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FUCHS' PROBLEM 34 FOR MIXED ABELIAN GROUPS

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Abstract. This paper investigates the extent to which an Abelian group A is determined by the homomorphism groups $\mathrm{Hom}(A, G)$. A class $\mathcal{C}$ of Abelian groups is a *Fuchs 34 class* if A and C in $\mathcal{C}$ are isomorphic if and only if $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all $G \in \mathcal{C}$. Two p-groups A and C satisfy $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all groups G if and only if they have the same n^{th}-Ulm-Kaplansky-invariants and the same final rank. The mixed groups considered in this context are the adjusted cotorsion groups and the class $\mathcal{G}$ introduced by Glaz and Wickless. While $\mathcal{G}$ is a Fuchs 34 class, the class of (adjusted) cotorsion groups is not.

1. Introduction

Fuchs' Problem 34 asks whether there exists a set $\mathcal{X}$ of Abelian groups such that $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all $G \in \mathcal{X}$ implies $A \cong B$. Goeters and the author showed in [3] that, for every set $\mathcal{X}$ of p-groups, there exist non-isomorphic totally projective p-groups A and C such that $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all $G \in \mathcal{X}$. Nevertheless, reduced p-groups A and C such that $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all cyclic groups G have the same n^{th}-Ulm-Kaplansky invariants for all $n < \omega$ provided that one assumes GCH [3].

A class $\mathcal{C}$ of Abelian groups is a *Fuchs 34 class* if A and C in $\mathcal{C}$ are isomorphic if and only if $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all $G \in \mathcal{C}$. Corollary 2.3 shows that the classes of torsion-complete and Σ-cyclic p-groups are Fuchs 34. On the other hand, the class of totally projective p-groups is not Fuchs 34: Reduced p-groups A and C satisfy $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all p-groups G if and only if they have the same n^{th}-Ulm-Kaplansky-invariants for all $n < \omega$ and the same final rank (Theorem 2.4). In particular, the Σ-cyclic p-groups are the largest Fuchs 34 class of p-groups which contains all cyclic groups and is closed with respect to direct sums. Theorem 2.7 shows that there exist non-isomorphic adjusted cotorsion groups A and C such that $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all cotorsion groups G. Thus, the class of (adjusted) cotorsion groups is not Fuchs 34. The author wants to thank K. Rangaswamy for suggesting the discussion of cotorsion groups in the context of Problem 34.

The class $\mathcal{G}$ of mixed Abelian groups introduced by Glaz and Wickless in [6] is a Fuchs 34 class (Corollary 3.5). Its elements are the pure subgroups of $\Pi_p G_p$ with finite torsion-free rank such that $|G_p| < \infty$ for all primes p and $\mathrm{Hom}(G, tG)$ is

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torsion. Homological properties of the groups in $\mathcal{G}$ were investigated in [1], and we will refer to the results of this paper frequently. As a consequence of our discussions, we obtain a complete set of numerical invariants for the groups in $\mathcal{G}$ (Theorem 3.3 and Corollary 3.4).

2. p-groups

In order to avoid the set-theoretic difficulties in [3], Fuchs suggested replacing the requirement $\mathrm{Hom}(A,G) \cong \mathrm{Hom}(C,G)$ for all $G \in \mathcal{X}$ by $\mathrm{Hom}(A, \oplus_I G) \cong \mathrm{Hom}(C, \oplus_I G)$ for all index sets I and all $G \in \mathcal{X}$ since cardinals α and β such that $\kappa^\alpha = \kappa^\beta$ for all infinite cardinals κ satisfy $\alpha = \beta$. The author wants to thank Gary Gruenhage for pointing out that this result holds in ZFC.

Every unbounded Σ-cyclic p-group B can be written as $B = \bigoplus_{n<\omega} B_n$ with $B_n = \bigoplus_{\kappa_n} \mathbb{Z}/p^{s_n}\mathbb{Z}$ for $n < \omega$ where κ_n is a non-zero cardinal and $0 < s_0 < s_1 < \dots$ are integers. By [5, Corollary 33.3], B is a basic subgroup of the torsion subgroup of $A = \prod_{n<\omega} B_n$. Choose a free subgroup F of A such that $B \oplus F$ is a p-basic subgroup of A. The rank of F was incorrectly given as $\prod_{n<\omega} \kappa_n$ in [5].

For any sequence $\{\alpha_n\}_{n<\omega}$ of cardinals, let $\pi(\{\alpha_n\}) = inf\{\prod_{i=n}^\infty 2\alpha_i \mid n < \omega\}$ and $\sigma(\{\alpha_n\}) = inf\{\sum_{i=n}^\infty \alpha_i \mid n < \omega\}$. There is $m_0 < \omega$, such that $\pi(\{\alpha_n\}) = \prod_{i=m}^\infty 2\alpha_i$ and $\sigma(\{\alpha_n\}) = \sum_{i=m}^\infty \alpha_i$ for all $m \geq m_0$. Finally, the symbol $r_0(G)$ denotes the torsion-free rank of the abelian group G.

