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STRICTLY SINGULAR NON-COMPACT OPERATORS ON HEREDITARILY INDECOMPOSABLE BANACH SPACES

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Abstract. An example is given of a strictly singular non-compact operator on a Hereditarily Indecomposable, reflexive, asymptotic ℓ_1 Banach space. The construction of this operator relies on the existence of transfinite c_0-spreading models in the dual of the space.

1. Introduction

A Banach space is said to be *Hereditarily Indecomposable* (H.I.) if for every pair Y, Z of subspaces of X with $Y \cap Z = \{\mathbf{0}\}$, the subspace $Y + Z$ is not closed (by a *subspace* of a Banach space we shall mean an infinite dimensional, closed linear subspace). The first example of an H.I. space was given by Gowers and Maurey [16] providing a negative solution to the famous unconditional basic sequence problem. The following important result was established in [16]: Every operator on a complex H.I. space is a strictly singular perturbation of a multiple of the identity (by the term *operator* we shall mean a bounded linear operator). Actually, the following characterization of complex H.I. spaces is given in [11]: X is H.I. if, and only if, every operator from a subspace of X into X is a strictly singular perturbation of the inclusion map. We recall that an operator on a Banach space is *strictly singular* if no restriction of it to a subspace is an isomorphism.

There has been an interest in investigating strictly singular operators on H.I. spaces because of their connection to the invariant subspace problem. Indeed, known results [10], [19] yield that if X is an H.I. space with the property that every strictly singular operator on X is compact, then every operator on X admits a non-trivial invariant subspace. It is therefore natural to investigate whether or not the known examples of H.I. spaces admit strictly singular, non-compact operators. Gowers [15] constructed an example of a strictly singular non-compact operator from a certain subspace of the Gowers-Maurey space into the whole space. An example of an operator (unpublished) with analogous properties was constructed by Argyros and Wagner on the Argyros-Deliyanni H.I. space [4].

Recently, Androulakis and Schlumprecht [3] gave an example of a strictly singular non-compact operator on the Gowers-Maurey space. We also note that examples of H.I. spaces admitting strictly singular non-compact operators were obtained by Argyros and Felouzis as a consequence of their deep dichotomy result [7].

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In the present paper we show that certain asymptotic ℓ_1 H.I. spaces constructed in [14] also admit strictly singular non-compact operators. This will be a consequence of the fact, established here, that their duals admit c_0^ω-spreading models. We recall the definition which requires the concept of the Schreier families $\{S_\xi\}_{\xi<\omega_1}$ [1] (defined in the next section).

Definition 1.1. Suppose that X is a Banach space with a basis (e_i). A semi-normalized block basis (x_i) of (e_i) is a c_0^ω (resp. ℓ_1^ω)-spreading model if there exists a constant $C>0$ such that the following property is satisfied: For every $j\in\mathbb{N}$, every finite subset F of $\mathbb{N}$ with $\min F\geq j$ and such that $(x_i)_{i\in F}$ is S_j-admissible, we have that $\|\sum_{i\in F}a_ix_i\|\leq C\max_{i\in F}|a_i|$ (resp.$\|\sum_{i\in F}a_ix_i\|\geq C\sum_{i\in F}|a_i|$) for every scalar sequence $(a_i)_{i\in F}$.

The Banach spaces discussed in this paper are Tsirelson-type spaces defined as the completion of c_{00} (the space of finitely supported real sequences) under norms given by suitable subsets of $\mathcal{P}$ (the set of finitely supported signed measures μ on $\mathbb{N}$ such that $|\mu(\{n\})|\leq 1$ for all $n\in\mathbb{N}$).

A subset $\mathcal{N}$ of $\mathcal{P}$ is said to be *norming* provided it satisfies the following:

(1) $e_n^*\in\mathcal{N}$, for all $n\in\mathbb{N}$, where e_n^* denotes the point mass measure at n.
(2) $\mathcal{N}$ is symmetric, that is, if $\mu\in\mathcal{N}$, then $-\mu\in\mathcal{N}$.
(3) $\mathcal{N}$ is closed under restriction to intervals, that is, if $\mu\in\mathcal{N}$, then $\mu|J\in\mathcal{N}$, for every interval J in $\mathbb{N}$.

The term norming is justified by the fact that one can define a norm $\|\cdot\|_{\mathcal{N}}$ on c_{00} in the following manner:

$$\Big\|\sum_{i=1}^{\infty}a_ie_i\Big\|_{\mathcal{N}}=\sup\Big\{\sum_{i=1}^{\infty}a_i\mu(\{i\}):\mu\in\mathcal{N}\Big\}$$

for every finitely supported scalar sequence (a_i). Of course, (e_i) is the natural basis of c_{00}. Letting $X_{\mathcal{N}}$ denote the completion of $(c_{00},\|\cdot\|_{\mathcal{N}})$, we see that (e_n) is a normalized, bimonotone basis for $X_{\mathcal{N}}$.

