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ON THE LOCAL SPECTRAL RADIUS OF POSITIVE OPERATORS

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ABSTRACT. We give some sufficient conditions for subadditivity and submultiplicativity of the local spectral radius of bounded positive linear operators.

1. INTRODUCTION

Let A be a bounded linear operator in a Banach space X. Denote by $r(A)$ and $r(A,x)$, $x \in X$, the spectral radius of A and the local spectral radius of A at x, respectively. Let us recall that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$$

and

$$r(A,x) = \limsup_{n\to\infty} \|A^n x\|^{1/n}.$$

Several interesting properties of $r(A)$ and $r(A,x)$ can be found for example in [1], [4], [5], [6], [7]. The aim of our paper is to establish some theorems on the local spectral radius of the sum and composition of two operators. The classical result states (see [5], [8]) that if A and B are bounded linear commutative operators, then $r(A+B) \le r(A) + r(B)$ and $r(AB) \le r(A)r(B)$. In [1], Daneš proved the analogous inequalities for local spectral radius. Precisely, he showed that if A and B are commutative, then for every $x \in X$

$$r(A+B,x) \le r(A,x) + r(B) \tag{1}$$

and

$$r(AB,x) \le r(A,x)r(B). \tag{2}$$

It is easy to show that the assumption of commutativity is essential, but it can be weakened in several ways. Some theorems of this type concerning operators in partially ordered Banach spaces can be found in [2], [9], [10].

It is of interest to know whether $r(B)$ can be replaced by $r(B,x)$ in (1) and (2), that is, if it is possible to obtain the inequalities

$$r(A+B,x) \le r(A,x) + r(B,x)$$

and

$$r(AB,x) \le r(A,x)r(B,x).$$

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Recall that $r(A,x) \le r(A)$ for all $x \in X$, $r(A) = \max\{r(A,x) : x \in X\}$ and the equality $r(A) = r(A,x)$ holds on the dense subset of X (see [1], [6], [7]). For the conditions ensuring $r(A) = r(A,x)$ we refer the reader to [3], [4]. Our purpose is to find conditions implying the above inequalities also in the case $r(A,x) < r(A)$ and $r(B,x) < r(B)$. It is well known that $r(AB) = r(BA)$ for any A and B. Since $r(AB,x)$ may be different from $r(BA,x)$, we will estimate both of them.

2. Main results

For the convenience of the reader, we begin this section with some definitions.

Definition 1. A nonempty subset K of a Banach space X is called a cone if K is closed and

(i) $\lambda x \in K$ for all $x \in K$ and $\lambda \ge 0$,
(ii) $x, y \in K$ implies $x + y \in K$,
(iii) $x, -x \in K$ implies $x = \theta$.

It is well known that K induces a partial order in X as follows: for $x, y \in X$ we say that $x \prec y$ if and only if $y - x \in K$ (see for example [5]).

Definition 2. A cone K is said to be normal if there exists $\gamma > 0$ such that if $\theta \prec x \prec y$, then $\|x\| \le \gamma \|y\|$.

Definition 3. Let K be a cone in X and let $A : X \to X$ be a bounded linear operator. We say that A is positive if $A(K) \subset K$.

Notice that $A(K) \subset K$ implies that A is increasing, that is, if $x \prec y$, then $Ax \prec Ay$.

Definition 4. Let K be a cone in X, $u_0 \in K$ and $u_0 \ne \theta$. We say that a bounded linear operator $A : X \to X$ is u_0-upper bounded if for every $x \in X$ there exists a nonnegative number $\alpha(x)$ such that $Ax \prec \alpha(x) u_0$.

Finally, recall some properties of local spectral radius which will be useful in our further considerations (see [1]):

$$r(A, \lambda x) = r(A,x) \text{ for all } x \in X \text{ and } \lambda \ne 0, \tag{3}$$

$$r(A, x+y) \le \max\{r(A,x), r(A,y)\} \text{ for all } x, y \in X, \tag{4}$$

$$r(A, A^k x) = r(A,x) \text{ for all } x \in X \text{ and } k \in \mathbb{N}. \tag{5}$$

Now, we state and prove a few theorems on subadditivity and submultiplicativity of local spectral radius of positive operators. From now on we assume that K is a normal cone in a Banach space X and A and B are bounded positive linear operators.