Lemma 2.1. *Let $A = \prod_{n<\omega} B_n$ be as in the previous two paragraphs with p-basic subgroup $F \oplus B$. Then, $r_0(F) = \pi(\{\kappa_n\})$.*

Proof. The proof of [5, Theorem 35.2] shows that $F/pF \cong A/(tA + pA)$. Since B is a p-basic subgroup of tA, we have $tA = B + p(tA)$, and hence $tA + pA = B + pA$. Choose m_0 such that $\pi(\{\kappa_n\}) = \prod_{i=m}^\infty 2\kappa_i$ and $\sigma(\{\kappa_n\}) = \sum_{i=m}^\infty \kappa_i$ for all $m \geq m_0$, and write $A = C \oplus D$ where $C = B_0 \oplus \dots \oplus B_{m_0}$ and $D = \prod_{n>m_0} B_n$. Then, $B' = B \cap D = \bigoplus_{n>m_0} B_n$, and

$$A/(B + pA) \cong D/[B' + pD] \cong (D/pD)/([B' + pD]/pD).$$

As a $\mathbb{Z}/p\mathbb{Z}$-vector-space, D/pD has dimension $\pi(\{\kappa_n\})$. On the other hand, $B' + pD/pD \cong B'/B' \cap pD = B'/pB'$ since B' is pure in D. But B'/pB' has $\mathbb{Z}/p\mathbb{Z}$-dimension $\sigma(\{\kappa_n\})$. Since $\sigma(\{\kappa_n\}) < \pi(\{\kappa_n\})$, the group $A/(tA+pA)$ has dimension $\pi(\{\kappa_n\})$. Hence, $r_0(F) = \pi(\{\kappa_n\})$. $\square$

Let μ and ν be cardinals. As in [5], define

$$d(\mu, \nu) = \begin{cases} \mu\nu & \text{if } \mu \text{ is finite,} \\ (2\nu)^\mu & \text{otherwise.} \end{cases}$$

Proposition 2.2. *The following are equivalent for reduced p-groups A and C:*

a) *$f_n(A) = f_n(C)$ for all $n < \omega$.*
b) *$\mathrm{Hom}(A,G) \cong \mathrm{Hom}(C,G)$ for all bounded p-groups G.*
c) *$\mathrm{Hom}(A,G) \cong \mathrm{Hom}(C,G)$ for all reduced p-groups G.*

Proof. From [5], we obtain that the basic subgroup B of a p-group A has the form $B = \bigoplus_{n<\omega}[\bigoplus_{f_n(A)} \mathbb{Z}/p^{n+1}\mathbb{Z}]$. Therefore, $fin\ r(B) = inf\{\sum_{i=n}^\infty f_i(A) | n < \omega\} = \sigma(\{\alpha_n\})$ with $\alpha_n = f_n(A)$. Here, $f_n(A)$ denotes the n^{th}-Ulm-Kaplansky invariant of A.

a) $\Rightarrow$ c) : Obviously, it suffices to consider the case that G is a reduced p-group. Let $\alpha_n = f_n(A) = f_n(C)$ and $\lambda_n = f_n(G)$ for $n < \omega$. Choose a basic subgroup B of

A with $\kappa = fin\ r(B)$. Every basic subgroup D of C is isomorphic to B. Finally, let $\lambda = fin\ r(G)$, and $\eta = fin\ r(H)$ where H is a basic subgroup of G. By [5, Theorem 46.4], $\mathrm{Hom}(A, G)$ is a p-adic algebraically compact group whose basic submodule is of the form $F = \bigoplus_{n<\omega}[\bigoplus_{\tau_n} \mathbb{Z}/p^{n+1}\mathbb{Z}] \oplus [\bigoplus_\rho J_p]$ where $\rho = d(\kappa, \eta)$ and

$$\tau_n = d\left(\alpha_n, \lambda + \sum_{k=n}^{\infty} \lambda_k\right) + d\left(\sum_{k=n+1}^{\infty} \alpha_k, \lambda_n\right).$$

Because none of the above invariants changes if A is replaced by C, the algebraically compact groups $\mathrm{Hom}(A, G)$ and $\mathrm{Hom}(C, G)$ have isomorphic basic submodules. This is only possible if they are isomorphic.

Since c) $\Rightarrow$ b) is obvious, it remains to show that b) implies a). For this, consider $n < \omega$, and let $G = \bigoplus_\kappa \mathbb{Z}/p^{n+2}\mathbb{Z}$. Choose a basic subgroup B of A and observe that

$$\mathrm{Hom}(A, G) = \mathrm{Hom}(A/p^{n+2}A, G) \cong \mathrm{Hom}(B/p^{n+2}B, G).$$

If $g_n(A) = \sum_{i=n+1}^{\infty} f_i(A)$, then

$$\mathrm{Hom}(B/p^{n+2}B, G) \cong \left[\bigoplus_{k=0}^{n}\left(\bigoplus_{d(f_k(A),\kappa)} \mathbb{Z}/p^{k+1}\mathbb{Z}\right)\right] \oplus \left[\bigoplus_{d(g_n(A),\kappa)} \mathbb{Z}/p^{n+2}\mathbb{Z}\right].$$

Therefore, $d(f_n(A), \kappa) = d(f_n(C), \kappa)$. In case that $f_n(A) < \infty$, choose $\kappa = 1$ so that $f_n(A) = d(f_n(A), 1) = d(f_n(C), 1) < \infty$. Consequently, $f_n(C)$ has to be finite, and $f_n(A) = f_n(C)$. On the other hand, if $f_n(A)$ is infinite, then $f_n(C)$ has to be infinite too. Let κ be infinite, and obtain $d(f_n(A), \kappa) = \kappa^{f_n(A)}$. Therefore, $\kappa^{f_n(A)} = \kappa^{f_n(C)}$ for all infinite cardinals κ, i.e. $f_n(A) = f_n(C)$ by the introductory remarks. □

