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APPROXIMATION OF MEASURABLE MAPPINGS BY SEQUENCES OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let X be a completely regular Hausdorff space, μ a positive, finite Baire measure on X, and E a separable metrizable locally convex space. Suppose $f : X \to E$ is a measurable mapping. Then there exists a sequence of functions in $C_b(X) \otimes E$ which converges to f a.e. $[\mu]$. If the function f is assumed to be weakly continuous and the measure μ is assumed to be τ-smooth, then a separability condition is not needed.

1. INTRODUCTION AND NOTATIONS

In [9], it is proved that for a metric space X with a Borel measure μ, a measurable mapping $f : X \to E$ is almost everywhere the limit of a sequence of continuous functions if E is a separable Banach space; this is an extension of an earlier result in [8]. Though the result is stated to be extended to a locally convex space E in [4] (Theorem 1, p. 1513), what is proved in that theorem ([4], Theorem 1) gives a sequence of continuous functions only when E is metrizable.

In this paper we give some extensions of this theorem by assuming X to be completely regular and E a separable metrizable locally convex space. If we assume the function f to be weakly continuous and the measure μ to be τ-smooth, then the separability condition on E is not needed. Our proofs are very different and the results follow easily from the regularity of measures.

For topological measure theory, notations and results from [7] and [10] will be used. All vector spaces are taken over K, the field of real or complex numbers (called scalars). For a completely regular Hausdorff space X, $C_b(X)$ will denote all bounded continuous on X. For locally convex spaces, notations and results of [6] will be used. For a locally convex space E, E' will denote its dual. For a completely regular Hausdorff space X and a locally convex space E, $C_b(X, E)$ will denote all bounded and continuous E-valued functions on X, and $C_b(X) \otimes E$ will denote the tensor product of $C_b(X)$ and E.

If, for $i = 1, 2$, X_i are sets and $\mathcal{A}_i$ are sigma-algebras on X_i, then a mapping $\rho : X_1 \to X_2$ is called measurable if $\rho^{-1}(B) \in \mathcal{A}_1$ for every $B \in \mathcal{A}_2$. If X_2 is a metrizable locally convex space, then the sigma-algebra $\mathcal{A}_2$ will always be the class of all Borel subsets of X_2. For measure theory we refer to [1]. If $(X, \mathcal{A}, \mu)$ is a finite measure space and E a metrizable locally convex space, a mapping $f : X \to E$

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will be called strongly measurable if there exists a sequence of simple measurable functions, converging to f a.e. $[\mu]$ ([1], Theorem 2, p. 99).

2. Main results

Lemma 1. *Let X be a completely regular Hausdorff space and μ a positive, finite Baire measure on X and E a separable normed space. Suppose $f : X \to E$ is a measurable mapping. Then there exists a sequence of functions in $C_b(X) \otimes E$ which converges to f a.e. $[\mu]$.*

Proof. For every $g \in E'$, $g \circ f$ is measurable. Since E is separable, this implies that f is strongly measurable ([1], Proposition 12, p. 93). Thus there exists a sequence $\{h_n\}$ of simple measurable functions such that $h_n \to f$ a.e. $[\mu]$. By regularity of Baire measures ([7], [10]), for a Baire subset $B \subset X$ and $c > 0$, there is a $\phi \in C_b(X)$ such that $\int |\chi_B - \phi| d\mu < c$. This means, for each h_n, there is $\bar{h_n} \in C_b(X) \otimes E$ such that $\int \|\bar{h_n} - h_n\| d\mu < \frac{1}{2^n}$. Thus there exists a subsequence of $\{\bar{h_n} - h_n\}$ which converges to 0, a.e. (μ). Thus $\bar{h_n} \to f$ a.e. (μ). This proves the results.

Now we prove the following theorem for separable metrizable locally convex spaces.

Theorem 2. *Let X be a completely regular Hausdorff space and μ a positive, finite Baire measure on X and E a separable metrizable locally convex space. Suppose $f : X \to E$ is a measurable mapping. Then there exists a sequence of functions in $C_b(X) \otimes E$ which converges to f a.e. $[\mu]$.*

Proof. Let $\|.\|_n$ be an increasing family of semi-norms generating the topology of E. By Lemma 1 and Egoroff's theorem ([1], Theorem 1, p. 94), there exists, for every k, a Baire set A_k with $\mu(A_k) < \frac{1}{2^k}$, and a $g_k \in C_b(X) \otimes E$ with $\|g_k - f\|_k < \frac{1}{2^k}$ on $X \setminus A_k$. From this it follows that $g_k \to f$ on $X \setminus A$, where $A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Since $\mu(A) = 0$, the result follows.

In the next theorem, we remove the condition of separability but put some additional conditions on the function f and the measure μ.

Theorem 3. *Let X be a completely regular Hausdorff space and μ a positive, finite τ-smooth measure on X and E a metrizable locally convex space. Suppose $f : X \to E$ is a weakly continuous mapping. Then there exists a sequence of functions in $C_b(X) \otimes E$ which converges to f a.e. $[\mu]$.*

Proof. Consider the measure $\lambda = \mu \circ f^{-1}$ on Borel subsets of E with weak topology. This measure is τ-smooth and so has a support Y, a weakly closed subset of E. Let $X_0 = f^{-1}(Y)$. X_0 is a closed subset of X and $\mu(X_0) = \mu(X)$. Suppose, for some $g \in E'$, $g \circ f = 0$ a.e. $[\mu]$. We claim that $g \circ f = 0$ on X_0. To prove this, let $V = \{t \in Y : |g(t)| > 0\}$ and $A = \{x \in X_0 : |g \circ f(x)| > 0\}$. It is easily verified that $A = f^{-1}(V)$. If V is non-void, then, being open in Y, $\lambda(V) > 0$ and so $\mu(A) > 0$. But this is a contradiction since $g \circ f = 0$ a.e. $[\mu]$. Thus A is a void set and so the claim is established. By [5] (Theorem 6, p. 813, see also [2], [3]), f is strongly meaurable. Since μ is τ-smooth, for any Borel subset $B \subset X$ and $c > 0$, there is a $\phi \in C_b(X)$ such that $\int |\chi_B - \phi| d\mu < c$ ([7], [10]); this is exactly the main result used in Lemma 1 for Baire measure μ and Baire subset $B \subset X$. Proceeding exactly as in Lemma 1 and Therorem 2, the result follows.

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