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THE NUMBER OF CONNECTED COMPONENTS IN DOUBLE BRUHAT CELLS FOR NONSIMPLY-LACED GROUPS

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Abstract. We compute the number of connected components in a generic real double Bruhat cell for series B_n and C_n and an exceptional group F_4.

1. Introduction and main result

Let G be a simply connected semisimple algebraic group. Let B and B_- be two $\mathbb{R}$-split opposite Borel subgroups, N and N_- their unipotent radicals, $H = B \cap B_-$ an $\mathbb{R}$-split maximal torus of G, and $W = \mathrm{Norm}_G(H)/H$ the Weyl group of G.

The group G has two *Bruhat decompositions*, with respect to B and B_- :

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- v B_- \ .$$

The *double Bruhat cells* $G^{u,v}$ are defined by $G^{u,v} = BuB \cap B_- v B_-$. The maximal torus H acts freely on $G^{u,v}$ by left (or right) translations. The quotient of $G^{u,v}$ by this action is called the *reduced double Bruhat cell* $L^{u,v} \subset G^{u,v}$ (see [SSVZ], [Z] for a more rigorous definition). Thus, $G^{u,v}$ is biregularly isomorphic to $H \times L^{u,v}$, and all properties of $G^{u,v}$ can be translated in a straightforward way into the corresponding properties of $L^{u,v}$ (and vice versa). In particular, Theorem 1.1 in [FZ] implies that $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space.

Let $L^{u,v}(\mathbb{R})$ denote the real part of $L^{u,v}$, that is, $L^{u,v}(\mathbb{R}) = L^{u,v} \cap G(\mathbb{R})$, where $G(\mathbb{R})$ is the real part of G. Consider the case when $u = e$ and $v = w_0$, the longest element in W. In this case $L^{u,v}$ is biregularly isomorphic to the intersection of two open opposite Schubert cells $C_{w_0} \cap w_0 C_{w_0}$, where $C_{w_0} = (Bw_0B)/B$ is the open Schubert cell in the flag variety G/B. These opposite cells appeared in the literature in various contexts (see e.g. [BFZ], [R1]). Let $\sharp$ denote the number of connected components in $L^{e,w_0}(\mathbb{R})$. Following [Z] we write $\sharp = \sharp(X_n)$, where $X_n = A_n, B_n, \ldots, G_2$ runs over all types of simple Lie groups in the Cartan–Killing classification.

The numbers $\sharp(A_n)$ were determined in [SSV97], [SSV98]: it turns out that $\sharp(A_1) = 2$, $\sharp(A_2) = 6$, $\sharp(A_3) = 20$, $\sharp(A_4) = 52$, and $\sharp(A_n) = 3 \cdot 2^n$ for $n \geqslant 5$. The numbers $\sharp(D_n)$ were determined in [Z]; namely, $\sharp(D_n) = 3 \cdot 2^n$ for $n \geqslant 4$. It is also

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shown in [Z] that $\sharp(E_n) = 3 \cdot 2^n$ for $n = 6, 7, 8$. The case G_2 was treated in [R2]: $\sharp(G_2) = 11$ (see also [Z] for another proof of this result). For nonsimply-laced series B_n and C_n, only the simplest case $n = 2$ is known; in this case $\sharp(B_2)(= \sharp(C_2)) = 8$ (see [R2], [Z]).

In this note we calculate $\sharp(X_n)$ for the remaining simple Lie groups of types B_n, C_n, and F_4, and thus provide a complete solution for the problem posed in [Z, Remark 5.3].

Theorem 1. *For any $n \geqslant 4$ one has $\sharp(B_n) = \sharp(C_n) = (n+5) \cdot 2^{n-1}$. Besides, $\sharp(B_3) = \sharp(C_3) = 30$, $\sharp(B_2) = \sharp(C_2) = 8$ and $\sharp(F_4) = 80$.*

In fact, we prove a more general result, and find the number of connected components of $L^{u,v}(\mathbb{R})$ for any generic pair $(u, v) \in W \times W$ (see Theorem 4 below).

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2. Proofs

We start by recalling the following important construction from [SSVZ], [Z]; in fact, this is not the original construction itself, but rather its version reduced modulo 2.