Theorem 1. *Suppose that A is u_0-upper bounded and*

$$BA^i B^j u_0 \prec A^i B^{j+1} u_0 \tag{6}$$

for $i = 1, 2, \ldots,$ and $j = 0, 1, 2, \ldots.$ Then

$$r(A+B, u_0) \le r(A, u_0) + r(B, u_0), \tag{7}$$

$$r(AB, u_0) \le r(A, u_0) r(B, u_0) \text{ and } r(BA, u_0) \le r(A, u_0) r(B, u_0). \tag{8}$$

Proof. The method of the proof is partly based on that used in [2] for the spectral radius. Since A is u_0-upper bounded and positive, for every $x \in K$ there exists $\alpha(x) \geq 0$ such that

$$\theta \prec A^n x \prec \alpha(x) A^{n-1} u_0$$

for all $n \in \mathbb{N}$. Hence

$$\|A^n x\| \leq \gamma \alpha(x) \|A^{n-1} u_0\|$$

which gives

$$r(A, x) \leq r(A, u_0) \tag{9}$$

for every $x \in K$. Let $\epsilon > 0$ and with $\delta_B = r(B, u_0) + \epsilon/2$ put

$$u = \sum_{i=0}^{\infty} \delta_B^{-(i+1)} B^i u_0, \tag{10}$$

evidently well defined. Observe that $u \in K$ and $u = \delta_B^{-1}(u_0 + Bu)$. Hence

$$Bu \prec \delta_B u \tag{11}$$

and

$$u_0 \prec \delta_B u. \tag{12}$$

From (12) it follows that $r(A, u_0) \leq r(A, u)$ which with (9) gives

$$r(A, u) = r(A, u_0). \tag{13}$$

Next, put $\delta_A = r(A, u_0) + \epsilon/2$ and define

$$v = \sum_{i=0}^{\infty} \delta_A^{-(i+1)} A^i u. \tag{14}$$

It is easy to see that $v \in K$, $u \prec \delta_A v$, $Av \prec \delta_A v$ and $u_0 \prec \delta_A \delta_B v$. Moreover, by (6) and (11) we have

$$Bv = \sum_{i=0}^{\infty} \delta_A^{-(i+1)} BA^i u \prec \sum_{i=0}^{\infty} \delta_A^{-(i+1)} A^i Bu \prec \sum_{i=0}^{\infty} \delta_A^{-(i+1)} \delta_B A^i u = \delta_B v.$$

Then

$$(A+B)u_0 \prec \delta_A \delta_B (A+B) v \prec \delta_A \delta_B (\delta_A + \delta_B) v,$$

$$(A+B)^2 u_0 \prec \delta_A \delta_B (\delta_A + \delta_B)(A+B) v \prec \delta_A \delta_B (\delta_A + \delta_B)^2 v,$$

and generally for all $n \in \mathbb{N}$

$$(A+B)^n u_0 \prec \delta_A \delta_B (\delta_A + \delta_B)^n v.$$

Thus

$$\|(A+B)^n u_0\| \leq \gamma \delta_A \delta_B (\delta_A + \delta_B)^n \|v\|,$$

and consequently

$$r(A+B, u_0) \leq \delta_A + \delta_B.$$

This gives (7), since $\epsilon > 0$ is arbitrary. To prove (8), observe that $ABv \prec \delta_A \delta_B v$ and $ABu_0 \prec \delta_A \delta_B v$. Thus,

$$(AB)^n u_0 \prec (\delta_A \delta_B)^n v$$

for all $n \in \mathbb{N}$. Hence

$$\|(AB)^n u_0\| \le \gamma(\delta_A \delta_B)^n \|v\|,$$

which gives $r(AB, u_0) \le \delta_A \delta_B$. The last inequality clearly implies $r(AB, u_0) \le r(A, u_0) r(B, u_0)$. In the same manner we can see that $(BA)^n u_0 \prec (\delta_A \delta_B)^n v$ and in consequence (8) holds. This completes the proof of Theorem 1.

It is worth mentioning that assumption (6) is fulfilled if for example $AB = BA$ or $BAx \prec ABx$ for every $x \in K$.

