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TRACE SPLITTINGS IN C^*-ALGEBRAS OF TILING SYSTEMS VIA COLOURINGS

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Abstract. Tiles of a hierarchical tiling system are coloured with given colours. The resulting system implements colour symmetries and prescribed frequencies and is itself a hierarchical system whose prototile types admit an elegant description. The frequencies of occurrence of colours is interpreted using the unique trace on the C^*-algebra of the given tiling system and the trace on the C^*-algebra of the coloured tiling system.

1. Introduction and preliminaries

In [EP] we obtained results demonstrating how one may colour the tiles of a hierarchical tiling (dynamical) system so that the resulting system is repetitive (i.e., has local isomorphism property) and has prescribed colour symmetries and also prescribed frequencies of colour occurrence. In this article, we show that the colouring obtained in [EP] is itself described by a hierarchical (coloured) tiling system. Then we describe the unique trace of the associated C^*-algebras and exhibit the colour frequencies using the trace. The precise statement of the result is found in Theorem 2.11. It says that projections in the C^*-algebra of the initial tiling system can be split into an orthogonal sum of projections in the C^*-algebra of the coloured tiling system respecting colour symmetries and such that when the trace is evaluated on both sides of the above splitting one finds decomposition in the desired frequency ratio. Furthermore, this can be done while preserving the partial order structure among self-adjoint elements.

A lot of simplification arises in the understanding of hierarchical tiling systems, their tiling space and the associated C^*-algebra (following [Co, II.3]), by the introduction of what we call in this article '*the relative position indexing set*'.

1.1. We may as well begin by recalling the standard features of a hierarchical tiling system: All the tilings considered are in some Euclidean space $\mathbb{R}^d$.

1.1.1. We have $(T_0, S_0), (T_1, S_1), (T_2, S_2), \ldots, (T_i, S_i), \ldots$ where T_i is a tiling of $\mathbb{R}^d$ by tiles congruent to one of the *prototiles* from the set S_i. We do not require that two tiles which are congruent to the same prototile are translates of each other, nor is it necessary that two tiles which are translates of each other must have the

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same prototile type. One more remark in the same vein: Possibly due to some symmetries one may 'visualize' more than one congruence of a tile to the same prototile, but only one congruence is legitimate.

1.1.2. Every prototile in S_{i+1} is subdivided into several subtiles each congruent to a prototile in S_i.

1.1.3. Any T_{i+1}-tile δ_{i+1} is the union of the T_i-tiles contained in δ_{i+1} and this subdivision depends only on the prototile type of δ_{i+1}. In fact if δ_{i+1} is congruent to an S_{i+1}-prototile τ, the subdivision of δ_{i+1} is congruent to the subdivision of τ given by (1.1.2).

1.1.4. All prototile sets S_i have the same cardinality n and the elements of each of these sets are enumerated by integers $1, 2, \dots, n$. The i-th S_ℓ-prototile is obtained from the i-th $S_{\ell-1}$-prototile by multiplication by a positive real number ξ greater than one (independent of i and ℓ). (Note: Our results also apply to more general inflation automorphisms, *e.g. self-affine* that have been considered in the literature. See [So] and the references therein.)

1.1.5. '*Multiplicity matrix*': There is an $n \times n$ matrix A with nonnegative integral entries $a_{i,j}$ such that the subdivision of the j-th S_ℓ-prototile given by (1.1.2) consists of $a_{1,j}$ copies congruent to the first $S_{\ell-1}$-prototile, $a_{2,j}$ copies congruent to the second $S_{\ell-1}$-prototile, etc., and $a_{n,j}$ copies congruent to the n-th $S_{\ell-1}$-prototile. *Notice that the matrix A is independent of ℓ.* As we already noted, two tiles may be translates of each other without being congruent to the same prototile. With this in mind, to avoid possible ambiguity, one should really say that $a_{1,j}$ subtiles in the subdivision of the j-th S_ℓ-prototile are 'marked' congruent to the first $S_{\ell-1}$-prototile, $a_{2,j}$ subtiles are 'marked' congruent to the second $S_{\ell-1}$-prototile, etc. (One can iterate on the subdivision and eventually every S_ℓ-prototile is subdivided into smaller subtiles each congruent to a prototile in S_0. We assume that for large enough ℓ every S_0-prototile type occurs in the subdivision of each S_ℓ-prototile.)

