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FARRELL SETS FOR HARMONIC FUNCTIONS

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Abstract. Let F denote a relatively closed subset of the unit ball B of $\mathbb{R}^n$. The purpose of this paper is to characterize those sets F which have the following property: any harmonic function h on B which satisfies $|h| \leq M$ on F (where $M > 0$) can be locally uniformly approximated on B by a sequence of harmonic polynomials which satisfy the same inequality on F. This answers a question posed by Stray, who had earlier solved the corresponding problem for holomorphic functions on the unit disc.

1. Introduction

Let B denote the open unit ball of Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. If h is a harmonic function on B, then it is an elementary fact that there is a sequence (p_m) of harmonic polynomials which converges to h locally uniformly on B: for example, we could take (p_m) to be the sequence of partial sums of the series expansion of h in terms of homogeneous polynomials (see Theorem 2.4.4 of [1]). Now let F be a relatively closed subset of B. If a harmonic function h on B satisfies $|h| \leq M$ on F for some positive number M, does it follow that we can choose the approximating sequence (p_m) such that $|p_m| \leq M$ on F for each m? The answer depends on the properties of F. A set F for which this implication holds will be called a *Farrell set for harmonic functions.* The problem of characterizing such sets was posed by Stray [17], who had previously solved the corresponding problem for holomorphic functions on the unit disc (see [14]). A complete characterization is given in this paper.

First we need some topological definitions. If ω is an open subset of B, then we say that the sphere ∂B is *accessible* from ω if there is a continuous function $\Gamma : [0, +\infty) \to \omega$ such that $\|\Gamma(t)\| \to 1$ as $t \to +\infty$. Given a relatively closed subset F of B, we define $F^{\sim}$ to be the union of F with the (connected) components of $B\backslash F$ from which ∂B is not accessible. Also, given a compact set K in $\mathbb{R}^n$, we define $K^{\wedge}$ to be the union of K with the bounded components of $\mathbb{R}^n \backslash K$. Our main result is as follows.

Theorem 1. *Let F be a relatively closed subset of B. Then F is a Farrell set for harmonic functions if and only if $\left(\overline{F} \cup K\right)^{\wedge} \cap B = (F \cup K)^{\sim}$ for every compact subset K of B.*

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It is interesting that the topological condition in Theorem 1 coincides with that obtained by Stray [14] in connection with Farrell sets for holomorphic functions, even though the two proofs have little in common and the theory of harmonic approximation differs significantly from its holomorphic counterpart (see [4]). This topological characterization also arises in connection with "Mergelyan sets" for holomorphic and harmonic functions (see [14] and [5]). We note that many authors have studied (an adapted notion of) Farrell sets for various classes of holomorphic and harmonic functions; see [2], [3], [6], [7], [8], [9], [10], [11], [12], [15] and [16].

2. Proof of Theorem 1

2.1. Suppose that $\widehat{(\overline{F} \cup K)} \cap B = \widetilde{(F \cup K)}$ for every compact subset K of B, let h be a harmonic function on B, and suppose that $|h| \leq M$ on F for some positive number M. Further, let $\varepsilon > 0$ and $0 < r < 1$, and let $B(\rho)$ denote the open ball of centre 0 and radius ρ. The "if" part of Theorem 1 will be established if we can show that there is a harmonic polynomial $q_{\varepsilon,r}$ such that

$$|h - q_{\varepsilon,r}| < \varepsilon \text{ on } B(r) \text{ and } |q_{\varepsilon,r}| \leq M + \varepsilon \text{ on } F; \tag{1}$$

for, if this were established, we could then define

$$p_m = \frac{M}{M + \varepsilon_m} q_{\varepsilon_m, 1-1/m},$$

where $\varepsilon_m \in (0, 1)$ is chosen small enough to satisfy

$$\left(1 + \sup\left\{|h(x)| : x \in B(1 - m^{-1})\right\}\right) < \frac{1}{m\varepsilon_m} \qquad (m \in \mathbb{N}).$$

This would then yield $|p_m| \leq M$ on F and

$$\begin{aligned} |h - p_m| &\leq \left|h - q_{\varepsilon_m, 1-1/m}\right| + \left|q_{\varepsilon_m, 1-1/m}\right| \left(1 - \frac{M}{M + \varepsilon_m}\right) \\ &< \varepsilon_m + \frac{1}{m\varepsilon_m} . \frac{\varepsilon_m}{M + \varepsilon_m} \\ &< \frac{1}{m}\left(1 + \frac{1}{M}\right) \qquad \text{on } B(1 - m^{-1}), \end{aligned}$$

so $p_m \to h$ locally uniformly on B as required.