Example 1. Let $X = c_0$ be a Banach space of all real sequences convergent to zero with supremum norm

$$\|x\| = \sup_i |x_i|.$$

Let $u_0 = \{1/4^i\}$ and $K = \{x \in c_0 : x = tu_0, t \ge 0\}$. Obviously, K is a normal cone in c_0. For $x \in c_0$, $x = \{x_i\}$, consider the operators $Ax = (x_2, x_3, x_4, ...)$, $Bx = \frac{1}{2}(x_3, x_4, x_5, ...)$. It is easy to show that A and B satisfy the assumptions of Theorem 1. Particularly, $B = \frac{1}{2}A^2$, so (6) is fulfilled. Furthermore, if $x \in K$, then $x = tu_0$, where $t \ge 0$. Hence $Ax = tAu_0 = \frac{1}{4}tu_0$, which means that A is u_0-upper bounded. Since $r(A, u_0) = \frac{1}{4}$ and $r(B, u_0) = \frac{1}{32}$, by Theorem 1 we get $r(A+B, u_0) \le \frac{9}{32}$ and $r(AB, u_0) \le \frac{1}{128}$. Observe that in this case we have $r(A) = 1$ and $r(B) = \frac{1}{2}$.

Theorem 2. *Suppose that there exists $u_0 \in K$ such that $Au_0 \prec r(A, u_0)u_0$ and*

$$BA^i B^j u_0 = A^i B^{j+1} u_0 \tag{15}$$

for $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$. Then (7) and (8) hold.

Proof. First, observe that $A^n u_0 \prec [r(A, u_0)]^n u_0$ for all $n \in \mathbb{N}$. For the element u defined by (10) we have

$$A^n u = \sum_{i=0}^{\infty} \delta_B^{-(i+1)} B^i A^n u_0 \prec [r(A, u_0)]^n u.$$

This gives $r(A, u) \le r(A, u_0)$. In consequence, $r(A, u) = r(A, u_0)$. The rest of the proof runs as before.

Theorem 3. *Suppose that for $u_0 \in K$, (6) is satisfied and*

$$Bu_0 \prec r(B, u_0)u_0. \tag{16}$$

Then the inequalities (7) and (8) hold.

Proof. From (16), we obtain for all $i \in \mathbb{N}$

$$B^i u_0 \prec [r(B, u_0)]^i u_0.$$

Therefore for u defined by (10) we have

$$u \prec \sum_{i=0}^{\infty} \delta_B^{-(i+1)} [r(B, u_0)]^i u_0 = \frac{2}{\epsilon} u_0,$$

which gives $r(A, u) \le r(A, u_0)$. We can now proceed analogously to the proof of Theorem 1.

Example 2. Let $X = c_0$ and $K = \{x \in c_0 : x_i \geq 0, i \in \mathbb{N}\}$. Obviously, K is a normal cone in c_0. Define $Ax = (x_3, x_4, x_5, ...)$ and $Bx = (x_2 + x_3, 2x_3, 2x_4, ...)$. For $u_0 = \{1/3^i\}$ we have $r(A, u_0) = \frac{1}{9}$, $r(B, u_0) = \frac{2}{3}$. It is clear that $Bu_0 \prec r(B, u_0)u_0$. Moreover, for $i = 1, 2, \dots,\ j = 0, 1, \dots,$ we get

$$BA^iB^ju_0 = 2^j\Big(\frac{1}{3^{j+2i+2}} + \frac{1}{3^{j+2i+3}}, \frac{2}{3^{j+2i+3}}, \frac{2}{3^{j+2i+4}}, \dots\Big)$$

and

$$A^iB^{j+1}u_0 = 2^{j+1}\Big(\frac{1}{3^{j+2i+2}}, \frac{1}{3^{j+2i+3}}, \frac{1}{3^{j+2i+4}}, \dots\Big).$$

It follows that (6) is satisfied. By Theorem 3, $r(A+B, u_0) \leq \frac{7}{9}$ and $r(AB, u_0) \leq \frac{2}{27}$. Notice that $r(A) = 1$ and $r(B) = 2$.

