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INVERSE LIMITS OF ALGEBRAS AS RETRACTS OF THEIR DIRECT PRODUCTS

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ABSTRACT. Inverse limits of modules and, more generally, of universal algebras, are not always pure in corresponding direct products. In this note we show that when certain set-theoretic properties are imposed, they even become direct summands.

Given a direct system $\{M_i\}_{i\in I}$ of modules, it is well known that $\varinjlim M_i$ is a pure quotient of the direct sum $\bigoplus_{i\in I} M_i$. In contrast, the dual statement that inverse limits are pure submodules of corresponding direct products is not always true: For each prime number p, we can construct a descending chain $\{A_n\}_{n\in\mathbb{N}}$ of divisible abelian groups whose intersection A is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (see [2, Exercise 6, p. 101]). Since divisibility is inherited by pure subgroups and direct products and since A is not divisible, it follows that the inverse limit A of the divisible groups A_n is not pure in $\prod_{n\in\mathbb{N}} A_n$. However, as we shall show in this note, when certain set-theoretic conditions are imposed on an inverse system of modules, the inverse limit is a *direct summand* of the corresponding direct product. This is motivated by the following observation: Let p be a prime number and let J_p be the p-adic group $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. As each $\mathbb{Z}/p^n\mathbb{Z}$ is finite, J_p is linearly, and hence algebraically compact. (See [1] and [2].) Since, as can easily be proved, J_p is pure in $\prod_n \mathbb{Z}/p^n\mathbb{Z}$, it follows that the canonical monomorphism $0 \to \varprojlim \mathbb{Z}/p^n\mathbb{Z} \to \prod_n \mathbb{Z}/p^n\mathbb{Z}$ splits.

The purpose of this note is to generalize this result in both set-theoretic and universal algebraic directions. We refer to [4] and [3] for the various undefined notions used here from the theory of large cardinals and universal algebra, respectively. Recall that a *tree* is a poset $(T,<)$ such that for each $t \in T$ the set $\{s \in T : s < t\}$ of the predecessors of t is well ordered by $<$. A subalgebra B of an algebra A is a *retract* of A if there exists a homomorphism $g : A \to B$ whose restriction to B is the identity on B; such a g is called a *retraction.* A directed set $\{I; \leq\}$ is λ-*directed* for some infinite cardinal λ, if every subset of I of size less than λ has an upper bound in I.

First, we need

Lemma 1. *A subalgebra B of an algebra A is a retract of A if and only if every system of equations over B and with a solution in A has a solution in B.*

Proof. (Cf. [2, Proposition 22.3].) Suppose B is a retract of A, with retraction g, and let Σ be a system of equations over B with a set of unknowns $\{x_s\}_{s\in S}$. If

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$\{a_s\}_{s\in S}$ is a solution of A in Σ, then, clearly, $\{g(a_s)\}_{s\in S}$ is a solution of Σ in B. Conversely, let Σ be the system over B

$$x_{f((a_i)_{i\in r(f)})} = f((x_{a_i})_{i\in r(f)}),$$
$$x_b = b$$

for any $a_i \in A, b \in B$ and any operation f on A (with arity $r(f)$), and where the unknowns are indexed by A. This system is solvable in A by $x_a = a$ $(a \in A)$. Thus, if $x_a = g(a)$ $(a \in A)$ is a solution of Σ in B, then the mapping $g : A \to B$ is a retraction. $\square$

Proposition 2. *Let α be a limit ordinal, κ be an infinite cardinal and $\{A_i; \sigma_i^j\}_{i\le j<\alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa < \mathrm{cf}(\alpha)$. Then the inverse limit $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$.*