1.1.6. We strengthen the hypotheses about the subdivion in (1.1.5) by making the following assumption about the relative position of the subdivided tiles inside the ambient tile: The inflation automorphism (multiplication by ξ) carries the i-th S_ℓ-prototile with its subdivision into the subdivison of the i-th $S_{\ell+1}$-prototile.

It is the combination of hypotheses (1.1.3) and (1.1.6) which qualifies the tiling structure to be called *'hierarchical'*.

1.2. Let us choose once and for all a finite set Λ of cardinality $\sum_{i,j=1}^{n} a_{i,j}$ equipped with two maps $\pi_1, \pi_2 : \Lambda \longrightarrow \{1, 2, \dots, n\}$ such that

$$\#\{\lambda \in \Lambda \mid \pi_1(\lambda) = i, \pi_2(\lambda) = j\} = a_{i,j}.$$

In view of hypothesis (1.1.5) above we can fix once and for all a bijection between the last-mentioned subset of Λ and the set of subtiles congruent to the i-th S_0-prototile in the subdivision of the j-th S_1-prototile. Putting together all these bijections ($\forall i, j$) an element $\lambda \in \Lambda$ points to a unique S_1-prototile τ (τ is the j-th prototile where $j = \pi_2(\lambda)$) and in addition a unique subtile δ in the subdivision of τ given by (1.1.2). If $i = \pi_1(\lambda)$, then this subtile δ is congruent to the i-th S_0-prototile. *We say that λ is the relative position of δ inside τ.* This is a bijective correspondence between Λ and the set of all subtiles in the subdivision (1.1.2) of

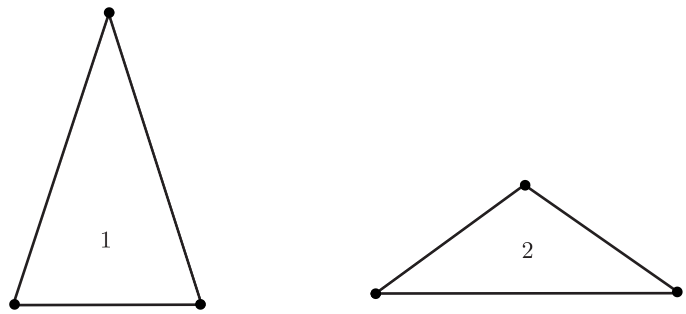

FIGURE 1.

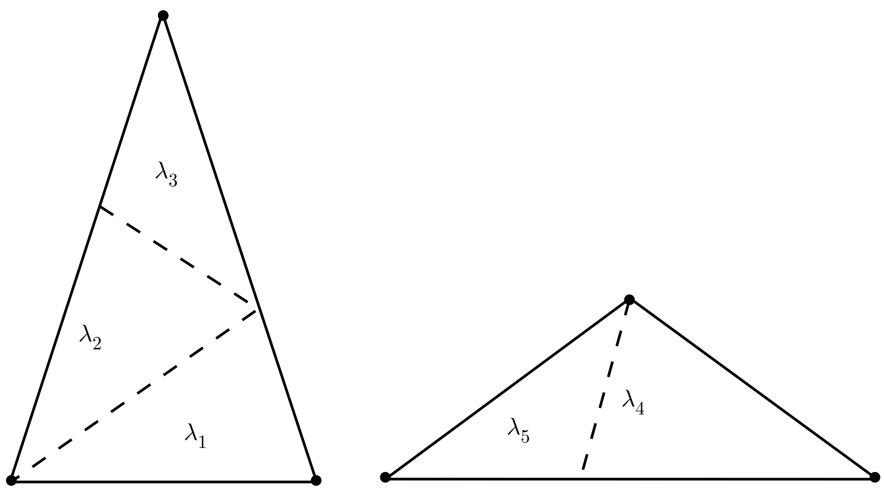

FIGURE 2.

the various S_1-prototiles. This set Λ will be called the '*relative position indexing set*'.

Example. The Penrose tiling has two prototile types 1 and 2 (*cf.* Figure 1). The T_1-prototile of type 1 decomposes as the union of two T_0-tiles of type 1 and one of type 2. The T_1-prototile of type 2 decomposes into the union of a T_0-prototile of type 1 and one of type 2.

Then it is easy to see that the multiplicity matrix A is given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The cardinality of the relative position index set Λ is 5. The picture shows the relative positions.

Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as shown in Figure 2. Then the two projections π_1, π_2 from Λ into $\{1, 2\}$ are defined as follows:

$$\begin{cases} \pi_1(\lambda_1) = \pi_1(\lambda_2) = \pi_1(\lambda_4) = 1, \\ \pi_1(\lambda_3) = \pi_1(\lambda_5) = 2, \\ \pi_2(\lambda_1) = \pi_2(\lambda_2) = \pi_2(\lambda_3) = 1, \\ \pi_2(\lambda_4) = \pi_2(\lambda_5) = 2. \end{cases}$$

□

Any T_0-tile δ is contained in a unique T_1-tile τ. In view of (1.1.3) the subdivision of τ into subtiles depends only on the S_1-prototile type of τ. Hence by the above paragraph preceding the Example, to the relative position of δ in τ there corresponds a unique relative position index $\lambda \in \Lambda$. More generally, any T_i-tile δ' is contained in a unique T_{i+1}-tile τ'. But by (1.1.3) and (1.1.6) via the (iterated) inflation automorphism ξ^i this inclusion is congruent to the image of the inclusion of a subtile δ in the subdivision (1.1.2) of a unique S_1-prototile τ. Hence again, to the relative position of δ' in τ' there corresponds a unique relative position index $\lambda \in \Lambda$.

1.3. **Tiling space.** A T_i-tile δ_i is contained in a unique T_{i+1}-tile δ_{i+1}. Thus the choice of a T_0-tile δ_0 gives rise to an infinite sequence $\delta_0, \delta_1, \dots$. If $\delta'_0, \delta'_1, \dots$ is such a sequence for another choice of δ_0, then in general $\delta_i = \delta'_i$ for sufficiently large i. We then say that the sequence T_i is *admissible*. This need not happen in exceptional cases. Let X be the space of all infinite sequences

$$X = \{z = (\lambda_1, \lambda_2, \dots, \lambda_i, \dots) \mid \lambda_i \in \Lambda, \pi_2(\lambda_i) = \pi_1(\lambda_{i+1})\}.$$

With the product topology, X is a compact space. The quotient of X by the equivalence relation

“$z = (\lambda_1, \lambda_2, \dots, \lambda_i, \dots)$
is equivalent to
$z' = (\lambda'_1, \lambda'_2, \dots, \lambda'_i, \dots)$
if $\lambda_i = \lambda'_i$ for all sufficiently large i ”

is what one calls the *‘tiling space’* of the given hierarchical tiling system. The quotient topology is not good. By [Co] one studies such ‘non-commutative spaces’ by associating a C^*-algebra to them. As already noted in [Co], for tiling spaces such as the ones encountered in this article this C^*-algebra turns out to be the C^*-inductive limit of a nested sequence of finite dimensional algebras $\mathcal{A}_k$ which we will make explicit shortly. But a rough indication of it is as follows: For each S_k-prototile τ consider the matrix algebra which has rows and columns parametrized by the subtiles of τ for its (k-fold) subdivision into tiles congruent to the S_0-prototiles (*cf.* (1.1.5)). The product of these algebras over all the S_k-prototiles is the algebra $\mathcal{A}_k$. The inclusion of $\mathcal{A}_k$ in $\mathcal{A}_{k+1}$ is given by the subdivision of S_{k+1}-prototiles into copies of tiles in S_k.

Next follows the more explicit description.

1.3.1. *The C^*-algebra of the tiling system.* Let X_k be the set of finite sequences $X_k = \{w = (\lambda_1, \lambda_2, \dots, \lambda_k) \mid \lambda_i \in \Lambda, \pi_2(\lambda_i) = \pi_1(\lambda_{i+1})\}$. (Later on it will be helpful to view X_k as the set of relative position indices for the inclusion of T_i-tiles inside a T_{i+k}-tile.)

Let $\mathcal{A}_k$ be the algebra of (complex-valued) scalar matrices $(a_{w,w'})_{w,w'\in X_k}$ satisfying $a_{w,w'} = 0$ if $\pi_2(\lambda_k) \neq \pi_2(\lambda'_k)$ where λ_k and λ'_k are the k-th components of w and w'. For $w, w' \in X_k$ denote by $\dot{w}, \dot{w}' \in X_{k-1}$ the truncation. The inclusion of $\mathcal{A}_{k-1}$ into $\mathcal{A}_k$ is defined as follows: Let $x = (a_{v,v'})_{v,v'\in X_{k-1}}$ be an element of $\mathcal{A}_{k-1}$. Let $w = (\lambda_1, \dots, \lambda_{k-1}, \lambda_k), w' = (\lambda'_1, \dots, \lambda'_{k-1}, \lambda'_k) \in X_k$. We define the image of x to be the matrix $(a_{w,w'})_{w,w'\in X_k}$ where

$$a_{w,w'} = \begin{cases} 0 & \text{if } \lambda_k \neq \lambda'_k, \\ a_{\dot{w},\dot{w}'} & \text{if } \lambda_k = \lambda'_k. \end{cases}$$

The C^*-inductive limit $\mathcal{A} = \lim_k \mathcal{A}_k$ for the above-defined inclusions $\mathcal{A}_k \to \mathcal{A}_{k+1}$ is the C^*-algebra of the space of tilings.