Let Π be the Coxeter graph of G, and let $s_i (i \in \Pi)$ be the system of simple reflections that generate W. A word $\mathbf{i} = (i_1, \dots, i_m)$ in the alphabet Π is a *reduced word* for $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$, and m is the smallest length of such a factorization. The length of any reduced word for w is called the *length* of w and denoted by $\ell(w)$.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $A = (a_{ij})$ be the Cartan matrix. Recall that for $i \neq j$ the indices i and j are adjacent in Π if and only if $a_{ij} a_{ji} \neq 0$; we shall denote this by $\{i, j\} \in \Pi$.

Let us consider the group $W \times W$. It corresponds to a graph $\tilde{\Pi}$ given by the union of two disconnected copies of Π. We identify the vertex set of $\tilde{\Pi}$ with $\{+1, -1\} \times \Pi$, and write a vertex $(\pm 1, i) \in \tilde{\Pi}$ simply as $\pm i$. For each $i \in \Pi$, we set $\varepsilon(\pm i) = \pm 1$ and $|\pm i| = i$. Thus, two vertices i and j of $\tilde{\Pi}$ are adjacent if and only if $\varepsilon(i) = \varepsilon(j)$ and $\{|i|, |j|\} \in \Pi$. In this notation, a reduced word for a pair $(u, v) \in W \times W$ is an arbitrary shuffle of a reduced word for u written in the alphabet $-\Pi$ and a reduced word for v written in the alphabet Π. The set of all reduced words for a given pair $(u, v) \in W \times W$ is denoted by $R(u, v)$.

Now let us fix a pair $(u, v) \in W \times W$, and let $d = \ell(u) + \ell(v)$. Let $\mathbf{i} = (i_1, \dots, i_d) \in R(u, v)$ be any reduced word for (u, v). We associate to $\mathbf{i}$ a $d \times d$ matrix (Ω_{kl}) over the two-element field $\mathbb{F}_2$ in the following way: set $\Omega_{kl} = 1$ if $|i_k| = |i_l|$ and $\Omega_{kl} = a_{|i_k|,|i_l|} \bmod 2$ if $|i_k| \neq |i_l|$.

Next, we associate with $\mathbf{i}$ a graph $\Sigma(\mathbf{i})$ on the set of vertices $[1, d] = \{1, 2, \dots, d\}$. For $l \in [1, d]$, we denote by $l^- = l^-_{\mathbf{i}}$ the maximal index k such that $1 \leqslant k < l$ and $|i_k| = |i_l|$; if $|i_k| \neq |i_l|$ for $1 \leqslant k < l$; then we set $l^- = 0$. The edges of $\Sigma(\mathbf{i})$ are now defined as follows.

A pair $\{k,l\} \subset [1,d]$ with $k<l$ is an edge of $\Sigma(\mathbf{i})$ if it satisfies one of the following three conditions:

(i) $k = l^-$;

(ii) $k^- < l^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and $\varepsilon(i_{l^-}) = \varepsilon(i_k)$;

(iii) $l^- < k^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and $\varepsilon(i_{k^-}) = -\varepsilon(i_k)$.

The edges of type (i) are called *horizontal*, and those of types (ii) and (iii) *inclined*. Each inclined edge corresponds to an edge of the graph Π. We shall write $\{k,l\} \in \Sigma(\mathbf{i})$ if $\{k,l\}$ is an edge of $\Sigma(\mathbf{i})$.

We now associate to each $r \in [1,d]$ a transvection $\tau_r\colon \mathbb{F}_2^d \to \mathbb{F}_2^d$ defined as follows: $\tau_r(\xi_1,\dots,\xi_d) = (\xi_1',\dots,\xi_d')$, where $\xi_k' = \xi_k$ for $k \neq r$, and

$$\xi_r' = \xi_r + \sum_{\{k,r\}\in\Sigma(\mathbf{i})} \Omega_{kr}\xi_k \tag{1}$$

(note that (1) coincides with the reduction modulo 2 of formula (2.2) in [Z]). We call an index $r \in [1,d]$ $\mathbf{i}$-*bounded* if $r^- > 0$. The set of all bounded indices (and corresponding vertices of $\Sigma(\mathbf{i})$) is denoted by B and its complement is denoted by C.