Corollary 2.3. *The classes of torsion-complete and Σ-cyclic p-groups are Fuchs 34 classes.*

Proof. The class of torsion-complete groups contains all bounded p-groups. Since Σ-cyclic and torsion-complete p-groups are determined up to isomorphism by their finite Ulm-Kaplansky-invariants, an application of Proposition 2.2 completes the proof. □

We now turn to the question of which other invariants of a p-group A are determined by $\mathrm{Hom}(A, G)$:

Theorem 2.4. *The following are equivalent for reduced p-groups A and C:*

a) i) $f_n(A) = f_n(C)$ *for all* $n < \omega$.
   ii) $fin\ r(A) = fin\ r(C)$.
b) $\mathrm{Hom}(A, E) \cong \mathrm{Hom}(C, E)$ *for all divisible groups* E.
c) $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all groups* G.
d) $Ext(A, G) \cong Ext(C, G)$ *for all torsion-free groups* G.

Proof. a) $\Rightarrow$ c): Because of Proposition 2.2, it remains to consider the case that G is a divisible p-group. We choose lower basic subgroups B and D of A and C respectively, i.e. $A/B \cong C/D \cong \bigoplus_\lambda \mathbb{Z}(p^\infty)$ where $\lambda = fin\ r(A) = fin\ r(C)$. The pure exact sequence $0 \to \mathrm{Hom}(A/B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Hom}(B,G) \to 0$ splits since $\mathrm{Hom}(A/B,G)$ is algebraically compact. Thus,

$$\begin{aligned}\mathrm{Hom}(A,G) &\cong \mathrm{Hom}(A/B,G)\oplus \mathrm{Hom}(B,G)\\ &\cong \mathrm{Hom}(C/D,G)\oplus \mathrm{Hom}(D,G)\cong \mathrm{Hom}(C,G).\end{aligned}$$

c) $\Rightarrow$ d): Let G be a torsion-free group, and $0 \to G \to \bigoplus_I \mathbb{Q} \to E \to 0$ an exact sequence with E torsion. It induces the sequence $0 = \mathrm{Hom}(A, \bigoplus_I \mathbb{Q}) \to \mathrm{Hom}(A,E) \to \mathrm{Ext}(A,G) \to \mathrm{Ext}(A, \bigoplus_I \mathbb{Q}) = 0$, from which

$$\mathrm{Ext}(A,G)\cong \mathrm{Hom}(A,E)\cong \mathrm{Hom}(C,E)\cong \mathrm{Ext}(C,G)$$

follows.

d) $\Rightarrow$ b): Let E be a divisible p-group, and consider an exact sequence $0 \to U \to \bigoplus_I \mathbb{Q} \to E \to 0$ for some suitable index-set I. Arguing as in c) $\Rightarrow$ d) yields the first and third isomorphism in $\mathrm{Hom}(C,E) \cong \mathrm{Ext}(C,U) \cong \mathrm{Ext}(A,U) \cong \mathrm{Hom}(A,E)$.

b) $\Rightarrow$ a): Consider lower basic subgroups B and D of A and C respectively. Then, $A/B \cong \bigoplus_{\lambda_A} \mathbb{Z}(p^\infty)$ and $C/D \cong \bigoplus_{\lambda_C} \mathbb{Z}(p^\infty)$ where $\lambda_A = fin\ r(A)$ and $\lambda_C = fin\, r(C)$. Write $B = \bigoplus_{n<\omega}[\bigoplus_{\alpha_n} \mathbb{Z}/p^{n+1}\mathbb{Z}]$ and $C = \bigoplus_{n<\omega}[\bigoplus_{\beta_n} \mathbb{Z}/p^{n+1}\mathbb{Z}]$, and consider a divisible p-group $E = \bigoplus_\kappa \mathbb{Z}(p^\infty)$. As in the proof of *a)* $\Rightarrow$ *c)*, we obtain $\mathrm{Hom}(A,E) = \mathrm{Hom}(A/B,E) \oplus \mathrm{Hom}(B,E)$. Observe that $\mathrm{Hom}(A/B,E)$ is torsion-free. Because of [5, Chapter 47],

$$\mathrm{Hom}(B,E)\cong \prod_{n<\omega}\prod_{\alpha_n}\left[\bigoplus_{\kappa}\mathbb{Z}/p^{n+1}\mathbb{Z}\right]=\prod_{n<\omega}\left[\bigoplus_{d(\alpha_n,\kappa)}\mathbb{Z}/p^{n+1}\mathbb{Z}\right].$$

By [5, Corollary 33.3], $f_n(\mathrm{Hom}(A,E)) = d(\alpha_n,\kappa)$. Similarly, we obtain

$$f_n(\mathrm{Hom}(C,E)) = d(\beta_n,\kappa).$$

Hence, $d(\alpha_n,\kappa) = d(\beta_n,\kappa)$ for all $n < \omega$. As in the proof of Proposition 2.2, one obtains $\alpha = \beta$.

If A has finite final rank, then A is bounded, and the same has to hold for C, since A and C have isomorphic basic subgroups by what has been shown so far. In particular, both A and C have final rank 0. Hence, we may assume that A and C are both unbounded. Choose E in such a way that κ is infinite.

