

SMALL COVERS OF THE DODECAHEDRON AND THE 120-CELL

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ABSTRACT. Let P be the right-angled hyperbolic dodecahedron or 120-cell, and let W be the group generated by reflections across codimension-one faces of P . We prove that if $\Gamma \subset W$ is a torsion free subgroup of minimal index, then the corresponding hyperbolic manifold \mathbb{H}^n/Γ is determined up to homeomorphism by Γ modulo symmetries of P .

1. INTRODUCTION

Let P be a right-angled bounded convex polytope in hyperbolic space \mathbb{H}^n , and let W be the group generated by reflections across codimension-one faces. For any torsion-free subgroup $\Gamma \subset W$ of finite index, the quotient \mathbb{H}^n/Γ is a closed hyperbolic manifold which is an orbifold cover of P . We call this manifold a *small cover of P* (as in [DJ]) if the index of Γ is minimal. Examples of small covers include the first closed hyperbolic 3-manifold to appear in the literature [L] as well as its generalizations [V].

Any symmetry of P induces an automorphism ϕ of W by permuting the generators, and if two small covers are of the form \mathbb{H}^n/Γ and $\mathbb{H}^n/\phi(\Gamma)$ they are isometric. The point of this note is to show that if P is regular, then the converse holds. In other words, if P is regular, then any small cover is uniquely determined up to isometry (and up to homeomorphism when $n \geq 3$) by Γ modulo the symmetry group of P . In fact, there are only two right-angled regular hyperbolic polytopes with dimension ≥ 3 , the dodecahedron and the 120-cell. We conclude the paper by showing that up to homeomorphism there are exactly 25 small covers of the dodecahedron, and that there is a unique small cover of the 120-cell with minimal complexity (in the sense of Section 4, below).

There are other hyperbolic manifolds based on the dodecahedron [SW] and the 120-cell [D] appearing in the literature, but these are obtained by identifying faces of a single copy of the polytope, and their constructions require that all dihedral angles be $2\pi/5$. Analogous constructions fail for right-angled realizations of these polytopes, and small covers provide a natural alternative.

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2. DEFINITIONS AND PRELIMINARY FACTS

Let P be an n -dimensional right-angled convex polytope in \mathbb{H}^n , and let \mathcal{F} denote the set of facets (i.e., codimension-one faces) of P . For each $F \in \mathcal{F}$, we let s_F denote the reflection across F , and we let W be the group generated by $\{s_F \mid F \in \mathcal{F}\}$. Defining relations for W are $(s_F)^2 = 1$ for all F and $s_F s_{F'} = s_{F'} s_F$ whenever $F \cap F' \neq \emptyset$. Following [DJ], we call an epimorphism $\lambda : W \rightarrow (\mathbb{Z}_2)^n$ a *characteristic function* if whenever F_1, \dots, F_n are facets that all meet at a vertex, the images $\lambda(s_{F_1}), \dots, \lambda(s_{F_n})$ form a \mathbb{Z}_2 -basis. The following proposition implies that any manifold which is an orbifold cover of P with 2^n sheets is of the form $\mathbb{H}^n / \ker(\lambda)$ for some characteristic function λ .

Proposition 2.1. *If Γ is a torsion free subgroup of W , then its index is $\geq 2^n$. If the index is equal to 2^n , then Γ is normal and is the kernel of a characteristic function $\lambda : W \rightarrow (\mathbb{Z}_2)^n$.*

Proof. We consider the reflection tiling of \mathbb{H}^n corresponding to P and W . If C is a codimension- k cell in this tiling, then there are exactly 2^k maximal cells containing C , and the stabilizer $\text{Stab}(C)$ is isomorphic to $(\mathbb{Z}_2)^k$. If Γ is torsion-free, then the natural map $\text{Stab}(C) \rightarrow W/\Gamma$ is injective; hence, the index of Γ is at least 2^n .