Let $\Gamma_{\mathbf{i}}$ denote the group of linear transformations of $\mathbb{F}_2^d$ generated by the transvections τ_r for all $\mathbf{i}$-bounded indices $r \in [1,d]$. The following result was conjectured in [SSVZ] for a simply laced case, and proved in [Z] in the general case (see also [SSV97] for the case of open cells for type A_n).

Theorem 2. *For every reduced word* $\mathbf{i} \in R(u,v)$*, the connected components of* $L^{u,v}(\mathbb{R})$ *are in a natural bijection with the* $\Gamma_{\mathbf{i}}$*-orbits in* $\mathbb{F}_2^d$*.*

This theorem, together with the description of orbits of groups generated by symplectic transvections presented in [SSV98], [SSVZ], form the basis of the enumerative results in the simply-laced case cited in the Introduction.

However, in the nonsimply-laced case, the transvections generating $\Gamma_{\mathbf{i}}$ are no longer symplectic. To handle this case, we have to extend several results of [SSV98], [SSVZ].

Let W^t, $t \in [1,n-1]$, be the Coxeter group with n generators $s_1,\dots,s_n$ and relations of the form $s_i^2 = 1$, $(s_is_j)^2 = 1$ for $j > i+1$, $(s_is_{i+1})^3 = 1$ for $i \neq t$, and $(s_ts_{t+1})^4 = 1$. Denote by Π^t the Coxeter graph of W^t. Finally, define the $n\times n$ matrix A^t as follows: $a_{t,t+1} = -2$, $a_{t+1,t} = -1$, $a_{i,i+1} = a_{i+1,i} = -1$ for any $i \in [1,n-1]$, $i \neq t$, $a_{ij} = 0$ for $|i-j| \neq 1$.

Fix a pair $(u,v) \in W^t \times W^t$, take an arbitrary reduced word $\mathbf{i}$ for the pair (u,v), and build the graph $\Sigma(\mathbf{i})$ and transvections τ_r exactly as above, with Π replaced by Π^t and A replaced by A^t. Observe that for $t=1$ the above construction describes the C_n case, for $t = n-1$ it describes the B_n case, and for $n=4$, $t=2$ the F_4 case.

Define Π_U^t to be the subgraph of Π^t induced by the vertices $\{1,2,\dots,t\}$ and Π_L^t to be the complement to Π_U^t in Π^t. In accordance with this partition of Π^t, we subdivide the vertex set of Σ into $U = \{k \in \Sigma\colon |i_k| \in \Pi_U^t\}$ and its complement L (we omit in the notation the dependence of Σ and other objects on the reduced word $\mathbf{i}$ which is assumed fixed). Together with the partition into bounded and unbounded vertices described above, this gives four subsets, which we denote B_U, B_L, C_U, and C_L; the subgraph of Σ induced by a subset $X \subseteq \Sigma$ is denoted Σ_X, and $\mathbb{F}_2^X$ is the linear subspace of $\mathbb{F}_2^d$ defined by the condition that all coordinates that correspond to $\Sigma \setminus X$ vanish. The subgroups Γ_U and Γ_L of Γ are defined in a natural way; clearly, Γ is generated by Γ_U and Γ_L.

For any vector $\nu \in \mathbb{F}_2^L$, the action of Γ_U preserves the affine subspace $\nu + \mathbb{F}_2^U$. Identifying $\nu + \mathbb{F}_2^U$ with $\mathbb{F}_2^U$ with the help of the shift $\xi \mapsto \xi - \nu$, we get an action of Γ_U on $\mathbb{F}_2^U$; slightly abusing notation, we call it the $\Gamma_U(\nu)$-*action on* $\mathbb{F}_2^U$. Note that for $\nu \neq 0$ the $\Gamma_U(\nu)$-action is not linear, but rather affine; the $\Gamma_U(0)$-action coincides with the usual linear action of Γ_U on $\mathbb{F}_2^U$.

It follows from [SSVZ, Proposition 6.1] that the number of fixed points of the $\Gamma_U(0)$-action equals 2^t; the number of nontrivial orbits of this action (those which are not fixed points) we denote by N_U. In a similar fashion, define the number N_L of nontrivial orbits of the action of Γ_L on $\mathbb{F}_2^L$; the number of fixed points of this action equals 2^{n-t}. Observe that one can also define the $\Gamma_L(\varkappa)$-action on $\mathbb{F}_2^L$ for any $\varkappa \in \mathbb{F}_2^U$, but this action does not depend on $\varkappa$ and coincides with the Γ_L-action.

