ON THE OSCILLATION AND PERIODIC CHARACTER
OF A THIRD ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. We prove that every positive solution of the following difference equation:

$$x_n = 1 + \frac{x_{n-2}}{x_{n-3}}, \quad n = 0, 1, \dots,$$

converges to a period two solution.

1. INTRODUCTION

The nonlinear difference equation

$$(1.1) \qquad x_n = F(x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots,$$

of order $k > 1$ has been receiving increasing attention in the last decade. The general assumption is that $F \in C[\mathbb{R}^k, R]$ and that a set, $\mathcal{S} = \{x_{-1}, \dots, x_{-k}\}$, of initial values is given. The basic theory of global behavior of solutions of Eq. (1.1) of order greater than one is still in its infancy. Most of the results about Eq. (1.1), $k > 1$, treat the case when F is a rational function. For a second order rational difference equation, see the monograph of Kulenovic and Ladas [6] and the references therein. For rational difference equations of order greater than two, several results have been obtained in [1, 2, 3, 4, 5]. Recently, in [7], G. Ladas conjectured that every positive solution of the third order rational difference equation

$$(1.2) \qquad x_n = 1 + \frac{x_{n-2}}{x_{n-3}}, \quad n = 0, 1, \dots,$$

converges to a period two solution.

Here we prove this conjecture using a general convergence result established in [3]. We also prove that every nontrivial solution of (1.2) oscillates.

2. PRELIMINARIES

For the proof of our main convergence result in Section 4, we will need the following global convergence theorem established in [3] (see also [2]).

Theorem A ([3, Theorem 1.1]). *Let $F \in C[I^k, I]$ for some interval I of positive real numbers and for some natural number k. Then every positive solution of Eq. (1.1) has a finite limit in I if the following statements are true:*

(1) $F(z_1, \ldots, z_k)$ is nondecreasing in each of its arguments;
(2) $F(z_1, \ldots, z_k)$ is strictly increasing in each of the arguments $z_{i_1}, \ldots, z_{i_e}$, where $i_1, \ldots, i_e$ are relatively prime;
(3) $F(c, \ldots, c) = c$ for every $c \in I$.

The proof of this theorem is based on the method of full limiting sequences developed by G. Karakostas [4], and it may be found in [3]. It was applied there to investigate the periodic character of the difference equation

$$x_n = \frac{A_1}{x_{n-1}} + \cdots + \frac{A_k}{x_{n-k}}.$$

In this paper, we use Theorem A to study the asymptotic behavior of the solutions of Eq. (1.2).

3. Oscillation

Consider Eq. (1.2). Once the positive values x_{-3}, x_{-2}, and x_{-1} are specified, the solution of (1.2) is uniquely determined. As in [1, p. 3258], we define the equilibrium or trivial solution, $\overline{x}$, to be the solution of (1.2) obtained by setting $x_n \equiv c$, $\forall n$. For Eq. (1.2),

(3.1) $$\overline{x} = 2.$$

We will prove that any nontrivial solution of (1.2) oscillates around $\overline{x} = 2$. That is, there will be a "string" of consecutive values of x_n such that $x_n \geq 2$. This set of elements will be called a positive semicycle. This will be followed by another "string" of consecutive solution values x_n such that $x_n < 2$. This set of elements will be designated as a negative semicycle. Oscillation means that for any N, there exist positive and negative semicycles with elements x_n where $n > N$.

We have the following theorem.

Theorem 3.1. *Any nontrivial solution of Eq.* (1.2) *oscillates about the equilibrium solution* $\overline{x} = 2$. *Positive and negative semicycles have length at most* 5. *Moreover, examples of positive and negative semicycles of length* 5 *do exist.*

Proof. We will prove oscillation by showing there is an upper bound of 5 for the maximum number of elements in both a positive and a negative semicycle. We consider positive semicycles first and argue various cases.

Suppose $x_{n-5} = 2$. Consider x_{n-4}. If $x_{n-4} < 2$, then x_{n-4} begins a negative semicycle. Suppose $x_{n-4} = 2$. Then $x_{n-3} > 2$, since otherwise x_n is the trivial solution. From (1.2), if $x_{n-5} = x_{n-4} = 2$, then $x_{n-2} = 2$. However, since $x_{n-2} = 2$ and $x_{n-3} > 2$, (1.2) implies $x_n < 2$. Thus x_n starts a negative semicycle.

Next, suppose $x_{n-4} > 2$ (and $x_{n-5} = 2$). Then (1.2) implies $x_{n-2} > 2$. If $x_{n-3} < x_{n-4}$, then $x_{n-1} < 2$ and starts a negative semicycle. Thus, suppose $x_{n-3} \geq x_{n-4}$. Then $x_{n-1} \geq 2$. So we have $x_{n-5} = 2$, $x_{n-3} \geq x_{n-4} > 2$, $x_{n-2} > 2$, and $x_{n-1} \geq 2$.

Case 1: $x_{n-2} < x_{n-3}$. Then (1.2) implies $x_n < 2$.
Case 2: $x_{n-2} \geq x_{n-3}$. This means

$$x_{n-2} = 1 + \frac{x_{n-4}}{x_{n-5}} \geq x_{n-3} \geq x_{n-4}$$

$$\Rightarrow 1 + \frac{x_{n-4}}{2} \geq x_{n-4}$$

$$\Rightarrow x_{n-4} \leq 2, \quad \text{a contradiction.}$$

Hence, if $x_{n-5} = 2$, there can be at most 4 additional consecutive elements with solution values ≥ 2.

