate a nilpotent subalgebra.
(3) Let L'' be the Lie subalgebra of A'' generated by s_i. If we substitute z_i with $l_i \in L'' \cap {A''}^{(q)}$ in f, we have $f(l_1, \ldots, l_N) = f'(l_1, \ldots, l_N)$ for some f' in $\mathcal{L}[[z_i]]$ of degree $> N$.

In [12] it is proved that if an algebra satisfies the above conditions (1)–(3), then it is finite dimensional. Using this result we obtain a contradiction since the algebra A'' was constructed to satisfy the Golod–Shafarevich condition. □

This Lemma shows that for each h the set $(L \times L)_h \cap (\bar{I}_p \times \bar{I}_p)$ is a Baire set of first category because it is a closed nowhere dense set. Next, we want to apply Baire's theorem to the family of sets $\{(L \times L)_h \cap (\bar{I}_p \times \bar{I}_p)\}$. But this family is not countable, because the set of all homogeneous polynomials is not countable. Therefore, we need to choose a countable sub-family whose union is still the whole set $(L \times L)$.

Lemma 5.2. *Let h be a homogeneous polynomial of degree N. Then*

$$(L \times L)_h \subset (L \times L)_{f_N},$$

where f_N is the element from Lemma 4.3.

Proof. From Remark 4.4 we know that f_N lies in the ideal of $\mathcal{L}[x, y]$ generated by h, i.e.,

$$f_N = \sum_{w = w_1 \ldots w_k} a_w h \operatorname{ad}(w_1) \operatorname{ad}(w_2) \ldots \operatorname{ad} w_k,$$

where the sum is over the words on letters x, y.

Let $(u, v) \in (L \times L)_h$. Then, there exists h' such that $h(u, v) = h'(u, v)$ and $\deg h' > N$. If we substitute this equality in the right side of the above expression for f_N, we obtain that $f_N(u, v) = f'(u, v)$, where $\deg f' > \deg f_N$. This shows that $(u, v) \in (L \times L)_{f_N}$. □

From the previous lemma together with assumption that G does not contain a free subgroup, we have that

$$L \times L = \bigcup_h (L \times L)_h = \bigcup_N (L \times L)_{f_N}.$$

But this is impossible by Baire's Category Theorem (see [10]), because $(\bar{I}_p \times \bar{I}_p)$ is a set of second category and by Lemma 5.1 the set $(\bar{I}_p \times \bar{I}_p) \cap (L \times L)_{f_N}$ is of first category for each N.

Thus, we have proven:

Theorem 5.3. *Let G be a pro-unipotent group which satisfies the Golod–Shafarevich condition. Then G contains a free pro-unipotent group.*

In fact we proved a slightly stronger result:

Corollary 5.4. *Let G be as above. Then a set of pairs (x, y) such that x, y does not generate a free pro-unipotent group inside G form a Baire set of second category.*

Using Magid's result ([7]) and Theorem 5.3 it is easy to obtain the following corollary.

Corollary 5.5. *Let G be a pro-unipotent group which satisfies the Golod–Shafarevich condition. Then G does not contain a linear (over any local field) subgroup of finite codimension.*

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