Lemma 1. *For any vector $\nu \in \mathbb{F}_2^L$ there are $2^t + N_U$ orbits of the $\Gamma_U(\nu)$-action on $\mathbb{F}_2^U$.*

Proof. Indeed, the $\Gamma_U(\nu)$-action on $\mathbb{F}_2^U$ is generated by affine transformations of the form $\theta_j(\xi) = \tau_j^U(\xi) + b_j$ for $j \in B_U$, $\xi \in \mathbb{F}_2^U$, where τ_j^U is the symplectic transvection with respect to the restriction of Ω to $\mathbb{F}_2^U$ and b_j depends only on ν. Assume that $\xi^* \in \mathbb{F}_2^U$ is a fixed point of this affine action. Then $\tau_j^U(\xi) - \xi^* = \tau_j^U(\xi - \xi^*)$ for any $j \in B_U$ and $\xi \in \mathbb{F}_2^U$, and hence the orbits of the $\Gamma_U(\nu)$-action are just the orbits of the $\Gamma_U(0)$-action shifted by ξ^*. Therefore, the number of affine orbits equals $2^t + N_U$, the number of $\Gamma_U(0)$-orbits.

It remains to check the existence of a fixed point of the affine action. Such a fixed point should satisfy the equation $M\xi = b(\nu)$ for some $b(\nu) \in \mathbb{F}_2^{B_U}$, where $M\colon \mathbb{F}_2^U \to \mathbb{F}_2^{B_U}$ is given by

$$(M\xi)_j = \xi_j + \sum_{\{k,j\} \in \Sigma_U} \xi_k.$$

The kernel of M consists of the fixed points of the $\Gamma_U(0)$-action. Therefore, its dimension equals $t = |C_U|$, which means that the image of M coincides with $\mathbb{F}_2^{B_U}$. Therefore, equation $M\xi = b$ can be solved for any b, and we are done. □

The number of Γ-orbits in $\mathbb{F}_2^d$ is determined as follows.

Theorem 3. *Assume that Σ_B is connected. Then the number of Γ-orbits in $\mathbb{F}_2^d$ equals $2^n + 2^{n-t}N_U + 2^t N_L$.*

Proof. First observe that the projections of the orbits of the Γ-action onto $\mathbb{F}_2^L$ are exactly the orbits of the Γ_L-action on $\mathbb{F}_2^L$. First consider Γ-orbits whose projections onto $\mathbb{F}_2^L$ are fixed points of this Γ_L-action. The number of fixed points of the Γ_L-action is 2^{n-t}, hence by Lemma 1 we see that the number of such Γ-orbits equals $(2^t + N_U)2^{n-t}$.

Next, consider Γ-orbits whose projections onto $\mathbb{F}_2^L$ are nontrivial orbits of the Γ_L-action on $\mathbb{F}_2^L$. We claim that the number of such Γ-orbits equals $2^t N_L$.

Indeed, let us fix a vector $\nu \in \mathbb{F}_2^L$ in such a Γ_L-orbit, and consider the $\Gamma_U(\nu)$-action on $\mathbb{F}_2^U$. As before, by Lemma 1 we get an affine action having $2^t + N_U$ orbits for this choice of ν. We shall show that the Γ_L-action can be used to glue these orbits into 2^t Γ-orbits differing only by the values on C_U. To achieve this, it is enough to show that one can change the value ξ_r for any given $r \in B_U$, and to keep all the other ξ_j, $j \in B$, unchanged. This is evidently true if $\tau_r(\xi) \neq \xi$, so in what follows we assume that $\tau_r(\xi) = \xi$.