Proof. We first show that every system of equations over $\varprojlim A_i$ and solvable in $\prod_{i<\alpha} A_i$ has a solution in $\varprojlim A_i$. Let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s\in S}$, let C be the set of all constants appearing in Σ, and suppose Σ is solvable in $\prod_{i<\alpha} A_i$ by $\{a_s\}_{s\in S}$, say. For each $i < \alpha$, let Σ^i be the system obtained from Σ by replacing each c in C by its i-th coordinate $c(i)$ in A_i. Fix s in S and let $T_s^i = \{\sigma_i^j(a_s(j)) : i \le j < \alpha\}$, where $a_s(j)$ is the j-th coordinate of a_s. Partial-order $T_s = \bigcup_{i<\alpha} T_s^i$ by setting $x < y$ when $x \in T_s^i, y \in T_s^j$ and $\sigma_i^j(y) = x$ for some $i < j < \alpha$. It is easy to see that $(T_s, <)$ is a tree of height α. For any $x = \sigma_i^j(a_s(j))$ in T_s^i, we have $x = a_s(i)$ or $x = \sigma_i^{i+1}\sigma_{i+1}^j(a_s(j)) \in \sigma_i^{i+1}(A_{i+1})$, so that $T_s^i \subseteq \{a_s(i)\} \cup \sigma_i^{i+1}(A_{i+1})$, and therefore $|T_s^i| < \kappa$. As α is a limit ordinal, T_s has a branch $(\mu_s(i))_{i<\alpha}$, by [5, Proposition 2.32, p. 304]. Since $\sigma_i^j(\mu_s(j)) = \mu_s(i)$ for all $i \le j < \alpha$, we obtain that $(\mu_s(i))_{i<\alpha} \in \varprojlim A_i$ for each $s \in S$. Now we have $c(i) = \sigma_i^j(c(j))$ for all $i \le j$, since $C \subseteq \varprojlim A_i$, so that for all $j \ge i$, $\{\sigma_i^j(a_s(j))\}_{s\in S}$ is a solution of Σ^i. By definition of T_s, for each $i < \alpha$, $\mu_s(i) = \sigma_i^j(a_s(j))$ for some $j \ge i$, i.e. $\{\mu_s(i)\}_{s\in S}$ is a solution of Σ^i. Since $(\mu_s(i))_{i<\alpha} \in \varprojlim A_i$ for all s in S, we infer that Σ is solvable in $\varprojlim A_i$, as required. It now follows that $\varprojlim A_i$ is not empty (choose Σ with $C = \varnothing$) and therefore is a subalgebra of $\prod_{i<\alpha} A_i$, and that it is a retract of $\prod_{i<\alpha} A_i$, by Lemma 1. $\square$

We next turn our attention to cardinals with the tree property stating that the cardinal satisfies König's Lemma. Recall that $\aleph_0$ and weakly compact (e.g. measurable) cardinals have the tree property, whereas $\aleph_1$ and singular cardinals do not.

Proposition 3. *Let α be a limit ordinal, κ be an infinite cardinal with the tree property, and $\{A_i; \sigma_i^j\}_{i\le j<\alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa \le \mathrm{cf}(\alpha)$. Then $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$.*

Proof. If $\kappa < \mathrm{cf}(\alpha)$, use Proposition 2. Suppose that $\kappa = \mathrm{cf}(\alpha)$ with $\alpha = \sum_{t<\kappa} \alpha_t$, where $\alpha_t < \alpha$. Then, using the tree property of κ and an argument similar to that of Proposition 2, we obtain that $\varprojlim A_i$ is a subalgebra of $\prod_{i<\alpha} A_i$, and that $\varprojlim A_{\alpha_t}$, the inverse limit of the inverse family $\{A_{\alpha_t}; \sigma_{\alpha_t}^{\alpha_s}\}_{t\le s<\kappa}$, is a retract of $\prod_{t<\kappa} A_{\alpha_t}$. Let $\varphi : \prod_{i<\alpha} A_i \to \prod_{t<\kappa} A_{\alpha_t}$ be the canonical projection. Then (see for example the proof of [3, Lemma 7, p. 133]), the restriction ψ of φ to $\varprojlim A_i$ is an isomorphism $\varprojlim A_i \to \varprojlim A_{\alpha_t}$ and we have $\varphi f = g\psi$, where $f : \varprojlim A_i \to \prod_{i<\alpha} A_i$ and $g : \varprojlim A_{\alpha_t} \to \prod_{t<\kappa} A_{\alpha_t}$ are the inclusions mappings. If $\pi : \prod_{t<\kappa} A_{\alpha_t} \to \varprojlim A_{\alpha_t}$ is such that πg is the identity, then $\psi^{-1}\pi\varphi f = \psi^{-1}\pi g\psi$ is the identity mapping on $\varprojlim A_i$, and so $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$. □

The conclusion of Proposition 3 can be arrived at for a wider class of inverse systems, provided κ is a compact cardinal. An infinite cardinal λ is *compact* if, for any set S, every λ-complete filter on S can be extended to a λ-complete ultrafilter. Thus $\aleph_0$ is compact, and it is known that uncountable compact cardinals are necessarily measurable. We have

Proposition 4. *Let $\{A_i; \sigma_i^j\}_{i\in I}$ be an inverse system of algebras and let κ be a compact cardinal such that $\{I; \leq\}$ is κ-directed and $|\bigcup_{j>i} \sigma_i^j(A_j)| < \kappa$, for every $i \in I$. Then $\varprojlim A_i$ is a retract of $\prod_{i\in I} A_i$.*