Since A and C have isomorphic basic subgroups B and D, we may rewrite these as $B \cong D \cong \bigoplus_{n<\omega}[\bigoplus_{\gamma_n} \mathbb{Z}/p^{s_n}\mathbb{Z}]$ where γ_n is a non-zero cardinal for all $n < \omega$, and $0 < s_0 < s_1 < \dots$ are integers. Then, $\mathrm{Hom}(B,E) \cong \prod_{n<\omega}[\bigoplus_{\lambda_n} \mathbb{Z}/p^{n+1}\mathbb{Z}]$ where $\lambda_n = d(\gamma_n,\kappa)$ is infinite. By Lemma 2.1, the torsion-free rank of any basic subgroup of $\mathrm{Hom}(B,E)$ is $\pi(\{\lambda_n\}) = \prod_{n=m}^\infty \lambda_n$ for some suitable $m < \omega$. Moreover, m can be chosen in such a way that $\sigma(\{\lambda_n\}) = \sum_{n=m}^\infty \gamma_n$.

On the other hand, $\mathrm{Hom}(A/B,E)$ is a torsion-free algebraically compact group which is the completion of $\bigoplus_{d(\lambda_A,\kappa)} J_p$ (see [5]). Therefore, every p-basic subgroup of $\mathrm{Hom}(A,E)$ has torsion-free rank $d(\lambda_A,\kappa) + \pi(\{\lambda_n\})$. In the same way, the torsion-free rank of a p-basic subgroup of $\mathrm{Hom}(C,E)$ is $d(\lambda_C,\kappa) + \pi(\{\lambda_n\})$. If γ_n is finite for some n, then $\lambda_n = \gamma_n\kappa = \kappa = \kappa^{\gamma_n}$. On the other hand, for infinite γ_n's, we obtain $\lambda_n = [2\kappa]^{\gamma_n} = \kappa^{\gamma_n}$. In either case, λ_n is infinite, and hence $\pi(\{\lambda_n\}) = \prod_{n=m}^\infty 2\lambda_n = \prod_{n=m}^\infty \kappa^{\gamma_n} = \kappa^{\sigma(\{\gamma_n\})}$ for some $m < \omega$. Since A and C are unbounded, λ_A and λ_C are infinite. Thus, $d(\lambda_A,\kappa) = \kappa^{\lambda_A}$ and $d(\lambda_C,\kappa) = \kappa^{\lambda_C}$. Hence, $\kappa^{\lambda_A} + \kappa^{\sigma(\{\gamma_n\})} = \kappa^{\lambda_C} + \kappa^{\sigma(\{\gamma_n\})}$ for all κ. Since $\sigma(\{\gamma_n\}) = fin\ r(B) \leq$

$fin\ r(A) = \lambda_A$, one obtains $\kappa^{\lambda_A} = \kappa^{\lambda_A} + \kappa^{\sigma(\{\gamma_n\})}$. By symmetry, $\kappa^{\lambda_A} = \kappa^{\lambda_C}$ for all infinite cardinals κ, and hence $\lambda_A = \lambda_C$. □

Corollary 2.5. *The class of Σ-cyclic p-groups is the largest Fuchs 34 class of p-groups which contains all the cyclic groups and is closed under direct sums.*

Proof. Obviously, the class of Σ-cyclic p-groups has the stated property by Theorem 2.4. Let $\mathcal{C}$ be any class with the above properties, and choose $C \in \mathcal{C}$. The basic subgroup A of C is Σ-cyclic and satisfies $f_n(A) = f_n(C)$ for all $n < \omega$ and $fin\ r(A) \leq fin\ r(C)$. Moreover, no generality is lost if we assume that $\lambda = fin\ r(C)$ is infinite. Then, $f_n(\bigoplus_\lambda A) = f_n(\bigoplus_\lambda C)$ for all $n < \omega$ and $fin\ r(\bigoplus_\lambda A) = \lambda = fin\ r(\bigoplus_\lambda C)$. Since $\mathcal{C}$ contains all cyclic p-groups and is closed with respect to direct sums, $\bigoplus_\lambda A \cong \bigoplus_\lambda C$. Thus, C is Σ-cyclic. □

Example 2.6. Let B be a basic subgroup of the generalized Prüfer group $H_{\omega+1}$. Then, $A = H_{\omega+1} \oplus B$ and $C = H_{\omega+1} \oplus H_{\omega+1}$ have final rank $\aleph_0$ and satisfy $f_n(A) = f_n(C) = 2$. Since $f_\omega(A) = 1$ and $f_\omega(C) = 2$, we have $A \ncong C$ although $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for all groups G.

The class of adjusted cotorsion groups consists of those reduced groups G with $Ext(\mathbb{Q}, G) = 0$ which have no non-zero torsion-free direct summands.

Theorem 2.7. *There exist non-isomorphic adjusted cotorsion groups A and C such that* $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ *for all cotorsion groups G.*

Proof. Choose adjusted cotorsion groups A and C whose torsion subgroups are countable p-groups of length $\omega + 1$ such that $tA \ncong tC$, but $f_n(tA) = f_n(tC)$ for all $n < \omega$ and $fin\ r(A) = fin\ r(C)$. Such groups exist by [5, Theorem 55.5] and Example 2.6. To show $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(C, G)$ for a cotorsion group G, write $G = H \oplus K \oplus D \oplus E$ where H is an adjusted cotorsion group, K is a reduced torsion-free algebraically compact group, D is torsion-free and divisible, and E is torsion and divisible. Observe that $\mathrm{Hom}(tA, E) \cong \mathrm{Hom}(tC, E)$ by Theorem 2.4.