Denote by $T(\xi)$ the set of all $j \in B_L$ such that $\tau_j(\xi) \neq \xi$; $T(\xi) \neq \varnothing$, since ν belongs to a nontrivial Γ_L-orbit. The connectivity of Σ_B implies the existence of a path joining r with the set $T(\xi)$. Moreover, since the set of all vertices q having the same height $|i_q|$ is connected in Σ_B, there exists a *monotone* path from r to $T(\xi)$, that is, one for which the height changes monotonously along the path. Let $P = (q_0 \in T(\xi), q_1, \dots, q_k = r)$ be a shortest monotone path between $T(\xi)$ and r; besides, let q_l be the first vertex at height t in this path. Note that since the path P is monotone, all the vertices q_j, $j \in [l,k]$, belong to B_U.

Assume first that $\tau_{q_j}(\xi) = \xi$ for $j \in [l,k]$. Consequently apply $\tau_{q_0}, \tau_{q_1}, \dots, \tau_{q_k}$; upon applying τ_{q_i}, the value ξ_{q_i} is changed, since the only edge of the type $\{q_i, q_j\}$, $j < i$, is the edge $\{q_i, q_{i-1}\}$ (otherwise the path is not the shortest possible). Hence, applying the whole sequence results in changing the value ξ_r. To restore the values ξ_{q_i}, $i \neq [0, k-1]$, consequently apply $\tau_{q_{l-1}}, \tau_{q_{l-2}}, \dots, \tau_{q_0}$ followed by $\tau_{q_l}, \tau_{q_{l+1}}, \dots, \tau_{q_{k-1}}$.

Otherwise, let q_m, $m \in [l,k]$, be the vertex of P closest to r for which $\tau_{q_m}(\xi) \neq \xi$. Consequently apply $\tau_{q_m}, \tau_{q_{m+1}}, \dots, \tau_{q_k}$ to change the value ξ_r. To restore the values ξ_{q_i}, $i \in [m, k-1]$, we have to solve the same problem as above, but now the length of a shortest monotone path to $T(\xi)$ equals $k-1$, and we are done by induction.

Proceeding in this way, we see that any Γ-orbit whose projection onto $\mathbb{F}_2^L$ does not coincide with a fixed point of the Γ_L-action on $\mathbb{F}_2^L$ contains a vector that vanishes at any point of B_U. Therefore, the only invariants of such an orbit are the values of ξ at the points of C_U. Since the number of these points equals t, we get 2^t Γ-orbits per each nontrivial Γ_L-orbit, which totals $2^t N_L$ Γ-orbits. □

To prove our main result, we need the following definition. Let $\Sigma = P$ be a path on m vertices $\{1, 2, \dots, m\}$. Define the Γ_P-*action* on $\mathbb{F}_2^P = \mathbb{F}_2^m$ as the action generated by symplectic transvections τ_j^P, $j \in [2, m]$, given by

$$\tau_j^P(\zeta) = \zeta + (\zeta_{j-1} + \zeta_{j+1}) e_j,$$

where $\{e_j\}$ is the standard basis of $\mathbb{F}_2^m$.

Lemma 2. *The number of orbits of the Γ_P-action equals $m+1$. Exactly two of these orbits are fixed points of the Γ_P-action.*

Proof. Let $\zeta \in \mathbb{F}_2^m$ be of the form

$$\zeta = (\underbrace{0\dots0}_{l_1}\underbrace{1\dots1}_{m_1}\dots\underbrace{0\dots0}_{l_k}\underbrace{1\dots1}_{m_k}\underbrace{0\dots0}_{l_{k+1}}),$$

where $l_1, l_{k+1} \geqslant 0$, $l_2, \dots, l_k, m_1, \dots, m_k > 0$; we put $c(\zeta) = k$. It is easy to see that $c(\tau_j^P(\zeta)) = c(\zeta)$ and that $(\tau_j^P(\zeta))_1 = \zeta_1$. Let us prove that if $c(\zeta) = k$ and $\zeta_1 = 1$ (resp., $\zeta_1 = 0$), then there exists $\gamma \in \Gamma_P$ such that

$$\gamma(\zeta) = (\underbrace{1, 0, 1, 0, \dots, 1, 0, 1}_{2k-1}, 0, \dots, 0)$$

(resp., $\gamma(\zeta) = (\underbrace{0, 1, 0, 1, \dots, 0, 1}_{2k}, 0, \dots, 0)$).