If the index is equal to 2^n , then we can fix a vertex v_0 of P and identify $\text{Stab}(v_0)$ with the set of cosets W/Γ . Let μ be the map

$$\mu : W \longrightarrow W/\Gamma = \text{Stab}(v_0) \cong (\mathbb{Z}_2)^n.$$

It is not *a priori* a homomorphism, but since the defining relations for W also hold for the images $\{\mu(s_F) \mid F \in \mathcal{F}\}$, there is an induced epimorphism $\lambda : W \rightarrow \text{Stab}(v_0)$ defined by $\lambda(s_F) = \mu(s_F)$, for all $F \in \mathcal{F}$. We will show that $\Gamma = \ker(\lambda)$. Let D be the union of the 2^n maximal cells containing v_0 , and let $N \subset W$ be the set

$$N = \{w \in W \mid P \cap w(P) \neq \emptyset\} = \bigcup_{C \subset P} \text{Stab}(C).$$

It is clear that D is a fundamental domain for Γ and can be shown that the induced facet-pairing of D is determined by the images $\mu(w)$ where $w \in N$. Moreover, the fact that μ and λ agree on $\text{Stab}(v_0)$ and on the generators of W implies that μ and λ must agree on N . Thus, Γ and $\ker(\lambda)$ define the same facet-pairing on D , so $\Gamma = \ker(\lambda)$. That λ is a characteristic function follows from the fact that $\text{Stab}(v) \rightarrow W/\Gamma$ must be a bijection for every v . □

Definition 2.2. Let $\lambda : W \rightarrow (\mathbb{Z}_2)^n$ be a characteristic function, and let $\Gamma_\lambda = \ker(\lambda)$. The *small cover* of P associated to λ is the closed hyperbolic manifold $M_\lambda = \mathbb{H}^n / \Gamma_\lambda$.

Small covers are functorial in the following sense.

Proposition 2.3. *Let M_λ be a small cover of P . For any face F of P , let $\mathbb{H}_F \subset \mathbb{H}^n$ be the hyperbolic subspace spanned by F and let M_F be the image of \mathbb{H}_F in M_λ (that is, $M_F = \mathbb{H}_F / (\Gamma_\lambda \cap \text{Stab}(\mathbb{H}_F))$). Then M_F is a totally geodesic submanifold which is itself a small cover of the face F .*

Proof. The stabilizer $\text{Stab}(F) \subset W$ fixes \mathbb{H}_F pointwise and is mapped isomorphically by λ onto a codimension- k subspace $V_F \subset (\mathbb{Z}_2)^n$ (where $k = \dim(F)$). Each facet of F can be expressed uniquely as the intersection $F' \cap F$ where F' is a facet of P that is orthogonal to F , and we let W_F denote the subgroup of W generated by

the corresponding reflections $s_{F'}$. The characteristic function λ induces a characteristic function $\bar{\lambda} : W_F \rightarrow (\mathbb{Z}_2)^n / V_F$ for the polytope F with kernel $\Gamma_\lambda \cap \text{Stab}(\mathbb{H}_F)$. It follows that $M_F = M_{\bar{\lambda}}$. \square

Let $A (= A(P))$ denote the symmetry group of P , and let $G = GL_n(\mathbb{Z}_2)$. We say that two small covers M_λ and M_μ are *equivalent* if $\lambda = g \circ \mu \circ a$ for some $a \in A$ and $g \in G$. Any small cover M_λ has a natural cell decomposition induced by the tiling of \mathbb{H}^n .