We shall next describe sufficient conditions on $\mathcal{N}$ in order for $X_{\mathcal{N}}^*$ to admit c_0^ω-spreading models. We shall be using two infinite subsets $M=(m_i)_{i=0}^\infty$ and $N=(n_i)_{i=0}^\infty$ of $\mathbb{N}$ satisfying the following requirements:

$$m_0>1 \text{ and there exists an increasing sequence of integers } (s_i)_{i=0}^\infty \text{ so that } m_j=\prod_{i<j}m_i^{s_i} \text{ for all } j\geq 1. \tag{1.1}$$

$$4f_j<n_j \text{ for all } j\geq 0, \text{ where } (f_j) \text{ is defined as follows: } f_0=1, \text{ while for } j\geq 1, f_j=\max\Big\{\sum_{i<j}\rho_in_i:\ \rho_i\in\mathbb{N}\cup\{0\}\,(i<j),\ \prod_{i<j}m_i^{\rho_i}<m_j\Big\}. \tag{1.2}$$

In order to state our result we need to introduce some notation.

Notation. (1) Given μ, ν in $\mathcal{P}$, we write $\mu<\nu$ if $\max\operatorname{supp}\mu<\min\operatorname{supp}\nu$.

(2) A finite subset A of $\mathcal{P}$ is S_p-admissible, $p\in\mathbb{N}$, if $A=\{\mu_1<\cdots<\mu_k\}$ and $\{\min\operatorname{supp}\mu_i:i\leq k\}\in S_p$.

(3) Given $\mathcal{N}\subset\mathcal{P}$ and $j\in\mathbb{N}\cup\{0\}$ we set $\mathcal{N}_j=\{(1/m_j)\sum_{\mu\in A}\mu\ :\ A\subset\mathcal{N}$ is $S_{n_j}-$admissible$\}$.

(4) $\mathcal{N}_\infty = \{\theta \sum_{i=1}^k \mu_i : k \in \mathbb{N}, \theta \in (0, 1/m_0], \mu_1 < \cdots < \mu_k, \exists \tau : \{1, \ldots, k\} \to \mathbb{N} \cup \{0\}, 1-1, \mu_i \in \mathcal{N}_{\tau(i)} \, (i \leq k)\}$.

The following definition will be important for our purposes.

Definition 1.2. A norming set $\mathcal{N}$ is said to be (M, N)-Schreier if the following properties are satisfied:

(1) $\mathcal{N} \subset \bigcup_{i=0}^\infty \mathcal{N}_i \cup \mathcal{N}_\infty \cup \{\pm e_n^* : n \in \mathbb{N}\}$.
(2) $\mathcal{N}_i \subset \mathcal{N}$, for $0 \leq i < \infty$.

The natural norming set of the mixed Tsirelson space $T(\frac{1}{m_i}, S_{n_i})_{i=0}^\infty$ [4] turns out to be (M, N)-Schreier. Further, one can check that if $N = (n_i)_{i=0}^\infty$ is M-good (this term is defined in [14]) ($M = (m_i)_{i=0}^\infty$) and $M^{(2)} = (m_{2i})_{i=0}^\infty$, $N^{(2)} = (n_{2i})_{i=0}^\infty$ satisfy conditions (1.1), (1.2), then the norming set $\mathcal{N}$ of the H.I. space $X_\mathcal{N}$ constructed in [14] is $(M^{(2)}, N^{(2)})$-Schreier.

The main result of this paper is the following:

Theorem 1.3. *Suppose $\mathcal{N}$ is (M, N)-Schreier. Then $X_\mathcal{N}^*$ admits a c_0^ω-spreading model (x_i^*). Moreover, for a suitably chosen infinite sequence of integers (l_i), there exists a non-compact operator T on $X_\mathcal{N}$ satisfying $Tx = \sum_{i=1}^\infty x_{l_i}^*(x) e_i$ for all $x \in X_\mathcal{N}$. In case $X_\mathcal{N}$ is H.I., then T can, in addition, be taken to be strictly singular.*

In fact, under the assumptions of Theorem 1.3, given any sequence (a_i) in ℓ_∞ the formula $Sx = \sum_{i=1}^\infty a_i x_{l_i}^*(x) e_i$, $x \in X_\mathcal{N}$, defines an operator on $X_\mathcal{N}$. It follows from this that the space of operators on $X_\mathcal{N}$ contains a subspace isomorphic to ℓ_∞ (cf. [3]).