2.2. Next we record a simple lemma which shows that B itself is a Farrell set for harmonic functions.

Lemma 1. *If h is a harmonic function on B and $|h| \leq M$ on B, then there is a sequence (p_m) of harmonic polynomials such that $p_m \to h$ locally uniformly on B and $|p_m| \leq M$ on B for each m.*

To see this, let $\varepsilon > 0$ and $0 < r < 1$, and define $h_\rho(x) = h(\rho x)$ when $\rho \in (0, 1)$ and $x \in B(\rho^{-1})$. Since $h_\rho \to h$ locally uniformly on B as $\rho \to 1-$, we can choose ρ_0 such that $|h_{\rho_0} - h| < \varepsilon/2$ on $B(r)$. Next we truncate the homogeneous polynomial expansion of h_{ρ_0} in $B(\rho_0^{-1})$ to obtain a harmonic polynomial q such that $|q - h_{\rho_0}| < \varepsilon/2$ on B, whence $|q| < M + \varepsilon/2$ on B and $|q - h| < \varepsilon$ on $B(r)$. Lemma 1 now follows in view of the observation in §2.1.

2.3. The first main step in proving (1) is to establish a representation formula for h on a neighbourhood of $(F \cup K)^{\sim}$, where $K = \overline{B(r)}$. Let $r_0 \in (r, 1)$ and $U = (F \cup K)^{\sim} \setminus (F \cup K)$, and define

$$V_0 = \{x \in B : |h(x)| < M + \varepsilon/5\} \cup U \cup B(r_0).$$

Then V_0 is an open subset of B which contains the relatively closed subset $(F \cup K)^{\sim}$ of B. We can now choose V to be an open set which is regular for the Dirichlet problem and satisfies $(F \cup K)^{\sim} \subset V \subseteq V_0$. In fact, we can arrange that $\partial V \cap B$ is smooth.

We extend h to $\overline{B}$ by assigning it the value 0 on ∂B and define

$$u = \begin{cases} h - H_h^V & \text{on } V, \\ 0 & \text{on } B \backslash V, \end{cases}$$

where H_f^V denotes the Perron-Wiener-Brelot solution of the Dirichlet problem on V with boundary data f. Since V is regular and h is bounded on ∂V and continuous on $\partial V \cap B$, the function u is continuous on B. Further, it follows from the generalized maximum principle (see Theorem 3.1.10 of [1]) that h is bounded on U, and so u is bounded on B. Thus u^+ and u^- are bounded subharmonic functions on B. By the Riesz decomposition theorem there are bounded Borel measurable functions u_*^+, u_*^- on ∂B and potentials w_1, w_2 on B such that $u^+ = I_{u_*^+} - w_1$ and $u^- = I_{u_*^-} - w_2$ on B, where I_f denotes the Poisson integral in B of a Borel measurable function f on ∂B.

We now define $u_* = u_*^+ - u_*^-$ and

$$v = \begin{cases} I_{u_*} & \text{on } B, \\ 0 & \text{on } \partial B. \end{cases}$$

Thus

$$u = v - w_1 + w_2 \quad \text{on } B. \tag{2}$$

Since V is regular for the Dirichlet problem and $w_2 - w_1$ is harmonic on V, and since $u = 0$ on $\partial V \cap B$, it follows from (2) that

$$0 = H_v^V - w_1 + w_2 = H_v^V + u - I_{u_*} \quad \text{on } V$$

(see Theorem 7.5.2 of [1]). Hence

$$h = u + H_h^V = I_{u_*} + H_{h-v}^V \quad \text{on } V, \tag{3}$$

which is the representation for h that we sought.

2.4. We will now construct the approximating polynomial $q_{\varepsilon,r}$ in (1). To begin with, we claim that there exists $r_1 \in (r_0, 1)$ such that

$$|h(x)| < M + \varepsilon/5 \quad (x \in V; \|x\| \geq r_1). \tag{4}$$

To see this, let h_0 denote the solution to the Dirichlet problem on U with boundary data

$$f(x) = \begin{cases} \dfrac{r_0 - \|x\|}{r_0 - r} h(x) & \text{on } \partial U \cap B(r_0), \\ 0 & \text{on } \partial U \backslash B(r_0) \end{cases}$$

(we note that $\partial U \cap B(r) = \emptyset$). Then $h_0(x) \to 0$ as $\|x\| \to 1$, where $x \in U$, so we can choose $r_1 \in (r_0, 1)$ such that

$$|h_0(x)| < \varepsilon/5 \quad (\|x\| \geq r_1, x \in U).$$

Since $\partial U \cap B \subseteq F \cup \partial B(r)$, we have

$$\limsup_{x \to z, x \in U} |h(x) - h_0(x)| \leq M$$

for all z in $\partial U \cap B$ apart from the polar set of irregular boundary points of U. It follows easily from the definition of U and the generalized maximum principle that $|h - h_0| \leq M$ on U, and so (4) holds.