Proposition 2.4. *Two small covers M_λ and M_μ are equivalent if and only if they are isomorphic as cell complexes.*

Proof. It is clear that equivalent small covers are isomorphic as cell complexes. Conversely, suppose $\phi : M_\lambda \rightarrow M_\mu$ is an isomorphism of cell complexes. Then the lift $\tilde{\phi} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is an automorphism of the tiling of \mathbb{H}^n such that $\Gamma_\lambda = \tilde{\phi} \Gamma_\mu \tilde{\phi}^{-1}$. Since the automorphism group of the tiling is a semi-direct product of W and A , there exists an $a \in A$ such that $\Gamma_\lambda = w \Gamma_{\mu \circ a} w^{-1}$ for some element $w \in W$. It follows that $\Gamma_\lambda = \Gamma_{\mu \circ a}$ (they are both normal), hence $\lambda = g \circ \mu \circ a$ for some $g \in G$. \square

3. THE MAIN THEOREM

Let P be a right-angled hyperbolic polytope as above. P is *regular* if its symmetry group acts transitively on the simplices in its barycentric subdivision. Any such simplex must be a hyperbolic Coxeter simplex, and by examining the standard list of these simplices (Bourbaki [B], p. 133) one can show that P must be either a polygon with > 4 sides, the dodecahedron, or the (4-dimensional) 120-cell. An important property of these regular polytopes is the following:

Lemma 3.1. *Let P be a regular right-angled hyperbolic polytope, and let L be the length of an edge in P . Then any path in P that connects non-adjacent facets has length $\geq L$ with equality holding only if the connecting path is an edge of P .*

Proof. Let α be a connecting path with endpoints lying on the (disjoint) facets F and F' . Because P is regular, it has a ‘‘cubical’’ decomposition obtained by cutting each edge with the orthogonal hyperplane through its midpoint (see Figure 1). We define two open neighborhoods B and D of the facet F as follows. B is the

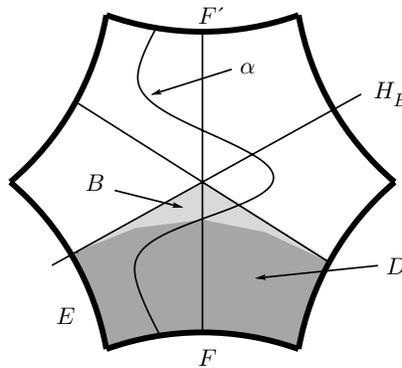


FIGURE 1.

union of the interiors of all cubes that intersect F , and D is the set of all points of P whose distance to F is less than $L/2$. We claim that $D \subset B$. To see this, it suffices to note that if E is an edge meeting F orthogonally at a vertex and H_E is the hyperplane bisecting E , then D lies entirely on one side of H_E (since the edge E , being a common perpendicular to H_E and F , realizes the shortest distance between the corresponding hyperplanes). Similarly, if B' and D' are the corresponding neighborhoods of F' , we have $D' \subset B'$. Since F and F' are not adjacent, $B \cap B' = \emptyset$ and, hence, $D \cap D' = \emptyset$. This means

$$l(\alpha) \geq l(\alpha \cap D) + l(\alpha \cap D') \geq L/2 + L/2 = L,$$

and since the edge E is the *unique* path of minimal length joining F to H_E , the equality $l(\alpha) = L$ is only possible if α is an edge of P . \square

Remark 3.2. To see that the lemma can fail without the regularity hypothesis, let P be any right-angled n -dimensional polytope and let $E \subset P$ be a minimal length edge. By gluing 2^{n-1} copies of P together around the edge E , one obtains a right-angled polytope Q with E now being an interior connecting path that is as short as any edge of Q .

We are now in a position to prove the main theorem. The argument is based on the fact that an isometry must preserve the set of minimal length closed geodesics. To simplify the exposition, we call a closed geodesic in a small cover an *edge loop* if it is of the form M_E (Proposition 2.3) for some edge $E \subset P$.