Theorem 4. *If* (15) *is satisfied and there exists* $k \in \mathbb{N}$*, such that*

$$B^{k+1}u_0 \prec r(B, u_0)B^ku_0, \tag{17}$$

then (7) *and* (8) *hold.*

Proof. For u defined by (10) we get from (17)

$$\begin{aligned} u &= \sum_{i=0}^{k} \delta_B^{-(i+1)} B^i u_0 + \sum_{j=1}^{\infty} \delta_B^{-(k+j+1)} B^{k+j} u_0 \\ &\prec \sum_{i=0}^{k} \delta_B^{-(i+1)} B^i u_0 + \sum_{j=1}^{\infty} \delta_B^{-(k+j+1)} [r(B, u_0)]^j B^k u_0 \\ &= \sum_{i=0}^{k} \delta_B^{-(i+1)} B^i u_0 + \frac{2}{\epsilon} \delta_B^{-(k+1)} r(B, u_0) B^k u_0. \end{aligned}$$

Put

$$v_0 = \sum_{i=0}^{k} \delta_B^{-(i+1)} B^i u_0 + \frac{2}{\epsilon} \delta_B^{-(k+1)} r(B, u_0) B^k u_0. \tag{18}$$

Since $A^nB^iu_0 = B^iA^nu_0$ for all $i, n \in \mathbb{N}$, we obtain

$$r(A, B^iu_0) \leq r(A, u_0).$$

Thus, by (3) and (4), $r(A, v_0) \leq r(A, u_0)$. Moreover, $u \prec v_0$ implies $r(A, u) \leq r(A, v_0)$. Therefore $r(A, u) \leq r(A, u_0)$ and the rest of the proof runs as before. □

In a similar way we can prove the following result.

Theorem 5. *If* (6) *and* (17) *are satisfied and* $Bu_0 \prec Au_0$*, then the inequalities* (7) *and* (8) *hold.*

Proof. It suffices to observe that in view of (6) and (17) we have $u \prec v_0$, where u and v_0 are defined by (10) and (18), respectively. Since $Bu_0 \prec Au_0$, we get from (6) $B^iu_0 \prec A^iu_0$ for all $i \in \mathbb{N}$. Thus

$$v_0 \prec \sum_{i=0}^{k} \delta_B^{-(i+1)} A^i u_0 + \frac{2}{\epsilon} \delta_B^{-(k+1)} r(B, u_0) A^k u_0.$$

By (3)–(5), we get $r(A, v_0) \leq r(A, u_0)$. As in the proof of Theorem 4, this gives $r(A, u) \leq r(A, u_0)$. The same type of analysis as that in the proof of Theorem 1 establishes the result.

Example 3. In the space c_0 consider the cone $K = \{x \in c_0 : x_i \geq 0, i \in \mathbb{N}\}$, the operators $Ax = (x_2 + x_3, x_3, x_4, ...)$ and $Bx = (x_3, x_4, x_5, ...)$ and the element $u_0 = (0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$. Observe that $BAx = (x_4, x_5, x_6, ...)$ and $ABx = (x_4 + x_5, x_5, x_6, ...)$, so $BAx \prec ABx$ for every $x \in K$ which implies (6). Furthermore, $r(A, u_0) = \frac{1}{2}$, $r(B, u_0) = \frac{1}{4}$, $B^2 u_0 \prec r(B, u_0) B u_0$ and $Bu_0 \prec Au_0$. By Theorem 5, $r(A+B, u_0) \leq \frac{3}{4}$ and $r(AB, u_0) \leq \frac{1}{8}$.

Finally, observe that u_0 can be replaced in (7) by any $x \in X$ such that

$$r(A + B, u_0) = r(A + B, x), \quad r(A, u_0) = r(A, x) \quad \text{and} \quad r(B, u_0) = r(B, x).$$

An analogous conclusion holds for (8). Particularly, by (3), the inequalities (7) and (8) hold for $x = \lambda u_0$, $\lambda \in \mathbb{R}$. We shall prove that (7) and (8) are also satisfied for the elements u and v defined in the proof of Theorem 1.

Corollary. *If the assumptions of Theorem* 1 *are satisfied, then* (7) *and* (8) *hold for* u *and* v *defined by* (10) *and* (14)*, respectively.*

Proof. From (12) it follows that A is u-upper bounded. It is easy to check that (6) holds with u_0 replaced by u. Thus, by Theorem 1, inequalities (7) and (8) hold for u. In the same manner we get the conclusion for v. □

Remark. Obviously (7) and (8) are also satisfied for any partial sum of the series (10) and (14). It is clear that the similar corollaries can be derived from Theorems 2–5. Moreover, observe that if (6) holds, then

$$r(A + B, u_0) \leq r(A) + r(B, u_0),$$

$$r(AB, u_0) \leq r(A) r(B, u_0) \text{ and } r(BA, u_0) \leq r(A) r(B, u_0).$$

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