An immediate consequence of Theorem 1.3 is our next corollary.

Corollary 1.4. *Let M, N be infinite subsets of $\mathbb{N}$ subject to conditions* (1.1) *and* (1.2). *Then the dual of the mixed Tsirelson space $T(\frac{1}{m_i}, S_{n_i})_{i=0}^\infty$ admits a c_0^ω-spreading model.*

Applying Theorem 1.3 to the H.I. space $X_\mathcal{N}$ constructed in [14] we obtain:

Corollary 1.5. *There exists an asymptotic ℓ_1, reflexive H.I. Banach space that admits a strictly singular non-compact operator and whose dual admits a c_0^ω-spreading model.*

Remark. In an earlier version of this paper we actually showed that the dual of every subspace of the H.I. space constructed in [14] admits a c_0^ω-spreading model.

Standard duality arguments yield that if $X_\mathcal{N}^*$ admits a c_0^ω-spreading model, then $X_\mathcal{N}$ admits an ℓ_1^ω-spreading model. However, the converse is not true in general. Thus, our approach provides a new method of showing that certain mixed Tsirelson spaces admit ℓ_1^ω-spreading models. This problem has been studied in [6] where it is shown that every subspace of certain regular mixed Tsirelson spaces [2] contains an ℓ_1^ω-spreading model. Their approach is based on the finite representability of c_0 in such spaces [5]. The method of constructing c_0^ω-spreading models in $X_\mathcal{N}^*$ relies on the existence of normalized functionals in $\mathcal{N}$ which belong simultaneously to different classes $\mathcal{N}_j$, $j \geq 0$. Our method is also applied in [6] in order to show that certain modified mixed Tsirelson spaces [5] also admit ℓ_1^ω-spreading models. We finally mention the result of D. Kutzarova and P.K. Lin [17] on the existence of ℓ_1-spreading models in Schlumprecht's space [22].

2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [18]. Let X be a Banach space. A sequence (x_n) in X is *semi-normalized* if there exists $\delta > 0$ such that $\delta \leq \|x_n\| \leq 1$, for all n.

Given any set D, we let $[D]$ (resp. $D^{<\infty}$) denote the set of its infinite (resp. finite) subsets. Given $M \in [\mathbb{N}]$, the notation $M = (m_i)$ indicates that $M = \{m_1 < m_2 < \cdots\}$. Let E and F be finite subsets of $\mathbb{N}$. We write $E < F$ if $\max E < \min F$.

Suppose now that X has a Schauder basis (e_n). A sequence (u_n) of non-zero vectors in X is a *block basis* of (e_n) if there exist successive subsets $F_1 < F_2 < \cdots$ of $\mathbb{N}$ and a scalar sequence (a_n) so that $u_n = \sum_{i \in F_n} a_i e_i$, for every $n \in \mathbb{N}$. We adopt the notation $u_1 < u_2 < \cdots$ to indicate that (u_n) is a block basis of (e_n). We let $\operatorname{supp} u_n$ denote the set $\{i \in F_n : a_i \neq 0\}$.

We shall next review the Schreier hierarchy $\{S_\xi\}_{\xi < \omega_1}$ [1]. Since we shall only be using the families $\{S_\xi\}_{\xi < \omega}$, we confine the definitions to the finite ordinal case.

The Schreier families. We let $S_0 = \big\{\{n\} : n \in \mathbb{N}\big\} \cup \{\emptyset\}$. Suppose S_ξ has been defined, $\xi < \omega$. We set

$$S_{\xi+1} = \{\bigcup_{i=1}^{n} F_i : n \in \mathbb{N},\, n \leq \min F_1, F_1 < \cdots < F_n,\, F_i \in S_\xi \,(i \leq n)\} \cup \{\emptyset\}.$$

An important property shared by the Schreier families is that they are *hereditary*: If $F \in S_\xi$ and $G \subset F$, then $G \in S_\xi$. Another important property is that they are *spreading*: If $\{p_1, \cdots, p_k\} \in S_\xi$, $p_1 < \cdots < p_k$, and $q_1 < \cdots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \cdots, q_k\} \in S_\xi$. It is not hard to check that if $F_1 < \cdots < F_n$ are members of S_α such that $\{\min F_i : i \leq n\}$ belongs to S_β, then $\bigcup_{i=1}^n F_i$ belongs to $S_{\alpha+\beta}$.