Indeed, if $\zeta = (\zeta_1, \dots, \zeta_{j-1}, \underbrace{1, \dots, 1}_{l}, 0, \zeta_{j+l+1}, \dots)$, then

$$\tau_{j+1}^P \cdots \tau_{j+l-1}^P(\zeta) = (\zeta_1, \dots, \zeta_{j-1}, 1, \underbrace{0, \dots, 0}_{l}, \zeta_{j+l+1}, \dots).$$

Similarly, if $\zeta = (\zeta_1, \dots, \zeta_{j-1}, \underbrace{0, \dots, 0}_{l}, 1, \zeta_{j+l+1}, \dots)$, then

$$(\tau^P_{j+2}\tau^P_{j+1}) \cdots (\tau^P_{j+l}\tau^P_{j+l-1})(\zeta) = (\zeta_1, \dots, \zeta_{j-1}, 0, 1, \underbrace{0, \dots, 0}_{l-1}, \zeta_{j+l+1}, \dots).$$

Combining transformations of these two types, we can eventually bring ζ to the required form.

Since the number of these forms equals $m+1$, and any two of them differ either at $c(\zeta)$ or at ζ_1 (or at both of them), we conclude that the number of Γ_P-orbits equals $m+1$. Evidently, if m is even, then $(0, 1, \dots, 0, 1)$ is a fixed point of the Γ_P-action, while if m is odd, the $(1, 0, \dots, 1, 0, 1)$ is such a fixed point. The only other fixed point is $(0, \dots, 0)$. □

Now we return to the cases B_n and C_n. We say that a pair (u, v) is *generic* if there exists $\mathbf{i} \in R(u, v)$ such that the subgraph $\Sigma_B(\mathbf{i})$ is connected, and the subgraph $\Sigma_{B_L}(\mathbf{i})$ (in the C_n case) or $\Sigma_{B_U}(\mathbf{i})$ (in the B_n case) is E_6-compatible. One can easily prove that almost all pairs (u, v) are generic, that is, the ratio of the number of generic pairs to the number of all pairs tends to 1 as n tends to ∞ (cp. with a similar result in the A_n-case proved in [SSV99]). Recall that in the C_n (respectively, B_n) case, the graph Σ_U (respectively, Σ_L) is a path. Let m denote the number of vertices in U for the C_n case, and the number of vertices in L for the B_n case. It is easy to see that this number depends only on the pair (u, v) and does not depend on the reduced word $\mathbf{i} \in R(u, v)$.

Theorem 4. *Let (u, v) be a generic pair. Then the number of connected components in $L^{u,v}(\mathbb{R})$ equals $(m+5) \cdot 2^{n-1}$ for both types B_n and C_n.*

Proof. Since the pair (u, v) is generic, there exists $\mathbf{i} \in R(u, v)$ such that the subgraph of $\Sigma(\mathbf{i})$ induced by B is connected. Hence, by Theorem 3, the number of $\Gamma_{\mathbf{i}}$-orbits for type C_n equals $2^n + 2^{n-1}N_U + 2N_L$, and for type B_n equals $2^n + 2^{n-1}N_L + 2N_U$. Besides, by [SSVZ, Th. 7.2], the E_6-compatibility condition implies that $N_L = 2^n$ for type C_n, and $N_U = 2^n$ for type B_n. Moreover, by Lemma 2, $N_U = m-1$ for type C_n, and $N_L = m-1$ for type B_n. Therefore, in both cases the total number of orbits equals $2^n + (m-1) \cdot 2^{n-1} + 2^{n+1} = (m+5) \cdot 2^{n-1}$. By Theorem 2, this number equals the number of connected components in $L^{u,v}(\mathbb{R})$. □

To prove Theorem 1 stated in the Introduction one has to check that the pair (e, w_0) is generic for $n \geqslant 4$. This fact follows immediately from Figure 1 presenting the graph $\Sigma(\mathbf{i})$ and its corresponding subgraphs for $n=4$ and $\mathbf{i} = 1234123412341234$.

Consider now the cases $n = 2, 3$. One can easily check that the subgraphs Σ_B remain connected, although the pair (e, w_0) is no longer generic; therefore, Theorem 3 remains valid. Besides, one gets $N_U = N_L = 1$ for types B_2 and C_2, $N_U = 2$, $N_L = 7$ for type C_3, and $N_U = 7$, $N_L = 2$ for type B_3. Thus, Theorem 3 yields $\sharp_2 = 4 + 2 + 2 = 8$ and $\sharp_3 = 8 + 8 + 14 = 30$.