Cotorsion groups are discussed in detail in [5]. In particular, since A is an adjusted cotorsion group, $A \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, tA)$, and A/tA is divisible. Therefore, $\mathrm{Hom}(A, K) = \mathrm{Hom}(A/tA, K) = 0 = \mathrm{Hom}(C, K)$. Moreover, we obtain an exact sequence

$$0 = \mathrm{Hom}(A/tA, H) \to \mathrm{Hom}(A, H) \to \mathrm{Hom}(tA, H) \to \mathrm{Ext}(A/tA, H) = 0$$

in which the last term vanishes since H is cotorsion. Therefore,

$$\mathrm{Hom}(A, H) \cong \mathrm{Hom}(tA, H) = \mathrm{Hom}(tA, tH).$$

By what has been shown so far,

$$\mathrm{Hom}(A, H) \cong \mathrm{Hom}(tA, tH) \cong \mathrm{Hom}(tC, tH) \cong \mathrm{Hom}(C, H).$$

To show $\mathrm{Hom}(A, D) \cong \mathrm{Hom}(C, D)$ and $\mathrm{Hom}(A, E) \cong \mathrm{Hom}(C, E)$, it suffices to establish that A and C have cardinality $2^{\aleph_0}$. Then, $A/tA \cong C/tC \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$, and $\mathrm{Hom}(A, D) \cong \mathrm{Hom}(A/tA, D) \cong \mathrm{Hom}(C, D)$. Moreover, the exact sequence $0 \to \mathrm{Hom}(A/tA, E) \to \mathrm{Hom}(A, E) \to \mathrm{Hom}(tA, E) \to 0$ splits since A/tA is torsion-free and divisible. Then,

$$\begin{aligned} \mathrm{Hom}(A, E) &\cong \mathrm{Hom}(A/tA, E) \oplus \mathrm{Hom}(tA, E) \\ &\cong \mathrm{Hom}(C/tC, E) \oplus \mathrm{Hom}(tC, E) \cong \mathrm{Hom}(C, E) \end{aligned}$$

by what has been established so far. To show that $|A| = 2^{\aleph_0}$, set $T = p^\omega A$, and observe that $B = tA/T$ is unbounded and a direct sum of cyclics as a countable separable group. Moreover, T is bounded since A has length $\omega+1$. The exact sequence $0 = \mathrm{Ext}(\mathbb{Q}, T) \to \mathrm{Ext}(\mathbb{Q}, tA) \to \mathrm{Ext}(\mathbb{Q}, B) \to 0$ yields $A/tA \cong \mathrm{Ext}(\mathbb{Q}, tA) \cong \mathrm{Ext}(\mathbb{Q}, B)$. If T_1 is the torsion-complete group with basic subgroup B, then T_1 is uncountable, and there exists an exact sequence $0 = \mathrm{Hom}(\mathbb{Q}, T_1) \to \mathrm{Hom}(\mathbb{Q}, T_1/B) \to \mathrm{Ext}(\mathbb{Q}, B)$, from which we obtain $|A/tA| \geq 2^{\aleph_0}$. On the other hand, since A/tA is torsion-free divisible, every basic subgroup of tA is a basic subgroup of the p-local group A. Because tA is countable, $|A| \leq \aleph_0^{\aleph_0}$ by [5, Theorem 34.3]. A similar argument yields $|C| = 2^{\aleph_0}$. □

In particular, the last result shows that the class of (adjusted/reduced) cotorsion groups is not a Fuchs 34 class.

3. Fuchs' problem 34 in $\mathcal{G}$

In [1], it was shown that every group $A \in \mathcal{G}$ can be written as $A = A_1 \oplus \ldots \oplus A_n$ where each A_i is essentially indecomposable. Moreover, the A_i's are unique up to quasi-isomorphism after a possible reordering. We denote the embeddings and projections associated with this decomposition by $\delta_j^A : A_j \to A$ and $\pi_j^A : A \to A_j$ for $j = 1, \ldots, n$ respectively. The superscripts referring to A are omitted, unless they are necessary to avoid confusion.

For a prime p, let $A_{(p)}$ denote the p-component of A in order to avoid confusion with the A_i's. Given a non-zero integer m, every group A in $\mathcal{G}$ can be written as $A = A_{(m)} \oplus A^{(m)}$ such that $A_{(m)} = A_{(p_1)} \oplus \ldots \oplus A_{(p_n)}$ where $p_1, \ldots, p_n$ are the primes dividing m, and multiplication by m is an automorphism of $A^{(m)}$. Observe that $A_{(m)}$ and $A^{(m)}$ are fully invariant subgroups of A.

Given Abelian groups A and G, the *A-socle of G* is $S_A(G) = \langle \phi(A) \mid \phi \in \mathrm{Hom}(A, G)\rangle$. The group G is *(finitely) A-generated* if it is an epimorphic image of $\bigoplus_I A$ for some (finite) index set I. Clearly, G is A-generated if and only if $S_A(G) = G$.