Theorem 3.3. *Let P be an n -dimensional right-angled regular hyperbolic polytope. Then two small covers of P are isometric if and only if they are equivalent.*

Proof. First we show that if M is any small cover of P , then any closed geodesic of minimal length must be an edge loop. Let α be a closed geodesic in M , and let F be a codimension-one cell in M that intersects α transversely. Lifting α to the universal cover \mathbb{H}^n , we obtain a geodesic segment $\tilde{\alpha}$ that connects two lifts \tilde{F} and \tilde{F}' of the cell F (see Figure 2). Let H and H' be the hyperplanes spanned by \tilde{F} and \tilde{F}' , respectively. Since H and H' are both mapped to the face F under the projection $\mathbb{H}^n \rightarrow \mathbb{H}^n/W = P$, they are hyperparallel, thus have a (unique) common perpendicular β . Moreover, β must pass through at least two copies of the tile P . It follows from Lemma 3.1 that the length of β is $\geq 2L$ with equality holding only if β is the lift of an edge. Since the geodesic $\tilde{\alpha}$ is at least as long as β with equality

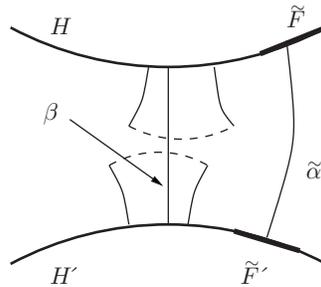


FIGURE 2.

holding only if $\tilde{\alpha} = \beta$, the closed geodesic α will have minimal length only if it is an edge loop.

Now suppose $\phi : M_\lambda \rightarrow M_\mu$ is an isometry. Since ϕ takes minimal length closed geodesics to minimal length closed geodesics, it must take edge loops to edge loops. The 0-cells in a small cover can be characterized as the points where edge loops intersect, thus ϕ must take 0-cells to 0-cells and, therefore, 1-cells to 1-cells. Since any cell of dimension ≥ 2 in a small cover can be characterized as the convex hull of its bounding 1-cells, the isometry ϕ must take cells to cells. Thus, by Proposition 2.4, M_λ and M_μ are equivalent. \square

Mostow rigidity gives the following:

Corollary 3.4. *If P is the dodecahedron or the 120-cell, then two small covers of P are homeomorphic if and only if they are equivalent.*

4. THE DODECAHEDRON AND 120-CELL

In this section, we describe an algorithm for enumerating equivalence classes of small covers for general P , and apply it to the dodecahedron and 120-cell. (The complexity of the full algorithm becomes unfeasible for the 120-cell, so instead we apply it to a restricted class of characteristic functions.)

The algorithm. Let P be n -dimensional, and let F_1, \dots, F_d be any ordering of the facets such that the first n facets F_1, \dots, F_n all meet at a vertex. Given any characteristic function $\lambda : W \rightarrow (\mathbb{Z}_2)^n$, we let $\lambda_i \in (\mathbb{Z}_2)^n$ be the image $\lambda(s_{F_i})$ of the fundamental reflection across the i th facet. By definition a d -tuple $\lambda = (\lambda_1, \dots, \lambda_d)$ determines a characteristic function if and only if the λ_i 's corresponding to facets meeting at any vertex form a basis for $(\mathbb{Z}_2)^n$. We call such a λ a *labeling of P* and each λ_i a *label*. We say a labeling is *normalized* if the first n labels form the standard basis for $(\mathbb{Z}_2)^n$, and we let $\Lambda(P)$ denote the set of normalized labelings of P .

The following algorithm uses a standard “branch-and-bound” method to determine $\Lambda(P)$. It sequentially assigns labels to the facets, checking that at each new facet F_i , the label λ_i is not a linear combination of previously assigned labels around a vertex of F_i . If no such λ_i exists, the algorithm backtracks, replacing the previous label λ_{i-1} with a new one and repeating the procedure. If the algorithm successfully assigns labels to all d facets, the complete labeling is added to $\Lambda(P)$ and the algorithm again backtracks, replacing the label λ_{d-1} with a new one and repeating the procedure. Since the algorithm is searching a finite tree, it will eventually terminate.

Algorithm 4.1.

Input: $FP =$ set of subsets $I \subset \{1, 2, \dots, d\}$ such that $\bigcap_{i \in I} F_i$ is a face of P .
 (FP is the set of all proper faces of P , with each face indexed by the facets that contain it.)