To treat the case of F_4, we first consider the graph $S = S(m)$ defined as follows: S contains vertices $\{1, \dots, 2m\}$ arranged into two levels, the lower (resp. upper) level is formed by odd-numbered (resp. even-numbered) vertices. Horizontal edges are of the form $(2i, 2i+2)$ and $(2i-1, 2i+1)$, and inclined edges are of the form $(2i+1, 2i+2)$ and $(2i, 2i+1)$, where i runs from 1 to $m-1$ (see Figure 1e). It is convenient to represent elements of $\mathbb{F}_2^S$ by vectors $\zeta = (\zeta_i)_{i=1}^{2m} \in \mathbb{F}_2^{2m}$. The Γ_S-action

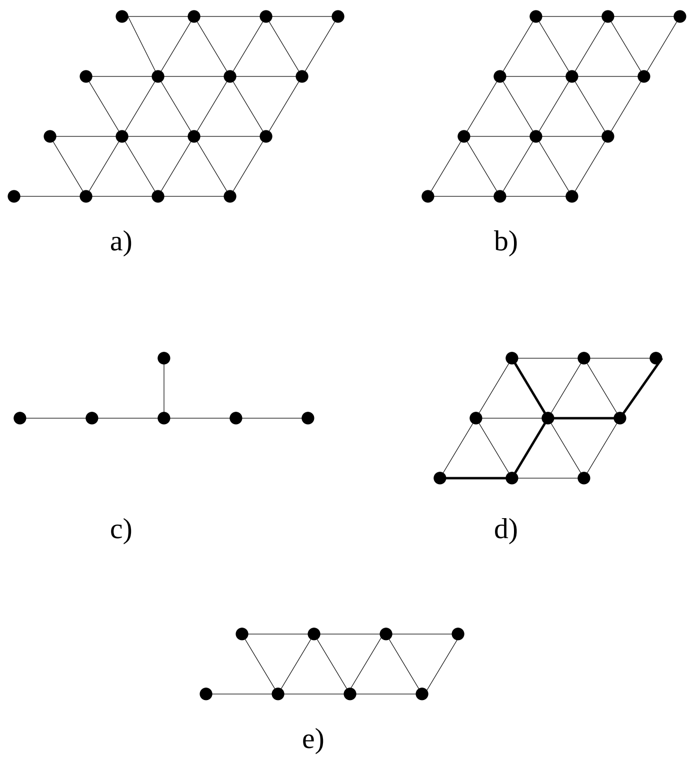

FIGURE 1. To the proof of Theorem 1: a) graph $\Sigma(\mathbf{i})$; b) graph $\Sigma_B(\mathbf{i})$; c) graph E_6; d) graph $\Sigma_{B_L}(\mathbf{i})$ and an induced E_6 in it; e) graph $S(4)$.

on $\mathbb{F}_2^S$ is generated by transvections τ_j^S, $j \in [3, 2m]$, defined by

$$\tau_j^S(\zeta) = \zeta + (\zeta_{j-2} + \zeta_{j-1} + \zeta_{j+1} + \zeta_{j+2})e_j, \tag{2}$$

where $\zeta_i = 0$ if $i > 2m$.

Lemma 3. *Let $m > 2$. Then every nontrivial orbit of the Γ_S-action contains either an element of the form $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, 0, \dots, 0)$ where not all ζ_i are equal to zero, or the element $\bar\zeta = (0, 0, 1, 1, 1, 0, \dots, 0)$.*

Proof. Let us fix a nontrivial orbit $\mathcal{O}$ of the Γ_S-action. To prove the statement, it suffices to show that for any $\xi \in \mathcal{O}$ of the form

$$\xi = (\xi_1, \dots, \xi_{j-1}, 1, \underbrace{0 \dots 0}_{2m-j})$$

such that $j > 4$ and $\xi \neq \bar\zeta$, there exists $\gamma \in \Gamma_S$ such that $\gamma(\xi)_i = 0$ for $i \geqslant j$.