Lemma 3.1. *Let A and G be in $\mathcal{G}$.*

a) *If G is a subgroup of an Abelian group H such that H/G is torsion, then $H = G + tH$.*
b) *If $W \subseteq S_A(G)$, then there is a finitely A-generated subgroup U of W such that $W = U + tW$.*
c) $t\,\mathrm{Hom}(A, G) = \mathrm{Hom}(A, tG)$ *and* $\mathrm{Hom}(A, G)/t\,\mathrm{Hom}(A, G)$ *has finite rank and is torsion-free divisible.*

Proof. a) For $x \in H$, there is a non-zero integer m such that $mx \in G$. Write $G = G_{(m)} \oplus G^{(m)}$, and choose $u \in G_{(m)}$ and $v \in G^{(m)}$ such that $mx = u + v$. Since multiplication by m is an automorphism of $G^{(m)}$, there is $w \in G^{(m)}$ with $v = mw$. Then, $m(x - w) = u \in G_{(m)} \subseteq tH$, and hence $x - w \in tH$.

b) Since $G \in \mathcal{G}$, the group W has finite torsion-free rank, and there is a finitely A-generated subgroup U of W such that W/U is torsion. By [1, Theorem 2.2(a)], $U \in \mathcal{G}$, and a) yields $W = U + tW$.

c) Since $G \in \mathcal{G}$, the group $S_A(tG)$ is a reduced A-generated torsion group and has the property that $\mathrm{Hom}(A, tG)$ is torsion by [1, Lemma 2.1(b)]. Therefore, $\mathrm{Hom}(A, tG) \subseteq t\,\mathrm{Hom}(A, G)$, while the other inclusion is obvious.

Consider the induced sequence

$$0 \to \mathrm{Hom}(A/tA, G/tG) \to \mathrm{Hom}(A, G/tG) \to \mathrm{Hom}(tA, G/tG) = 0$$

which yields that $\mathrm{Hom}(A, G/tG)$ is torsion-free and divisible of finite rank. Moreover, because of the sequence

$$0 \to \mathrm{Hom}(A, tG) \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(A, G/tG)$$

and in view of the fact that $\mathrm{Hom}(A, tG)$ is torsion, we obtain

$$r_0(\mathrm{Hom}(A, G)) \leq r_0(\mathrm{Hom}(A, G/tG)) < \infty.$$

Finally, let $\alpha \in \mathrm{Hom}(A, G)$ and m be a non-zero integer. Write $A = A_{(m)} \oplus A^{(m)}$ and $G = G_{(m)} \oplus G^{(m)}$, and obtain $\alpha = \alpha_m + \alpha^{(m)}$ such that $\alpha_m(A^{(m)}) = 0 = \alpha^{(m)}(A_{(m)})$. Since multiplication by m is an automorphism of $G^{(m)}$, there is $\beta \in \mathrm{Hom}(A^{(m)}, G^{(m)})$ with $\alpha^{(m)}|_{A^{(m)}} = m\beta$. We extend β to a map $\tilde{\beta} : A \to G$ by $\tilde{\beta}(A_{(m)}) = 0$, and obtain $\alpha - m\tilde{\beta} = \alpha_m \in t\,\mathrm{Hom}(A, G)$. Thus, $\mathrm{Hom}(A, G)/t\,\mathrm{Hom}(A, G)$ is divisible. $\square$

The endomorphism ring of an Abelian group A will be denoted by $E(A)$ or simply E if there is no possibility for confusion. Given an E-module M, the symbol $\overline{M}$ denotes the $\overline{E} = E/tE$-module M/tM. Finally, $N(R)$ is the nilradical of a ring R.

Lemma 3.2. *Let $A \in \mathcal{G}$ and $\alpha : A_i \to A_j$ be a map. Suppose that $tE \subseteq N$ is an ideal of E such that $\overline{N} = N(\overline{E})$.*

a) *If A_i and A_j are not quasi-isomorphic, then $\delta_j \alpha \pi_i \in N$.*
b) *If $\ker\,\alpha$ is not bounded, then $\delta_j \alpha \pi_i \in N$.*
c) *A map $\alpha : A_i \to A_j$ with a bounded kernel is a quasi-isomorphism.*

Proof. a) Assume $\delta_j \alpha \pi_i \notin N$, and observe that it has to have infinite order. Suppose there is $\sigma \in E$ such that $\overline{\pi_i \sigma \delta_j \alpha} \notin N(\overline{E(A_i)})$. Since $\overline{E(A_i)}$ is local, there is a map $\beta \in E(A_i)$ such that $\overline{\beta \pi_i \sigma \delta_j \alpha} = \overline{1}_{A_i}$. Hence, we can choose a non-zero integer k such that $k\beta\pi_i\sigma\delta_j\alpha = k1_{A_i}$. If we write $A_i = (A_i)_{(k)} \oplus A_i^{(k)}$, then $\beta|_{A_i^{(k)}} : A_i^{(k)} \to A_i^{(k)}$, and $\beta|_{A_i^{(k)}}(\pi_i\sigma\delta_j\alpha)|_{A_i^{(k)}} = 1_{A_i^{(k)}}$ since multiplication by k is an automorphism of $A_i^{(k)}$. Therefore, $(\beta\pi_i\sigma\delta_j)|_{A_j^{(k)}}$ is a splitting map for $\alpha|_{A_i^{(k)}}$, and $A_j^{(k)} = \alpha(A_i^{(k)}) \oplus U$ for some subgroup U of $A_j^{(k)}$. Moreover, $\alpha|_{A_i^{(k)}}$ is one-to-one. Since $\delta_j\alpha\pi_i$ has infinite order and $(A_i)_{(k)}$ is finite, $\alpha(A_i^{(k)})$ cannot be bounded. Hence, U has to be finite because A_j is essentially indecomposable. Consequently, $\alpha|_{A_i^{(k)}}$ is a quasi-isomorphism, and $A_i \sim A_j$, a contradiction.