Output: $\Lambda =$ list of normalized labelings $(\lambda_1, \dots, \lambda_d)$ for P .

Initialization: Set the following:

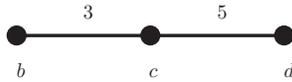
- $\lambda_1 \leftarrow (1, 0, \dots, 0), \quad \lambda_2 \leftarrow (0, 1, \dots, 0), \quad \dots \quad \lambda_n \leftarrow (0, 0, \dots, 1),$
- $\Lambda \leftarrow \emptyset,$
- $S \leftarrow$ list of nonzero elements of $(\mathbb{Z}_2)^n,$
- $i \leftarrow n + 1.$

Procedure:

- (1) If $i = n$ then STOP.
- (2) Set $S_i \leftarrow S$.
- (3) For all $I \in FP$ of the form $I = \{i_1, \dots, i_k\} \cup \{i\}$ with $1 \leq i_1 \leq \dots \leq i_k < i$, remove the vector $\lambda_{i_1} + \dots + \lambda_{i_k}$ from the list S_i .
- (4) If $S_i = \emptyset$ then set $i \leftarrow i - 1$ and go to (1).
- (5) Set $\lambda_i \leftarrow S_i[1]$ (where $S_i[1]$ denotes the first element of the list S_i).
- (6) Remove λ_i from the list S_i (maintaining the list order).
- (7) If $i = d$ then add the labeling $(\lambda_1, \dots, \lambda_d)$ to the list Λ and go to (4).
- (8) If $i < d$ then set $i \leftarrow i + 1$ and go to (1).

The group of symmetries $A(P)$ acts on $\Lambda(P)$, and the equivalence classes of small covers are in bijection with $A(P)$ -orbits. To determine these orbits, we choose a set of generators for $A(P)$ and form the graph whose vertices are elements of $\Lambda(P)$ and whose edges join any pair of vertices that differ by a generator of $A(P)$. The $A(P)$ -orbits then correspond to the connected components of this graph, so any standard graph algorithm can be applied to determine them.

The dodecahedron. Let P^3 be the regular dodecahedron. The symmetry group $A(P^3)$ is the Coxeter group with diagram \mathbf{H}_3 :



Using the algorithm, we obtain 2165 normalized labelings of P^3 . Forming the graph on this set $\Lambda(P^3)$ with respect to the standard generating set $\{b, c, d\}$ for $A(P^3)$, we obtain 25 connected components. Hence, by Corollary 3.4, the dodecahedron has precisely 25 small covers up to homeomorphism. Representative labelings for these small covers are given in Table 1. To keep the data concise, we denote a point $(\epsilon_1, \epsilon_2, \epsilon_3) \in (\mathbb{Z}_2)^3$ by the decimal equivalent $4 \cdot \epsilon_1 + 2 \cdot \epsilon_2 + \epsilon_3 \in \mathbb{Z}$. The ordering we use for the facets is shown in the following figure:

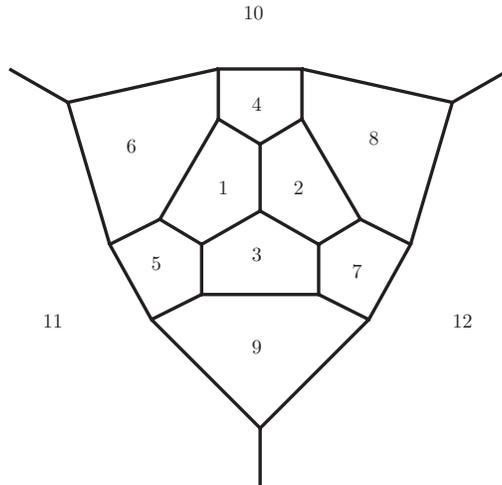


TABLE 1. Small covers of the dodecahedron

$\lambda = (\lambda_1, \dots, \lambda_{12})$	A_λ	$\lambda = (\lambda_1, \dots, \lambda_{12})$	A_λ
(1, 2, 4, 4, 2, 6, 1, 7, 7, 1, 3, 5)	1	(1, 2, 4, 4, 2, 6, 3, 5, 5, 3, 1, 7)	1
(1, 2, 4, 4, 2, 6, 3, 5, 5, 7, 3, 1)	1	(1, 2, 4, 4, 2, 6, 7, 1, 1, 3, 7, 5)	1
(1, 2, 4, 4, 2, 7, 1, 7, 7, 1, 3, 5)	1	(1, 2, 4, 4, 2, 7, 1, 7, 7, 1, 4, 2)	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3$
(1, 2, 4, 4, 2, 7, 3, 5, 5, 2, 6, 1)	1	(1, 2, 4, 4, 2, 7, 3, 5, 5, 6, 4, 7)	1
(1, 2, 4, 4, 2, 7, 3, 7, 5, 2, 6, 1)	1	(1, 2, 4, 4, 2, 7, 3, 7, 5, 5, 3, 1)	\mathbb{Z}_2
(1, 2, 4, 4, 2, 7, 3, 7, 5, 5, 6, 1)	1	(1, 2, 4, 4, 2, 7, 7, 1, 1, 2, 6, 5)	\mathbb{Z}_2
(1, 2, 4, 4, 2, 7, 7, 1, 5, 6, 3, 4)	1	(1, 2, 4, 4, 2, 7, 7, 3, 5, 5, 3, 1)	$\mathbb{Z}_2 \times \mathbb{Z}_2$
(1, 2, 4, 4, 2, 7, 7, 3, 5, 5, 6, 1)	1	(1, 2, 4, 4, 3, 6, 3, 5, 5, 7, 2, 1)	1
(1, 2, 4, 4, 3, 6, 3, 7, 5, 5, 2, 1)	\mathbb{Z}_2	(1, 2, 4, 4, 3, 6, 3, 7, 6, 5, 2, 1)	1
(1, 2, 4, 4, 3, 6, 5, 3, 6, 5, 2, 1)	S_3	(1, 2, 4, 4, 3, 6, 7, 3, 2, 5, 7, 1)	\mathbb{Z}_2
(1, 2, 4, 4, 3, 6, 7, 3, 5, 1, 2, 6)	\mathbb{Z}_2	(1, 2, 4, 4, 3, 6, 7, 3, 5, 5, 2, 1)	\mathbb{Z}_2
(1, 2, 4, 4, 3, 7, 7, 3, 2, 5, 6, 1)	1	(1, 2, 4, 4, 3, 7, 7, 3, 5, 6, 2, 1)	\mathbb{Z}_2
(1, 2, 4, 5, 3, 7, 7, 3, 5, 4, 2, 1)	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_6$		

Also included in Table 1 is the stabilizer subgroup $A_\lambda \subset A(P^3)$ for the given labeling λ . This subgroup acts via isometries on the corresponding small cover M_λ as does the group $W/\Gamma_\lambda = (\mathbb{Z}_2)^3$. It follows from Theorem 3.3 that M_λ admits no further symmetries. In other words, the isometry group of M_λ is the semi-direct product of A_λ and $(\mathbb{Z}_2)^3$. In particular, the small cover on the bottom left has the largest symmetry group (with order 192).

Remark 4.2. It can be shown that a normalized labeling defines an orientable small cover if and only if all of the labels λ_i are in the set $\{1, 2, 4, 7\}$, thus the third small cover in the second column of Table 1 is the only orientable one. (This orientable small cover is one of the examples constructed in [V], and some of the nonorientable examples were discussed in [A].)