If the set $T = \{i \colon 3 \leqslant i \leqslant j, \tau_i^S(\xi) \neq \xi\}$ is not empty (this is clearly the case for $j = 2m$), we denote by k the largest element in T and define γ as the product of τ_i^S along any shortest path from k to j. Then $\gamma(\xi)_i = 0$ for $i \geqslant j$.

Otherwise, $T = \varnothing$ and the smallest i such that $\tau_i^S(\xi) \neq \xi$ is equal either to $j+1$ or to $j+2$. In the first case, ξ has to be of the form

$$\xi = (\xi_1, \dots, \xi_{j-5}, 0, 1, 0, 0, 1, 0, \underbrace{0 \dots 0}_{2m-j-1}).$$

Define $\gamma = \tau_{j+1}^S \tau_j^S \tau_{j-2}^S \tau_{j-1}^S \tau_{j+1}^S$. Then

$$\gamma(\xi) = (\xi_1, \dots, \xi_{j-5}, 0, 1, 1, 1, \underbrace{0 \dots 0}_{2m-j+1}).$$

In the second case, either $\xi = \bar{\zeta}$ and we are done, or

$$\xi = (\xi_1, \dots, \xi_{j-6}, 0, 0, 0, 1, 1, 1, 0, 0, \underbrace{0 \dots 0}_{2m-j-2}),$$

in which case we put $\gamma = \tau_{j+1}^S \tau_{j-1}^S \tau_{j-3}^S \tau_{j+2}^S \tau_j^S \tau_{j-1}^S \tau_{j+1}^S \tau_{j+2}^S$. Then

$$\gamma(\xi) = (\xi_1, \dots, \xi_{j-5}, 0, 0, 1, 1, 1, \underbrace{0 \dots 0}_{2m-j+1}).$$

This finishes the proof. □

Corollary. *If $m > 2$, then the number of orbits of the Γ_S-action is equal to* 12*. Four of these orbits are fixed points of the action.*

Proof. It follows from (2) that for every choice of $\alpha, \beta \in \mathbb{F}_2$ there is exactly one fixed point of the Γ_S-action with $\zeta_{2n-1} = \alpha$, $\zeta_{2n} = \beta$. Thus, we have four orbits that are fixed points of the action.

By the previous lemma, any other orbit is either the orbit through $\bar{\zeta}$ or the orbit through an element of the form $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, 0, \dots, 0)$, where ζ_i cannot all be equal to zero. It is easy to see that if $\zeta_3 \neq \zeta_2$, then $\tau_4^S(\zeta) \neq \zeta$; moreover, either $\tau_3^S(\zeta) \neq \zeta$, or $\tau_3^S \tau_4^S(\zeta) \neq \tau_4^S(\zeta)$. Besides, if $\zeta_3 = \zeta_2$ and $\zeta_4 = \zeta_1 + \zeta_2$, then $\tau_3^S(\zeta) = \tau_4^S(\zeta) = \zeta$. This means that the number of nontrivial orbits does not exceed 8.

A nonhomogeneous quadratic form

$$Q_S(\xi) = \sum_{i \in S} \xi_i + \sum_{(i,j) \in S} \xi_i \xi_j$$

is an invariant of the Γ_S-action (see [SSVZ]) along with the values of ξ_1, ξ_2. Now, to finish the proof it is sufficient to notice that the triple $(\xi_1, \xi_2, Q_S(\xi))$ takes different values on the following eight elements:

$$(1,1,1,1,0,\dots,0), (1,1,1,0,0,\dots,0), (1,0,0,1,0,\dots,0), (1,0,0,0,0,\dots,0),$$
$$(0,1,1,1,0,\dots,0), (0,1,1,0,0,\dots,0), (0,0,1,0,0,\dots,0), (0,0,1,1,1,0,\dots,0).$$

□

We are now in a position to finish the proof of Theorem 1.

Theorem 5. $\sharp(F_4) = 80$.

Proof. Recall that $\mathbf{i} = (1234)^6$ is a reduced word for w_0 in the Weyl group that corresponds to F_4. We can use Theorem 3 again. In this case, $n = 4$, $t = 2$ and both subgraphs Σ_L and Σ_U coincide with $S(6)$. Then, by Theorem 3 and Corollary to Lemma 3, $\sharp(F_4) = 2^4 + 2 * 2^2 * 8 = 80$. □

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