Therefore, $\overline{\pi_i\sigma\delta_j\alpha} \in N(\overline{E(A_i)})$ which is nilpotent. There is $m < \omega$ such that $(\pi_i\sigma\delta_j\alpha)^m \in tE(A_i)$. Then, $(\sigma\delta_j\alpha\pi_i)^{m+1} = \sigma\delta_j\alpha(\pi_i\sigma\delta_j\alpha)^m\pi_i \in tE$. Consider the left ideal $I = E\delta_j\alpha\pi_i$ of E, and observe that $\overline{I}$ is a nil ideal by what has been shown so far. Since $\overline{E}$ is Artinian, $\overline{I}$ is nilpotent and $\overline{I} \subseteq N(\overline{E(A)}) = \overline{N}$, a contradiction.

b) Suppose that α has an unbounded kernel. Assume $\delta_j\alpha\pi_i \notin N$, and consider $\sigma \in E$. If $\overline{\pi_i\sigma\delta_j\alpha} \notin N(\overline{E(A_i)})$, then there is a map $\beta \in E(A_i)$ such that $\overline{\beta\pi_i\sigma\delta_j\alpha} = \overline{1}_{A_i}$. As in a), we obtain that $\alpha|_{A_i^{(k)}}$ splits for some non-zero integer k. In particular, α has to have a bounded kernel, a contradiction. Thus, $\overline{\pi_i\sigma\delta_j\alpha} \in N(\overline{E(A_i)})$, and we obtain a contradiction arguing as in a).

c) Select an integer m such that $m \ker\ \alpha = 0$ and write $A_i = (A_i)_{(m)} \oplus A_i^{(m)}$. Then, $\alpha|_{A_i^{(m)}} : A_i^{(m)} \to A_i^{(m)}$ is a monomorphism which quasi-splits by [4, Theorem 2.2], say $\beta\alpha|_{A_i^{(m)}} = k1_{A_i^{(m)}}$ for some non-zero integer k. Therefore, $A_i^{(km)}$ is a direct summand of $A_i^{(m)}$ with finite complement, and $A_i^{(km)} = \alpha(A_i^{(km)}) \oplus U$ for some subgroup U of A_i. Since A_i is essentially indecomposable, U is finite, and $A_i/\alpha(A_i)$ is finite, establishing that α is a quasi-isomorphism. □

Theorem 3.3. *Groups A and B in $\mathcal{G}$ such that $tA \cong tB$ are isomorphic if*

$$r_0(\mathrm{Hom}(A, G)) = r_0(\mathrm{Hom}(B, G))$$

for all $G \in \mathcal{G}$.

Proof. If A and B are torsion, then there is nothing to prove. Thus, assume that $r_0(A) > 0$, and observe that B cannot be torsion since every torsion group in $\mathcal{G}$ is finite while tA is infinite. There are essentially indecomposable groups $A_1, \dots, A_n$ and $B_1, \dots, B_m$ such that $A = A_1 \oplus \dots \oplus A_n$, $B = B_1 \oplus \dots \oplus B_m$, and $r_0(A_i), r_0(B_j) > 0$ for all i and j. We induct on $n+m$ with $n+m = 0$ being trivial as already mentioned. Thus, assume $n+m > 0$, from which we immediately obtain $n > 0$ and $m > 0$.

Observe $r_0(E(B)) = r_0(\mathrm{Hom}(A, B)) = r_0(\mathrm{Hom}(A, S_A(B)))$. While $S_A(B)$ may not be in $\mathcal{G}$, it contains a finitely A-generated subgroup $H \in \mathcal{G}$ by Lemma 3.1 such that $S_A(B) = H + tB$ since $tA \cong tB$ guarantees $tB \subseteq S_A(B)$. But $S_A(B)/H = (H+tB)/H \cong tB/(H \cap tB)$ is an A-generated reduced torsion group. In particular, $\mathrm{Hom}(A, S_A(B)/H)$ is torsion by [1, Lemma 2.1(b)]. Hence, the last term in the sequence

$$0 \to \mathrm{Hom}(A, H) \to \mathrm{Hom}(A, S_A(B)) \to \mathrm{Hom}(A, S_A(B)/H)$$

is torsion, and $r_0(\mathrm{Hom}(A, S_A(B)) = r_0(\mathrm{Hom}(A, H))$. Therefore,

$$\begin{aligned} r_0(E(B)) &= r_0(\mathrm{Hom}(A, H)) = r_0(\mathrm{Hom}(B, H)) \\ &\leq r_0(\mathrm{Hom}(B, S_A(B))) \leq r_0(E(B)) < \infty. \end{aligned}$$

Therefore, $I = \mathrm{Hom}(B, S_A(B))$ is a right ideal of $E(B)$ such that $E(B)/I$ is torsion as an Abelian group. On the other hand, since $tB \subseteq S_A(B)$, we have $tE(B) \subseteq I$ and $E(B)/I \cong \overline{E(B)}/\overline{I}$ is torsion-free and divisible by Lemma 3.1(c). Therefore, $I = E(B)$ from which $B = S_A(B)$ follows. By symmetry, $S_B(A) = A$.