Proof. We first show that $\varprojlim A_i$ is non-empty. For each $i \in I$, let $p_i \in A_i, \pi_i : \prod_{j\in I} A_j \to A_i$ be the i-th canonical projection, and let $T_i = \{\sigma_i^j(p_j) : i, j \in I, i \leq j\}$. For every $J \in [I]^{<\kappa} = \{S \subseteq I : |S| < \kappa\}$, let $X_J = \{x \in \prod_{i\in I} T_i : \sigma_i^j(p_j) = p_i$, for all $i, j \in J$ and $i \leq j\}$. Since I is κ-directed and κ is regular (compact cardinals are regular), $\varnothing \subset X_{\bigcup_{\tau<\lambda} J_\tau} \subseteq \bigcap_{\tau<\lambda} X_{J_\tau}$, whenever $J_\tau \in [I]^{<\kappa}$ and λ is a cardinal less than κ. It follows that the set $\{X_J\}_{J\in[I]^{<\kappa}}$ generates on $\prod_{i\in I} T_i$ a κ-complete proper filter, which, as κ is compact, can be extended to a κ-complete ultrafilter U. For each $Y \in U$, let $Y_i = \{x_i \in T_i : x = (x_i)_{i\in I} \in Y\}$ and let $U_i = \{Y_i : Y \in U\}$. As in the proof of [3, Theorem 1, p. 132], we obtain that U_i is a κ-complete ultrafilter on T_i. By hypothesis $|T_i| < \kappa$, so that U_i is a principal generated by a singleton $\{y_i\}$, say. Now, for all $i, j \in I$, $\pi_i^{-1}(\{y_i\}), \pi_j^{-1}(\{y_j\})$ and $X_{\{i,j\}}$ are in U, so that $\pi_i^{-1}(\{y_i\}) \cap \pi_j^{-1}(\{y_j\}) \cap X_{\{i,j\}} \in U$. Therefore, if $i \leq j$, there exists $x = (x_i)_{i\in I} \in X_{\{i,j\}}$ such that $\sigma_i^j(y_j) = \sigma_i^j(x_j) = x_i = y_i$. This proves that $\varprojlim A_i$ is non-empty, and thus a subalgebra of $\prod_{i\in I} A_i$. Next, let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s\in S}$ and constants $\{c\}_{c\in C}$, and suppose it is solvable in $\prod_{i\in I} A_i$ by $\{a_s\}_{s\in S}$. For each $i \in I$, let Σ^i be the system obtained from Σ by replacing each c in C by its i-th coordinate $c(i)$ in A_i. Fix s in S, and set $B_i^s = \{\sigma_i^j(a_s(j)) : j \in I, i \leq j\}$. It is easy to see that $\{B_i^s\}_{i\in I}$ can be regarded as an inverse system of non-empty sets with bonding maps σ_i^j $(i \leq j)$. By the first part of this proof, $\varprojlim B_i^s$ is non-empty. Clearly, if $\mu_s \in \varprojlim B_i^s$, then $\{\mu_s\}_{s\in S}$ is a solution of Σ in $\varprojlim A_i$. Now use Lemma 1. □

Remarks. The first part of the foregoing proof (that $\varprojlim A_i$ is non-empty) is a straightforward adaptation of an argument of Grätzer [3, Theorem 1, p. 132], where he used ultrafilters to prove the classical theorem that inverse limits of finite non-empty sets are non-empty. Indeed, since $\aleph_0$ is compact, Proposition 4 generalizes both [3, Theorem 1, p. 132] (and hence König's Graph Lemma) and the observation on J_p mentioned above. As was pointed out by the referee, Proposition 2 and its proof provide an alternative proof of [3, Theorem 1, p. 132] for inverse systems $\{A_i; \sigma_i^j\}_{i<\alpha}$: If $\mathrm{cf}(\alpha) = \omega$, then, for some countable cofinal subset $\{i_n : n \in \mathbb{N}\}$ of α, $\varprojlim A_i$ is isomorphic to $\varprojlim A_{i_n}$ which, using the proof of Proposition 2 with König's Lemma, is non-empty. If $\mathrm{cf}(\alpha) > \omega$, then the same follows again by using $\kappa = \omega$ in Proposition 2.

Corollary 5. *Let κ be a compact cardinal and let $\{A_i\}_{i\in I}$ be an inverse system of algebras such that $\{I; \leq\}$ is κ-directed and $|A_i| < \kappa$ for all $i \in I$. Then $\varprojlim A_i$ is a retract of $\prod_{i\in I} A_i$.* □

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