It must be pointed out that in the construction of the tiling space and the associated C^*-algebra the choice of a specific sequence of tilings $T_i, i = 0, 1, 2, \dots,$ does not play a crucial role; another sequence T_i^* with the same prototiles and subdivision data would yield the same tiling space and the same C^*-algebra since the prototile and subdivision data determines the relative position indexing set and the maps π_1, π_2.

From now on, we will talk of a 'hierarchical tiling system' by simply specifying the set of prototile types (inclusive of the inflation rule), the relative position indexing set Λ and the maps π_1 and π_2.

1.4. The following operation produces a new tiling system from the given (T_i, S_i), $i = 0, 1, 2, \dots,$ but evidently does not change the associated C^*-algebra. For any positive integer k, we consider the sequence of tilings $(T_i, S_i), i = 0, k, 2k, \dots,$ $ik, \dots$. The multiplicity matrix A is replaced by A^k; the relative position index set Λ is replaced by X_k.

Yet another way of producing a new tiling system from a given one is to fix a relative position index λ_0 and then to declare all tiles in relative position λ_0 as belonging to a new prototile type. This increases by one the cardinality of the prototile set. If Λ' denotes the new relative position indexing set and π'_1, π'_2: $\Lambda' \longrightarrow \{1, 2, \dots, n+1\}$ denote the analogues of π_1 and π_2, then there are maps

$$\Phi : \Lambda' \longrightarrow \Lambda,$$
$$\phi : \{1, 2, \dots, n+1\} \longrightarrow \{1, 2, \dots, n\},$$

which are compatible with π'_1, π'_2 and π_1, π_2. Here, $\phi(n+1) = \pi_1(\lambda_0)$ and $\phi(i) = i$ otherwise. Furthermore, Φ is a bijection from ${\pi'_2}^{-1}\{1, 2, \dots, n\}$ onto Λ and Φ is also a bijection from ${\pi'_2}^{-1}(n+1)$ onto ${\pi_2}^{-1}(\pi_1(\lambda_0))$. If X'_k denotes the analogue of X_k the map Φ induces maps from X'_k to X_k. This gives rise to embeddings of $\mathcal{A}_k$ into $\mathcal{A}'_k$ and $\mathcal{A}$ into $\mathcal{A}'$. Recall that $\mathcal{A}_k$ (resp. $\mathcal{A}'_k$) is a product of full matrix algebras indexed by $\{1, 2, \dots, n\}$ (resp. $\{1, 2, \dots, n+1\}$). Under the above imbedding a minimal idempotent of $\mathcal{A}_k$ in the matrix algebra indexed by $\pi_1(\lambda_0)$ is split into a sum of two minimal idempotents in $\mathcal{A}'_k$ belonging to the matrix algebras indexed by $\pi_1(\lambda_0)$ and $n+1$.

The above construction introduces a new prototile type. But the new prototile type occurs less often: We just mention without proof a fact relating these frequencies. Let μ be the maximal eigenvalue of the multiplicity matrix A and suppose the prototile types $1, 2, \dots, n$ occur in the original tiling in the ratio $x_1, x_2, \dots, x_n$ and that the prototile types $1, 2, \dots, n, n+1$ occur in the new tiling in the ratio

$x'_1, x'_2, \dots, x'_n, x'_{n+1}$. Then $x'_i = x_i$, if i is different from both $\pi_1(\lambda_0)$ and $n+1$, whereas $x'_{n+1} = \mu^{-1} x_{\pi_2(\lambda_0)}$ and $x'_{\pi_1(\lambda_0)} = x_{\pi_1(\lambda_0)} - x'_{n+1}$.