A finite collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is said to be S_ξ-admissible, $\xi < \omega$, if there exists an enumeration $\{I_k : k \leq n\}$ of $\mathcal{F}$ such that $I_1 < \cdots < I_n$ and the set $\{\min I_k : k \leq n\}$ is a member of S_ξ. A finite block basis $u_1 < \cdots < u_n$ in a Banach space with a basis is S_ξ-admissible if $\{\operatorname{supp} u_i : i \leq n\}$ is also. A Banach space X with a basis (e_n) is *asymptotic* ℓ_1 [20] if there exists $\delta > 0$ such that every S_1-admissible block basis $(u_i)_{i=1}^p$ of (e_n) satisfies $\| \sum_{i=1}^p u_i\| \geq \delta \sum_{i=1}^p \|u_i\|$.

3. Tree representations of functionals in $\mathcal{N}$

In this section we describe tree representations of members of $\mathcal{N}$ which turn out to be very useful in estimating the norm of certain functionals in $\mathcal{N}$.

We recall that a *tree* is a partially ordered finite set $(\mathcal{T}, \leq)$, such that for every $\alpha \in \mathcal{T}$, the set $\{\beta \in \mathcal{T} : \beta \leq \alpha\}$ is well ordered. The elements of $\mathcal{T}$ are called *nodes*. A node of $\mathcal{T}$ is *terminal* if it has no successors in $\mathcal{T}$. Given $\alpha \in \mathcal{T}$ which is not terminal, we denote by $D_\alpha(\mathcal{T})$ the set of the immediate successors of α in $\mathcal{T}$. A tree $\mathcal{T}$ is *rooted* if it has a unique node α_0 (the root) such that $\alpha_0 \leq \alpha$ for all $\alpha \in \mathcal{T}$. A *branch* of $\mathcal{T}$ is a maximal, under inclusion, well ordered subset. The *height* $o(\mathcal{T})$ of $\mathcal{T}$ is the cardinality of its longest branch.

In the sequel, $\mathcal{N}$ is a (M, N)-Schreier set of measures (see Definition 1.2).

Definition 3.1. A functional tree in $\mathcal{N}$ is a subset $(x^*_\alpha)_{\alpha \in \mathcal{T}}$ of $\mathcal{N}$ indexed by a rooted tree $\mathcal{T}$ and such that the following are satisfied:

(1) $\operatorname{supp} x^*_\beta \subset \operatorname{supp} x^*_\alpha$ whenever $\alpha < \beta$ in $\mathcal{T}$.

(2) If $\alpha \in \mathcal{T}$ is non-terminal, then $(x^*_\beta)_{\beta\in D_\alpha(\mathcal{T})}$ is, under an appropriate enumeration, a finite block basis of (e^*_n).

Definition 3.2. Let $x^* \in \mathcal{N}$. A tree representation of x^* is a functional tree $(x^*_\alpha)_{\alpha\in\mathcal{T}}$ in $\mathcal{N}$ together with a function $\psi\colon \mathcal{T} \to [m_0,\infty)\cup\{0\}$ so that the following properties hold:

(1) $x^*_{\alpha_0} = x^*$, where α_0 is the root of $\mathcal{T}$.
(2) $\psi(\alpha) = 0$ if, and only if, α is terminal. In that case $x^*_\alpha = \pm e^*_{p_\alpha}$ for some $p_\alpha \in \mathbb{N}$.
(3) If $\alpha \in \mathcal{T}$ is non-terminal, then $x^*_\alpha = (1/\psi(\alpha))\sum_{\beta\in D_\alpha(\mathcal{T})} x^*_\beta$.
(4) Every non-terminal $\alpha \in \mathcal{T}$ is either of type I or of type II. Specifically, α is of type I if $\psi(\alpha) = m_j$ and $(x^*_\beta)_{\beta\in D_\alpha(\mathcal{T})}$ is S_{n_j}-admissible for some $j \geq 0$. α is of type II if $D_\alpha(\mathcal{T})$ consists of type I nodes and $\psi|D_\alpha(\mathcal{T})$ is $1-1$.

Notation. We set $\psi_\alpha = \prod_{\beta<\alpha}\psi(\beta)$ if α is not the root of $\mathcal{T}$, and $\psi_\alpha = 1$ if α is the root of $\mathcal{T}$.

It is not hard to see that every member of $\mathcal{N}$ admits a (not necessarily unique) tree representation.

Remark. Suppose that $(x^*_\alpha)_{\alpha\in\mathcal{T}}$ is a tree representation for x^* with associated function ψ. The following facts can be easily established by induction on $o(\mathcal{T})$:

(1) Let A be a subset of $\mathcal{T}$ consisting of pairwise incomparable nodes. Then $(x^*_\alpha)_{\alpha\in A}$ is, under an appropriate enumeration, a block basis of (e^*_n).
(2) If A is additionally assumed to intersect every branch of $\mathcal{T}$, then $x^* = \sum_{\alpha\in A}\frac{1}{\psi_\alpha}x^*_\alpha$.