Suppose $x_{n-5} > 2$. If $x_{n-4} < x_{n-5}$, then $x_{n-2} < 2$. Thus, we assume $x_{n-4} \geq x_{n-5}$ (> 2). If $x_{n-3} < x_{n-4}$, then $x_{n-1} < 2$. So, assume $x_{n-3} \geq x_{n-4}$, which means $x_{n-1} \geq 2$. Then

$$x_n = 1 + \frac{x_{n-2}}{x_{n-3}} \leq 1 + \frac{x_{n-2}}{x_{n-4}} = 1 + \frac{1}{x_{n-4}} + \frac{1}{x_{n-5}} < 1 + \frac{1}{2} + \frac{1}{2} = 2,$$

which means x_n starts a negative semicycle.

In summary, if $x_{n-5} \geq 2$, there can be at most four consecutive additional solution elements with values ≥ 2.

Next we consider negative semicycles and begin by assuming $x_{n-5} < 2$. If $x_{n-4} \geq x_{n-5}$, then (1.2) implies $x_{n-2} \geq 2$, which marks the beginning of a positive semicycle. Thus, we may suppose $x_{n-4} < x_{n-5}$. In a similar fashion, we may assume $x_{n-3} < x_{n-4}$, which means $x_{n-1} < 2$. Then

$$x_n = 1 + \frac{x_{n-2}}{x_{n-3}} > 1 + \frac{x_{n-2}}{x_{n-4}} = 1 + \frac{1}{x_{n-4}} + \frac{1}{x_{n-5}} > 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Accordingly, if $x_{n-5} < 2$, there can be at most 4 additional consecutive solution elements with values < 2.

The previous arguments imply that any nontrivial solution of (1.2) oscillates and that positive and negative semicycles can have length at most 5. To exhibit a positive semicycle of length 5, let $x_{n-5} = 3$, $x_{n-4} = 4$, and $x_{n-3} = 4$. To exhibit a negative semicycle of length 5, let $x_{n-5} = 3/2$, $x_{n-4} = 5/4$, and $x_{n-3} = 9/8$. This completes the proof of Theorem 3.1. $\qquad\square$

4. PERIOD TWO SOLUTIONS

The main result in this section is the following convergence theorem.

Theorem 4.1. *Every positive solution of Eq.* (1.2) *converges to a period two solution.*

Proof. Let $\{x_n\}$ denote a positive solution of Eq. (1.2). By Eq. (1.2), we may write

$$\frac{x_n}{x_{n-2}} = \frac{1}{x_{n-2}} + \frac{1}{x_{n-3}}.$$

Using various iterates of Eq. (1.2), we obtain

$$x_{n+2} = 1 + \frac{x_n}{x_{n-1}} = 1 + \frac{x_n}{x_{n-2}} \cdot \frac{x_{n-2}}{x_{n-1}}$$

$$= 1 + \left(\frac{1}{x_{n-2}} + \frac{1}{x_{n-3}}\right) \frac{x_{n-2}}{x_{n-1}}$$

$$= 1 + \frac{1}{x_{n-1}} + \frac{1}{x_{n-3}(x_{n+1} - 1)}.$$

By substituting $(n-3)$ for $(n+2)$ in the last expression and utilizing Eq. (1.2) again, we obtain for $n \geq 5$ that

$$(4.1) \qquad x_n = 1 + \frac{x_{n-2}}{x_{n-3}} = 1 + \frac{x_{n-2}}{1 + \dfrac{1}{x_{n-6}} + \dfrac{1}{x_{n-8}(x_{n-4} - 1)}}.$$

Equation (4.1) is now in a form where we can consider n as being either even or odd. Specifically, for $q = 0, 1$, set

(4.2) $$n = 2s + q, \qquad y_s = x_{2s+q} - 1.$$

Then Eq. (4.1) becomes

(4.3) $$y_s = \frac{1 + y_{s-1}}{1 + \dfrac{1}{1 + y_{s-3}} + \dfrac{1}{y_{s-2}(y_{s-4} + 1)}}.$$

Observe that Eq. (4.3) has the same form as Eq. (1.1) with

$$F(z_1, z_2, z_3, z_4) = \frac{1 + z_1}{1 + \dfrac{1}{z_3 + 1} + \dfrac{1}{z_2(1 + z_4)}}$$

and $I = (0, \infty)$. It is clear that $F(z_1, z_2, z_3, z_4)$ is strictly increasing in all its arguments and

$$F(c, c, c, c) = c.$$

Therefore by Theorem A, y_n converges. Hence, by (4.2), it follows that the sequences $\{x_{2s}\}$ and $\{x_{2s+1}\}$ are both convergent, which completes the proof. $\qquad\square$

Corollary 4.2. *Equation* (1.2) *has infinitely many period 2 solutions. More precisely, every positive period two solution of Eq.* (1.2) *is generated by the initial conditions*

(4.4) $$x_{-1} = x_{-3} = a, \qquad x_{-2} = \frac{a}{a - 1}$$

for some $a > 1$.

Proof. It is easy to verify that Eq. (1.2) with the initial conditions (4.4), for any $a > 1$, generates a period two solution of Eq. (1.2). On the other hand, let $\{x_n\}$ be a period two solution of Eq. (1.2), *i.e.*,

(4.5) $$x_{2s} = a, \qquad x_{2s-1} = b \qquad \text{for every } s \geq -1.$$

Then, by Eq. (1.2), with $n = 2s$, we obtain

$$a = 1 + \frac{a}{b} > 1 \qquad \text{and} \qquad b = \frac{a}{a - 1},$$

which completes the proof. $\qquad\square$

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