Next, for each ρ in $(\frac{1}{2}, 1)$, we define a function f_ρ on ∂V by

$$f_\rho(x) = \begin{cases} (h-v)(x) & \text{on } \partial V \cap B(2\rho - 1), \\ \dfrac{\rho - \|x\|}{1-\rho}(h-v)(x) & \text{on } \partial V \cap (B(\rho) \backslash B(2\rho-1)), \\ 0 & \text{on } \partial V \backslash B(\rho), \end{cases}$$

and extend it to $\overline{V}$ by solving the Dirichlet problem on V. It follows from dominated convergence that the bounded harmonic functions f_ρ converge pointwise on V to H^V_{h-v} as $\rho \to 1-$. Let $r_2 \in (r_1, 1)$ and define

$$s_m = \begin{cases} \left|H^V_{h-v} - f_{1-1/m}\right| & \text{on } V, \\ 0 & \text{on } B \backslash V \end{cases} \qquad (m \in \mathbb{N}).$$

Then, for all sufficiently large m, the function s_m is subharmonic on $B(r_2)$, and $s_m \to 0$ pointwise on B as $m \to \infty$. Let S_k denote the upper semicontinuous regularization of $\sup_{m \geq k} s_m$. Then (S_k) is a decreasing sequence of subharmonic functions on $B(r_2)$ which has limit 0 quasi-everywhere (that is, outside some polar set), and hence everywhere on this ball. It follows from Dini's theorem that this sequence of upper semicontinuous functions converges uniformly on the compact set $\overline{B(r_1)}$. Thus we can choose r_3 in the interval $(r_1, 1)$ such that

$$\left|H^V_{h-v} - f_{r_3}\right| < \varepsilon/5 \quad \text{on } V \cap \overline{B(r_1)}. \tag{5}$$

Let σ denote surface area measure on ∂B. By Lusin's theorem (see, for example, Theorem 2.23 of Rudin [13]) there is a continuous function $g : \partial B \to \mathbb{R}$ such that $\sup_{\partial B} |g|$ does not exceed the essential supremum of $|u_*|$ and such that $\sigma(\{z \in \partial B : u_*(z) \neq g(z)\})$ is arbitrarily small. In particular, we can choose g such that

$$|I_{u_*} - w_g| < \varepsilon/5 \text{ on } \overline{B(r_3)}, \tag{6}$$

where

$$w_g = \begin{cases} g & \text{on } \partial B, \\ I_g & \text{on } B. \end{cases}$$

We note that $w_g + f_{r_3}$ is harmonic on V, which coincides with $\left(\overline{V}\right)^\circ$, and continuous on $\overline{V}$. Our choice of V ensures that $\mathbb{R}^n \backslash \overline{V}$ is non-thin at each point of $\partial\overline{V}$. (We refer to Chapter 7 of [1] for an account of the notion of thinness.) Thus we can appeal to the Keldyš-Deny approximation theorem (Theorem 7.9.5 of [1], or Theorem 1.3 of [4]) to see that there is a harmonic function H on some neighbourhood of $\overline{V}$ such that

$$|(w_g + f_{r_3}) - H| < \varepsilon/5 \quad \text{on } \overline{V}. \tag{7}$$

In particular, H is harmonic on an open set containing the closure of $(F \cup K)^\sim$, and hence the set $\left(\overline{F} \cup K\right)^\frown$, by our hypothesis. Since this latter set has a connected complement in $\mathbb{R}^n$, we can apply Walsh's theorem (Corollary 2.6.5 of [1], or Theorem 1.7 of [4]) to see that there is a harmonic polynomial $q_{\varepsilon,r}$ such that

$$|H - q_{\varepsilon,r}| < \varepsilon/5 \quad \text{on} \quad \left(\overline{F} \cup K\right)^{\widehat{}}. \tag{8}$$

If we now combine (5) - (8) with (3), we obtain

$$|h - q_{\varepsilon,r}| = \left|I_{u_*} + H_{h-v}^V - q_{\varepsilon,r}\right| < \varepsilon \quad \text{on} \quad \left(\overline{F} \cup K\right)^{\widehat{}} \cap \overline{B(r_1)}. \tag{9}$$

Since $K = \overline{B(r)}$ and $r < r_1$, the first inequality in (1) is established.

2.5. It remains to establish the second inequality in (1). From (9) we see that

$$|q_{\varepsilon,r}| \leq M + \varepsilon \quad \text{on} \quad F \cap \overline{B(r_1)}. \tag{10}$$

To establish the same inequality on $F \backslash \overline{B(r_1)}$ we will make use of the minimal fine topology (on $\overline{B}$), an account of which may be found in Chapter 9 of [1].