The proof of Theorem 1.3 requires the following lemma.

Lemma 3.3. *Let $(x^*_\alpha)_{\alpha\in\mathcal{T}}$ be a functional tree in $\mathcal{N}$ and let $\phi\colon \mathcal{T}\to\mathbb{N}$ be a function such that if $\alpha \in \mathcal{T}$ is non-terminal, then $(x^*_\beta)_{\beta\in D_\alpha(\mathcal{T})}$ is $S_{\phi(\alpha)}$-admissible. Then, for every subset A of $\mathcal{T}$ consisting of pairwise incomparable nodes, the collection $(x^*_\alpha)_{\alpha\in A}$ is S_p-admissible, where $p = \max\{\sum_{\beta<\alpha}\phi(\beta) : \alpha \in A\}$.*

Proof. The proof of the lemma is done by induction on $o(\mathcal{T})$. If $o(\mathcal{T}) = 1$ the assertion of the lemma is trivial. Assuming the assertion true when $o(\mathcal{T}) < k$, $k > 1$, let $\mathcal{T}$ be such that $o(\mathcal{T}) = k$. Let α_0 be the root of $\mathcal{T}$ and set $\mathcal{T}_\alpha = \{\beta \in \mathcal{T} : \alpha \leq \beta\}$, for all $\alpha \in D_{\alpha_0}(\mathcal{T})$. Our assumptions yield that $(x^*_\alpha)_{\alpha\in D_{\alpha_0}(\mathcal{T})}$ is $S_{\phi(\alpha_0)}$-admissible. We can assume that $|A| \geq 2$ and set $A_\alpha = \mathcal{T}_\alpha \cap A$, for all $\alpha \in D_{\alpha_0}(\mathcal{T})$. Since $o(\mathcal{T}_\alpha) < k$ our induction hypothesis implies that $(x^*_\beta)_{\beta\in A_\alpha}$ is S_{p_α}-admissible, where $p_\alpha = \max\{\sum_{\alpha\leq\beta<\gamma}\phi(\beta) : \gamma \in A_\alpha\}$, for all $\alpha \in A$. Note that $p_\alpha \leq p - \phi(\alpha_0)$, for all $\alpha \in A$. This completes the inductive step as $(x^*_\alpha)_{\alpha\in D_{\alpha_0}(\mathcal{T})}$ is $S_{\phi(\alpha_0)}$-admissible. □

4. Proof of Theorem 1.3

The existence of a c_0^ω-spreading model (x^*_k) in $X^*_\mathcal{N}$ follows after establishing Lemma 4.1 and Corollary 4.4. The former shows that a natural candidate for (x^*_k) satisfies an upper c_0^ω-estimate. The latter implies that this particular sequence (x^*_k) is semi-normalized, based on a decomposition result (Lemma 4.3) for the elements of $\mathcal{N}$, and thus it is indeed a c_0^ω-spreading model. It will be crucial for the entire proof that $x^*_k \in \mathcal{N}_j$, for all $j \leq k$.

Remark. Observe that if $x_1^* < \cdots < x_p^*$ is an $S_{n_i+n_j}$-admissible sequence in $\mathcal{N}$, for some $i \geq 0$, $j \geq 0$, then $\frac{1}{m_i m_j}\sum_{k=1}^p x_k^* \in \mathcal{N}$. Indeed, we may decompose $\{1,\dots,p\}$ into successive subsets $F_1 < \cdots < F_t$ so that $(x_k^*)_{k\in F_r}$ is S_{n_j}-admissible for every $r \leq t$, while $(x^*_{\min F_r})_{r=1}^t$ is S_{n_i}-admissible. If we set $y_r^* = \frac{1}{m_j}\sum_{k\in F_r} x_k^*$, then $y_r^* \in \mathcal{N}$ for all $r \leq t$, by condition (2) of Definition 1.2. Clearly, $(y_r^*)_{r=1}^t$ is S_{n_i}-admissible and thus $\frac{1}{m_i}\sum_{r=1}^t y_r^* \in \mathcal{N}$, again by condition (2) of Definition 1.2.

A similar inductive argument now implies the following: Let $q = \sum_{i=0}^t a_i n_i$ for some $t \in \mathbb{N}\cup\{0\}$ and $a_i \in \mathbb{N}\cup\{0\}$, $i \leq t$. Assume that $(x_l^*)_{l=1}^k$ is an S_q-admissible collection of functionals in $\mathcal{N}$. Then $(1/\prod_{i\leq t} m_i^{a_i})\sum_{l=1}^k x_l^* \in \mathcal{N}$.