Choose a two-sided ideal N_B of $E(B)$ containing $tE(B)$ such that $\overline{N_B} = N(\overline{E(B)})$. Since $E(B)/N_B \cong \overline{E(B)}/\overline{N_B}$ is torsion-free and divisible as an Abelian group, the group $B/N_B B \cong [E(B)/N_B] \otimes_{E(B)} B$ has these properties too. Moreover, $B \neq N_B B$ since the fact that $N(\overline{E(B)})$ is nilpotent yields $N_B^k \subseteq tE(B)$ for some $k < \omega$. Consequently, $B = N_B B$ would yield $B = N_B^k B \subseteq tB$ contradicting the fact that B is not torsion. Since $B = S_A(B)$, there is a map $\phi : A \to B$ such that $\phi(A) \not\subseteq N_B B$. From $\phi(A) = \phi(A_1) + \dots + \phi(A_n)$, we obtain $\phi(A_i) \not\subseteq N_B B$ for some $i \in \{1, \dots, n\}$. No generality is lost if we assume $\phi(A_1) \not\subseteq N_B B$. Since $A_1 = S_B(A_1) = \sum_{j=1}^m S_{B_j}(A_1)$, one can find an index $j \in \{1, \dots, m\}$ such that $\phi(S_{B_j}(A_i)) \not\subseteq N_B B$. Without loss of generality, $j = 1$. Hence, there is a map $\gamma : B_1 \to A_1$ such that $\phi\gamma(B_1) \not\subseteq N_B B$. Choose $x_0 \in B_1$ such that $\phi\gamma(x_0) \notin N_B B$. Since, $\phi\gamma(x) = \sum_{j=1}^m \delta_j^B \pi_j^B \phi\gamma(x)$ for all $x \in B_1$, there is $j \in \{1, \dots, m\}$ such that $\delta_j^B \pi_j^B \phi\gamma(x_0) \notin N_B B$.

If B_1 and B_j are not quasi-isomorphic, then the map $\pi_j^B\phi\gamma : B_1 \to B_j$ satisfies $\delta_j^B\pi_j^B\phi\gamma\pi_1^B \in N_B$ by Lemma 3.2(a). Hence, $\delta_j^B\pi_j^B\phi\gamma\pi_1^B(x_0) \in N_B B$, a contradiction. Therefore, B_1 and B_j are quasi-isomorphic. By part b) of Lemma 3.2, ker $\pi_j^B\phi\gamma$ has to be bounded using the same arguments as before. Part c) of Lemma 3.2 gives that $\pi_j^B\phi\gamma : B_1 \to B_j$ is a quasi-isomorphism. But then, the map $\gamma : B_1 \to A_1$ quasi-splits. Since A_1 is essentially indecomposable, ker γ and $A_1/\gamma(B_1)$ are finite, and $A_1 \sim B_1$. If $\alpha_1 : A_1 \to B_1$ and $\beta_1 : B_1 \to A_1$ satisfy $\alpha_1\beta_1 = k1_{B_1}$ and $\beta_1\alpha_1 = k1_{A_1}$ for some non-zero integer k, write $A_1 = (A_1)_{(k)} \oplus A_1^{(k)}$ and $B_1 = (B_1)_{(k)} \oplus B_1^{(k)}$. In particular, $\alpha_1|_{A_1^{(k)}} : A_1^{(k)} \to B_1^{(k)}$ is an isomorphism. Observe that $A = A_{(k)} \oplus A^{(k)}$ and $B = B_{(k)} \oplus B^{(k)}$ yield $A^{(k)} = A_1^{(k)} \oplus \ldots \oplus A_n^{(k)}$ and $B^{(k)} = B_1^{(k)} \oplus \ldots \oplus B_m^{(k)}$. Because $tA \cong tB$ gives $A_{(k)} \cong B_{(k)}$, we may assume that $A_1 \cong B_1$. Then, $t(A_2 \oplus \ldots \oplus A_n) \cong t(B_2 \oplus \ldots \oplus B_m)$ since $A_{(p)}$ and $B_{(p)}$ are finite for all primes p. Finally, setting $A' = A_2 \oplus \ldots \oplus A_n$ and $B' = B_2 \oplus \ldots \oplus B_m$ yields

$$\begin{aligned} r_0(\mathrm{Hom}(A', G)) &= r_0(\mathrm{Hom}(A, G)) - r_0(\mathrm{Hom}(A_1, G)) \\ &= r_0(\mathrm{Hom}(B, G)) - r_0(\mathrm{Hom}(B_1, G)) \\ &= r_0(\mathrm{Hom}(B', G)) \end{aligned}$$

for all $G \in \mathcal{G}$ since $A_1 \cong B_1$. By induction, $A' \cong B'$, and we are done. □

Corollary 3.4. *Let A and B be in $\mathcal{G}$. Then, $A \cong B$ if and only if*

i) $r_0(\mathrm{Hom}(A, G)) = r_0(\mathrm{Hom}(B, G))$ *for all* $G \in \mathcal{G}$.

ii) $f_n(A_{(p)}) = f_n(B_{(p)})$ *for all* $n < \omega$ *and all primes* p. □

Corollary 3.5. $\mathcal{G}$ *is a Fuchs* 34 *class.*

Proof. Let A and B be in $\mathcal{G}$ such that $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(B, G)$ for all $G \in \mathcal{G}$. In particular, $\mathrm{Hom}(A, G)$ and $\mathrm{Hom}(B, G)$ have the same torsion-free rank for all $G \in \mathcal{G}$. In order to be able to apply Theorem 3.3, it remains to show that $tA \cong tB$. Let p be a prime, and choose $n < \omega$ such that $p^n A_{(p)} = p^n B_{(p)} = 0$. Select a group $G \in \mathcal{G}$ such that $G_{(p)} \cong \mathbb{Z}/p^n\mathbb{Z}$. Observe that $A_{(p)} \cong \mathrm{Hom}(A, G_{(p)}) = \mathrm{Hom}(A, G)_{(p)} \cong \mathrm{Hom}(B, G)_{(p)} \cong B_{(p)}$ because of part c) of Lemma 3.1. □

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