The first step is to establish a suitable bound on I_{u_*}. We note from (4) that h is bounded on V, so we can define $M_1 = \sup_V |h|$. Then $|u| \leq 2M_1$ on B by the definition of u, and so we can arrange that $|u_*| \leq 2M_1$ on ∂B, since the Fatou-Naïm-Doob theorem shows that u_*^+ (respectively u_*^-) is determined at σ-almost every point z of ∂B by the minimal fine limit of the bounded subharmonic function u^+ (respectively u^-) at z. Hence, by (3),

$$|h - I_{u_*}| = \left|H_{h-v}^V\right| \leq 3M_1 R_1^{B\backslash V} \quad \text{on} \quad V,$$

where $R_1^{B\backslash V}$ denotes the reduced function (réduite) of the constant function 1 with respect to $B \backslash V$ and superharmonic functions on B. (Our choice of V ensures that this reduced function is already superharmonic, and so regularization is unnecessary.) The greatest harmonic minorant of $R_1^{B\backslash V}$ in B is given by I_χ, where χ is the function valued 1 at the points of ∂B where $B \backslash V$ is not minimally thin and 0 elsewhere on the sphere (see Lemma 9.3.6 of [1]). At σ-almost every point of ∂B where $B \backslash V$ is minimally thin, $R_1^{B\backslash V}$ thus has minimal fine limit 0 by the Fatou-Naïm-Doob theorem, and the same must therefore be true of $h - I_{u_*}$. Hence $|u_*| \leq M + \varepsilon/5$ at almost every such point, in view of (4). On the other hand, the definition of u ensures that $u_* = 0$ at σ-almost every point z in ∂B where $B \backslash V$ is not minimally thin. Thus $|I_{u_*}| \leq M + \varepsilon/5$ on B and $|g| \leq M + \varepsilon/5$ on ∂B.

Now we will use this bound to estimate $w_g + f_{r_3}$ on $V \backslash \overline{B(r_1)}$. From (5), (6), (3) and then (4) we see that

$$|w_g + f_{r_3}| \leq |h| + 2\varepsilon/5 < M + 3\varepsilon/5 \quad \text{on} \quad V \cap \partial B(r_1). \tag{11}$$

If $x \in \partial V \cap (B(r_3) \backslash B(r_1))$, and if $f_{r_3}(x)$ and $I_{u_*}(x)$ have opposite signs and satisfy $|f_{r_3}(x)| \leq |I_{u_*}(x)|$, then

$$|f_{r_3}(x) + I_{u_*}(x)| \leq |I_{u_*}(x)| \leq M + \varepsilon/5 \tag{12}$$

by the previous paragraph. At other points x of $\partial V \cap (B(r_3) \backslash B(r_1))$ we observe from the definition of f_{r_3} and (4) that

$$|f_{r_3}(x) + I_{u_*}(x)| \leq |(h - v + I_{u_*})(x)| = |h(x)| \leq M + \varepsilon/5. \tag{13}$$

From (12), (13) and (6) we thus have

$$|w_g + f_{r_3}| \leq M + 2\varepsilon/5 \quad \text{on} \quad \partial V \cap (B(r_3) \backslash B(r_1)). \tag{14}$$

On $\partial V \backslash B(r_3)$ we have $f_{r_3} = 0$. Thus

$$|w_g + f_{r_3}| = |w_g| \leq M + \varepsilon/5 \quad \text{on} \quad \partial V \backslash B(r_3), \tag{15}$$

since $|g| \leq M + \varepsilon/5$. From (11), (14), (15) and the maximum principle we see that

$$|w_g + f_{r_3}| \leq M + 3\varepsilon/5 \qquad \text{on} \;\; V \backslash \overline{B(r_1)}.$$

Combining this with (7) and (8), and then (10), we obtain the second inequality of (1), as required. The proof of the "if" part of Theorem 1 is now complete.

2.6. The "only if" part of Theorem 1 is more straightforward. Let F be a Farrell set for harmonic functions and suppose that there exists a compact subset K of B such that $\left(\overline{F} \cup K\right)\widehat{\ } \cap B \neq (F \cup K)\widetilde{\ }$. Then there is a bounded component W of $\mathbb{R}^n \backslash \left(\overline{F} \cup K\right)$ from which ∂B is accessible. Let $x_0 \in W$. By Corollary 3.1.11 of [1] there is a harmonic function h on B such that $|h| \leq 1$ on $B \backslash W$ and $|h(x_0)| > 1$. Since F is a Farrell set, we can find a sequence of harmonic polynomials (q_m) such that $|q_m| \leq 1$ on F and $q_m \to h$ uniformly on $K \cup \{x_0\}$. However, $\partial W \subseteq \overline{F} \cup K$ and $|h| \leq 1$ on K, so we arrive at a contradiction to the maximum principle. It follows that $\left(\overline{F} \cup K\right)\widehat{\ } \cap B = (F \cup K)\widetilde{\ }$ for all compact subsets K of B.

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