Notation. We set $p_k = \sum_{i<k} s_i n_i$, for all $k \in \mathbb{N}$ (see (1.1)). Note that $p_k \leq 2f_k$ by the choice of N.

Lemma 4.1. *Let $F_1 < F_2 < \cdots <$ be successive subsets of $\mathbb{N}$ such that $F_k \in S_{p_k}$ and set $x_k^* = \frac{1}{m_k}\sum_{i\in F_k} e_i^*$, for all $k \in \mathbb{N}$. Assume $(x_k^*)_{k=1}^\infty$ is semi-normalized. Then $(x_k^*)_{k=1}^\infty$ is a c_0^ω-spreading model.*

Proof. We first observe that for all $l \leq k$ we can find $z_1^* < \cdots < z_t^*$ in $\mathcal{N}$, S_{p_l}-admissible so that $x_k^* = \frac{1}{m_l}\sum_{i=1}^t z_i^*$. Indeed, we may write $F_k = \bigcup_{i=1}^t H_i$, where $H_1 < \cdots < H_t$ belong to $S_{p_k - p_l}$ and $(H_i)_{i=1}^t$ is S_{p_l}-admissible. Set $z_i^* = (m_l/m_k)\sum_{r\in H_i} e_r^*$, $i \leq t$. But since $p_k - p_l = \sum_{l\leq j<k} s_j n_j$, and $z_i^* = \frac{1}{\prod_{l\leq j<k} m_j^{s_j}}\sum_{r\in H_i} e_r^*$, $i \leq t$, our preceding remark yields that $(z_i^*)_{i=1}^t$ is an S_{p_l}-admissible family in $\mathcal{N}$ which clearly satisfies $x_k^* = (1/m_l)\sum_{i=1}^t z_i^*$.

Now let $k_0 \in \mathbb{N}$. Let $F \in \mathbb{N}^{<\infty}$ with $\min F \geq k_0$ and so that $(x_k^*)_{k\in F}$ is $S_{f_{k_0}}$-admissible. According to our initial observation, for each $k \in F$ there exists an $S_{p_{k_0}}$-admissible family $(y_i^*)_{i\in G_k}$ in $\mathcal{N}$, so that $x_k^* = \frac{1}{m_{k_0}}\sum_{i\in G_k} y_i^*$. Note that $p_{k_0} \leq 2f_{k_0}$. We now obtain, since $4f_{k_0} < n_{k_0}$, that $(y_i^*)_{i\in G}$ is $S_{n_{k_0}}$-admissible, where $G = \bigcup_{k\in F} G_k$. Of course, $\sum_{k\in F} x_k^* = \frac{1}{m_{k_0}}\sum_{i\in G} y_i^*$. Therefore, $\sum_{k\in F} x_k^* \in \mathcal{N}$, by condition (2) of Definition 1.2. The proof is now complete since $n \leq f_n$ for all $n \in \mathbb{N}$. □

We shall also make use of the following numerical result.

Lemma 4.2. *Assume that $(a_i)_{i=0}^{k-1}$ are positive integers satisfying $\prod_{i<k} m_i^{a_i} < m_k$. Then $\sum_{i<k} a_i n_i < \sum_{i<k} s_i n_i$.*

Proof. By induction on k. The case $k = 1$ is trivial since $a_0 < s_0$. Assume the assertion holds for some $k \geq 1$ and let the integers $(a_i)_{i=0}^k$ satisfy $\prod_{i<k+1} m_i^{a_i} < m_{k+1}$. Observe that $m_{k+1} = m_k^{s_k+1}$. Clearly, $a_k \leq s_k$. We shall distinguish between two cases: $a_k = s_k$ and $a_k < s_k$. If the former, then $\prod_{i<k} m_i^{a_i} < m_k$. By the induction hypothesis we obtain $\sum_{i<k} a_i n_i < \sum_{i<k} s_i n_i$. The assertion now follows as $a_k = s_k$.

When $a_k < s_k$, we obtain that $\prod_{i<k} m_i^{a_i} < m_k^{s_k - a_k + 1}$. Our assumptions on N allow us to deduce that $\sum_{i<k} a_i n_i \leq 2(s_k - a_k + 1)f_k$. It follows that $\sum_{i<k+1} a_i n_i \leq 2(s_k - a_k + 1)f_k + a_k n_k$. But also, $2(s_k - a_k + 1)f_k + a_k n_k < s_k n_k$. Indeed, the latter inequality follows easily as $4f_k < n_k$. Hence, $\sum_{i<k+1} a_i n_i < s_k n_k$ from which the assertion follows. The inductive step as well as the proof of the lemma are now complete. □