The 120-cell. Let P^4 be the 120-cell. It has 120 dodecahedral facets, and the symmetry group $A(P^4)$ is the Coxeter group with diagram \mathbf{H}_4 :



In theory, our algorithm can be used to search for all possible labelings of P^4 with labels in the set $(\mathbb{Z}_2)^4 - \{0\}$, but the running time is too large to make the computation feasible. It is not hard to show, however, that a labeling of P^4 must have at least 5 labels, and that if it has exactly 5, they must be (modulo G) 1, 2, 4, 8, 15 (decimal equivalents again). Applying Algorithm 4.1 with the list of possible labels set to $S = [1, 2, 4, 8, 15]$, the search is effective and returns exactly 10 labelings. Another computation shows that the stabilizer of one of these labelings has index 10; hence, all of these labelings are equivalent. A representative labeling λ is given in Table 2. The facets are indexed by their barycenters, which were determined using the standard 4-dimensional geometric representation for $A(P^4)$. This is the contragradient representation (see [B], Chapter V, Section 4), so the generating involutions are linear reflections across the coordinate hyperplanes in \mathbb{R}^4 and are orthogonal with respect to the inverse of the Coxeter matrix of type \mathbf{H}_4 .

TABLE 2. The unique small cover of the 120-cell using 5 labels.