Lemma 4.3. *Let $k \in \mathbb{N}$ and $x^* \in \mathcal{N}$ such that $k \leq \min \operatorname{supp} x^*$. Then there exist $p \in \mathbb{N}$, a partition $\{I_1, I_2\}$ of $\{1, \dots, p\}$, functionals $x_1^* < \cdots < x_p^*$ in $\mathcal{N}$ and scalars $(\lambda_i)_{i=1}^p$ in $(0,1]$ so that the following are satisfied:*

(1) $x^* = \sum_{i=1}^p \lambda_i x_i^*$.
(2) $x_i^* = \pm e_{j_i}^*$ *for all* $i \in I_1$ *and* $\{j_i : i \in I_1\} \in S_{p_k - 1}$.
(3) $\lambda_i \leq 1/m_k$ *for all* $i \in I_2$.

Proof. Let $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ be a tree representation for x^* with associated function ψ. Let $\mathfrak{B}$ denote the set of all branches of $\mathcal{T}$. Given $b \in \mathfrak{B}$ let $\alpha(b)$ denote the smallest node $\beta \in b$ such that $\psi_\beta \geq m_k$, or, if such a β does not exist, let $\alpha(b)$ be the terminal node of b.

We let $A = \{\alpha(b) : b \in \mathfrak{B}\}$. It is not hard to check that A consists of pairwise incomparable nodes of $\mathcal{T}$. Since A intersects all branches of $\mathcal{T}$, we have that $x^* = \sum_{\alpha \in A} \frac{1}{\psi_\alpha} x_\alpha^*$. We now set $A_1 = \{\alpha \in A : \psi_\alpha < m_k\}$ and $A_2 = A \setminus A_1$. It is clear that A_1 consists of terminal nodes of $\mathcal{T}$. Let $(x_i^*)_{i=1}^p$ be an enumeration of $(x_\alpha^*)_{\alpha \in A}$ such that $x_1^* < \cdots < x_p^*$. We now define $I_r = \{i \leq p : x_i^* \in (x_\alpha^*)_{\alpha \in A_r}\}$ for $r \leq 2$. We finally set $\lambda_i = 1/\psi_\alpha$ $(i \leq p)$ if $x_i^* = x_\alpha^*$ for some $\alpha \in A$.

We need only show that $(x_i^*)_{i \in I_1}$ is S_{p_k-1}-admissible. The rest of the required properties are straightforward. To this end, we set $\mathcal{R} = \bigcup_{\alpha \in A_1} \{\beta \in \mathcal{T} : \beta \leq \alpha\}$. The key point is that if $\beta \in \mathcal{R}$ is of type II in $\mathcal{T}$, then $|D_\beta(\mathcal{R})| \leq k$. Indeed, $\psi | D_\beta(\mathcal{R})$ is 1-1 as β is of type II. On the other hand $\psi(\gamma) < m_k$ and $\psi(\gamma) \in M$ for all $\gamma \in D_\beta(\mathcal{R})$. It follows that $|D_\beta(\mathcal{R})| \leq k$. In particular, $(x_\gamma^*)_{\gamma \in D_\beta(\mathcal{R})}$ is S_1-admissible.

Define $\phi \colon \mathcal{R} \to \mathbb{N}$ by

$$\phi(\beta) = \begin{cases} n_i, & \text{if } \beta \text{ is of type I and } \psi(\beta) = m_i, \text{ for some } i \geq 0, \\ n_0, & \text{if } \beta \text{ is of type II,} \\ 1, & \text{if } \beta \in A_1. \end{cases}$$

We now have that $(x_\gamma^*)_{\gamma \in D_\beta(\mathcal{R})}$ is $S_{\phi(\beta)}$-admissible, for every non-terminal $\beta \in \mathcal{R}$. Since $\prod_{\beta < \alpha} \psi(\beta) < m_k$, for all $\alpha \in A_1$, Lemma 4.2 yields $\sum_{\beta < \alpha} \phi(\beta) < p_k$, for all $\alpha \in A_1$. Therefore, $(x_\alpha^*)_{\alpha \in A_1}$ is S_{p_k-1}-admissible by Lemma 3.3. This completes the entire proof. □

Corollary 4.4. *For every $x^* \in \mathcal{N}$ and $k \in \mathbb{N}$ we have that $\{i \in \mathbb{N}, i \geq k : |x^*(e_i)| \geq 2/m_k\}$ belongs to S_{p_k-1}.*

Proof. We can assume that $\min \operatorname{supp} x^* \geq k$. We next apply Lemma 4.3 to conclude that $\{i \in \mathbb{N}, i \geq k : |x^*(e_i)| \geq 2/m_k\}$ is contained in $\{j_i : i \leq p\}$ which belongs to S_{p_k-1}. □

Proof of Theorem 1.3. We first show that given k, l in $\mathbb{N}$ there exists $F_k \in S_{p_k}$, $l < \min F_k$, so that letting $x_k^* = \frac{1}{m_k} \sum_{i \in F_k} e_i^*$ we have that $1/3 \leq \|x_k^*\| \leq 1$. Once this is accomplished and since we can take the F_k's to be successive, it will follow from Lemma 4.1 that (x_k^*) is a c_0^ω-spreading model.

A standard property of the repeated averages hierarchy [9] (Lemma 2.3 of [13]) allows us to find $F_k \in S_{p_k}$ with $\max\{k, l\} < \min F_k$, positive scalars $(a_i)_{i \in F_k}$ with $\sum_{i \in F_k} a_i = 1$ and such that $\sum_{i \in G} a_i < \frac{1}{m_k}$ for every $G \in S_{p_k-1}$. It follows readily from Corollary 4.4 that for every $x^* \in \mathcal{N}$, the set $\{i \in F_k : |x^*(e_i)| \geq 2/m_k\}$ belongs to S_{p_k-1}. Letting $x_k = \sum_{i \in F_k} a_i e_i$, we conclude that $\|x_k\| \leq \frac{3}{m_k}$. But since $x_k^* \in \mathcal{N}$ and $x_k^*(x_k) = 1/m_k$, we have that $\frac{1}{3} \leq \|x_k^*\| \leq 1$, as desired.

For the moreover assertions, we may assume, by passing to a subsequence of the x_k^*'s, that if $F \in \mathbb{N}^{<\infty}$, $\min F > j$, and $(x_i^*)_{i\in F}$ is S_{n_j}-admissible, then $\|\sum_{i\in F} x_i^*\| \le 1$.

We define a linear map $T\colon c_{00} \to c_{00}$ by the formula $Tx = \sum_i x_i^*(x)e_i$. We are going to show that there exists a constant $C > 0$ such that $\|Tx\|_{\mathcal{N}} \le C\|x\|_{\mathcal{N}}$, for all $x \in c_{00}$. It will then follow that T extends to an operator on $X_{\mathcal{N}}$, still denoted by T, which satisfies $Tx = \sum_i x_i^*(x)e_i$, for all $x \in X_{\mathcal{N}}$. T is the desired operator. Indeed, T is non-compact because (x_i^*) is semi-normalized. On the other hand if $\mathrm{Ker}(T)$ is infinite-dimensional (which is the case if $\mathbb{N} \setminus \bigcup_i \operatorname{supp} x_i^*$ is infinite) and $X_{\mathcal{N}}$ is H.I., then T is strictly singular.

It thus remains to establish that T is bounded on c_{00} with respect to the $\|\cdot\|_{\mathcal{N}}$ norm. To this end, fix a normalized $x \in X_{\mathcal{N}}$ and let $x^* \in \mathcal{N}$. If $x^* = \pm e_n^*$ for some $n \in \mathbb{N}$, then we easily see that $|x^*(Tx)| \le 1$. If the support of x^* contains at least two elements, set $H_1 = \{i \in \operatorname{supp} x^* : \frac{2}{m_1} \le |x^*(e_i)| \le \frac{1}{m_0}\}$ and $H_k = \{i \in \operatorname{supp} x^* : \frac{2}{m_k} \le |x^*(e_i)| < \frac{2}{m_{k-1}}\}$, for $k \ge 2$. Note that $\operatorname{supp} x^* = \bigcup_{k=1}^\infty H_k$. We also put $G_k = \{i \in H_k : i \ge k\}$. Corollary 4.4 yields that $G_k \in S_{n_k}$, as $p_k < n_k$ for all $k \ge 1$, and therefore $\sum_{i\in G_k} |x_i^*(x)| \le 2$. We now have

$$\begin{aligned}|x^*(Tx)| &= \left|\sum_{k=1}^\infty \sum_{i\in H_k} x_i^*(x)x^*(e_i)\right| \\ &\le \sum_{i\in H_1} |x_i^*(x)||x^*(e_i)| + \sum_{k=2}^\infty \left(\sum_{i\in G_k} |x_i^*(x)| + \sum_{i\in H_k\setminus G_k} |x_i^*(x)|\right)\frac{2}{m_{k-1}} \\ &\le \frac{2}{m_0} + \sum_{k=2}^\infty \left(\frac{4}{m_{k-1}} + 2\frac{k-1}{m_{k-1}}\right) \le \sum_{k\ge 0} \frac{4+2k}{m_k} = D.\end{aligned}$$

To complete the proof of the theorem we need only take $C = \max\{D, 1\}$. □

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