center of facet F	$\lambda(F)$	center of facet F	$\lambda(F)$
$\pm(2, 0, 0, 0)$	1	$\pm(2, 1 + \sqrt{5}, -3 - \sqrt{5}, 1 + \sqrt{5})$	1
$\pm(-2, 2, 0, 0)$	2	$\pm(-2, 3 + \sqrt{5}, -3 - \sqrt{5}, 1 + \sqrt{5})$	8
$\pm(0, -2, 2, 0)$	4	$\pm(2, 1 + \sqrt{5}, 0, -1 - \sqrt{5})$	4
$\pm(0, 0, -2, 1 + \sqrt{5})$	8	$\pm(-2, 3 + \sqrt{5}, 0, -1 - \sqrt{5})$	8
$\pm(0, 0, 1 + \sqrt{5}, -1 - \sqrt{5})$	8	$\pm(3 + \sqrt{5}, -3 - \sqrt{5}, 2, 0)$	1
$\pm(0, 1 + \sqrt{5}, -1 - \sqrt{5}, 2)$	4	$\pm(3 + \sqrt{5}, -1 - \sqrt{5}, -2, 1 + \sqrt{5})$	8
$\pm(0, 1 + \sqrt{5}, 0, -2)$	15	$\pm(1 + \sqrt{5}, -3 - \sqrt{5}, 0, 1 + \sqrt{5})$	4
$\pm(1 + \sqrt{5}, -1 - \sqrt{5}, 0, 2)$	2	$\pm(3 + \sqrt{5}, -1 - \sqrt{5}, 1 + \sqrt{5}, -1 - \sqrt{5})$	8
$\pm(1 + \sqrt{5}, -1 - \sqrt{5}, 1 + \sqrt{5}, -2)$	15	$\pm(1 + \sqrt{5}, -3 - \sqrt{5}, 3 + \sqrt{5}, -1 - \sqrt{5})$	2
$\pm(1 + \sqrt{5}, 0, -1 - \sqrt{5}, 1 + \sqrt{5})$	15	$\pm(3 + \sqrt{5}, 0, -1 - \sqrt{5}, 2)$	2
$\pm(1 + \sqrt{5}, 0, 2, -1 - \sqrt{5})$	2	$\pm(1 + \sqrt{5}, 0, -3 - \sqrt{5}, 3 + \sqrt{5})$	4
$\pm(1 + \sqrt{5}, 2, -2, 0)$	8	$\pm(3 + \sqrt{5}, 0, 0, -2)$	1
$\pm(3 + \sqrt{5}, -2, 0, 0)$	4	$\pm(1 + \sqrt{5}, 0, 1 + \sqrt{5}, -3 - \sqrt{5})$	15
$\pm(-1 - \sqrt{5}, 0, 0, 2)$	1	$\pm(1 + \sqrt{5}, 1 + \sqrt{5}, -1 - \sqrt{5}, 0)$	15
$\pm(-1 - \sqrt{5}, 0, 1 + \sqrt{5}, -2)$	15	$\pm(2 + 2\sqrt{5}, -1 - \sqrt{5}, 0, 0)$	15
$\pm(-1 - \sqrt{5}, 1 + \sqrt{5}, -1 - \sqrt{5}, 1 + \sqrt{5})$	15	$\pm(-3 - \sqrt{5}, 0, 2, 0)$	4
$\pm(-1 - \sqrt{5}, 1 + \sqrt{5}, 2, -1 - \sqrt{5})$	4	$\pm(-3 - \sqrt{5}, 2, -2, 1 + \sqrt{5})$	2
$\pm(-1 - \sqrt{5}, 3 + \sqrt{5}, -2, 0)$	1	$\pm(-1 - \sqrt{5}, -2, 0, 1 + \sqrt{5})$	8
$\pm(-3 - \sqrt{5}, 1 + \sqrt{5}, 0, 0)$	8	$\pm(-3 - \sqrt{5}, 2, 1 + \sqrt{5}, -1 - \sqrt{5})$	1
$\pm(0, -1 - \sqrt{5}, 0, 1 + \sqrt{5})$	15	$\pm(-1 - \sqrt{5}, -2, 3 + \sqrt{5}, -1 - \sqrt{5})$	8
$\pm(0, -1 - \sqrt{5}, 3 + \sqrt{5}, -1 - \sqrt{5})$	1	$\pm(-3 - \sqrt{5}, 3 + \sqrt{5}, -1 - \sqrt{5}, 2)$	4
$\pm(2, -3 - \sqrt{5}, 1 + \sqrt{5}, 0)$	8	$\pm(-1 - \sqrt{5}, 1 + \sqrt{5}, -3 - \sqrt{5}, 3 + \sqrt{5})$	1
$\pm(-2, -1 - \sqrt{5}, 1 + \sqrt{5}, 0)$	2	$\pm(-3 - \sqrt{5}, 3 + \sqrt{5}, 0, -2)$	2
$\pm(0, 2, -3 - \sqrt{5}, 3 + \sqrt{5})$	2	$\pm(-1 - \sqrt{5}, 1 + \sqrt{5}, 1 + \sqrt{5}, -3 - \sqrt{5})$	15
$\pm(2, -2, -1 - \sqrt{5}, 3 + \sqrt{5})$	1	$\pm(-1 - \sqrt{5}, 2 + 2\sqrt{5}, -1 - \sqrt{5}, 0)$	15
$\pm(-2, 0, -1 - \sqrt{5}, 3 + \sqrt{5})$	4	$\pm(0, -3 - \sqrt{5}, 2, 2)$	1
$\pm(0, 2, 1 + \sqrt{5}, -3 - \sqrt{5})$	1	$\pm(0, -1 - \sqrt{5}, -2, 3 + \sqrt{5})$	2
$\pm(2, -2, 3 + \sqrt{5}, -3 - \sqrt{5})$	4	$\pm(0, -3 - \sqrt{5}, 3 + \sqrt{5}, -2)$	4
$\pm(-2, 0, 3 + \sqrt{5}, -3 - \sqrt{5})$	2	$\pm(0, -1 - \sqrt{5}, 2 + 2\sqrt{5}, -3 - \sqrt{5})$	15
$\pm(0, 3 + \sqrt{5}, -1 - \sqrt{5}, 0)$	2	$\pm(0, 0, -3 - \sqrt{5}, 2 + 2\sqrt{5})$	8

With respect to this representation the vector $(2, 0, 0, 0)$ is the barycenter of the first facet and the remaining barycenters are the translates of this point under the action of $A(P^4)$.

It follows from Theorem 3.3 that up to homeomorphism M_λ is the unique small cover that uses only 5 labels, and that no small cover using more than 5 labels is homeomorphic to M_λ . Since $A(P^4)$ has order 14400, the isometry group of M_λ has order $16 \times 1440 = 23040$.

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