2. The trace splitting

The construction of new hierarchical tiling systems starting from a given one which we are going to describe now is more complex than those in subsection 1.4. In this construction new prototile types are again introduced, but, unlike the previous construction we decide the prototile type of a T_i-tile by looking at a single subtile (with known prototile type) in the subdivision of the chosen tile; using this information we decide the prototile types of the other T_{i-1}-tiles in the subdivision. In contrast to the earlier construction the new prototile types occur more uniformly.

Let $(T_i, S_i), i = 0, 1, 2, \dots,$ be a given tiling system with other notation associated as in the last section: In particular, all S_i have the same cardinality n; Λ is the relative position indexing set for the subdivision of T_i-tiles into T_{i-1}-tiles; the multipliciy of prototile types for the above subdivision is given by the $n \times n$ matrix $A = (a_{i,j})$; if δ is a subtile in relative position λ in the subdivision of a tile τ, then $\pi_1(\lambda)$ is the prototile type of δ and $\pi_2(\lambda)$ is the prototile type of τ.

2.1. Let $\mathfrak{S}$ be a finite group. Let $\psi : \Lambda \to \mathfrak{S}$ be any function. We assume that $\psi(\Lambda)$ generates the group $\mathfrak{S}$. The next proposition describes the prototile set, the relative position indexing set and the analogues of the maps π_1, π_2 for the new hierarchical tiling system we are going to construct from $\mathfrak{S}$ and ψ. So far we have been enumerating the elements of the prototile sets S_ℓ by the integers $1, 2, \dots, n$. It is convenient to denote the latter set by S and refer to it as the prototile set, when there is no likelihood of confusion.

2.2. **Proposition.** *Let $\overline{S}$ be the set $S \times \mathfrak{S}$. Let*

$$\overline{\Lambda} = \left\{ \{(s,\sigma), \lambda, (s',\sigma')\} \in \overline{S} \times \Lambda \times \overline{S} \mid \pi_1(\lambda) = s, \pi_2(\lambda) = s', \psi(\lambda) = {\sigma'}^{-1}\sigma \right\}.$$

Then $\overline{S}$ and $\overline{\Lambda}$ are respectively the prototile set and relative position indexing set for a hierarchical tiling system. The analogues $\overline{\pi}_1, \overline{\pi}_2$ of π_1, π_2 are given by

$$\overline{\pi}_1(\{(s,\sigma), \lambda, (s',\sigma')\}) = (s,\sigma),$$
$$\overline{\pi}_2(\{(s,\sigma), \lambda, (s',\sigma')\}) = (s',\sigma').$$

The multiplicity matrix $\overline{A}$ is given by

$$a_{(s,\sigma),(s',\sigma')} = \#\{\lambda \in \Lambda \mid \pi_1(\lambda) = s, \pi_2(\lambda) = s', \psi(\lambda) = {\sigma'}^{-1}\sigma\}.$$

Proof. Observe that if $\{(s,\sigma), \lambda, (s',\sigma')\} \in \overline{\Lambda}$ and $g \in \mathfrak{S}$, then $\{(s, g\sigma), \lambda, (s', g\sigma')\} \in \overline{\Lambda}$. In addition, if $h \in \mathfrak{S}, \lambda \in \Lambda$ are given, then it is clear that there is a unique choice of $s, s' \in S$ and $h' \in \mathfrak{S}$ such that $\{(s,h), \lambda, (s',h')\} \in \overline{\Lambda}$. For, from the definition of $\overline{\Lambda}, s = \pi_1(\lambda), s' = \pi_2(\lambda)$ and $h' = h(\psi(\lambda))^{-1}$. Similarly, if $h' \in \mathfrak{S}, \lambda \in \Lambda$, are given, then there is a unique choice of $s, s' \in S$ and $h \in \mathfrak{S}$ such that $\{(s,h), \lambda, (s',h')\} \in \overline{\Lambda}$. For, from the definition of $\overline{\Lambda}, s = \pi_1(\lambda), s' = \pi_2(\lambda)$ and $h = h'\psi(\lambda)$.

2.2.1. To explain the construction of the new tiling system required in the proposition, first we describe the subdivision of the new prototiles. By definition, any new prototile $\overline{\tau} \in \overline{S}_i$ is of the form (τ, h') where $\tau \in S_i$ and $h' \in \mathfrak{S}$. We regard the subdivision of τ in the original tiling system as a subdvision of $\overline{\tau}$ in the new tiling system by declaring the following prototile identifications: If δ is a subtile in the subdivision of τ in relative position λ in the context of the original tiling system, then we regard this subtile δ as a subtile $\overline{\delta}$ of the subdvision of $\overline{\tau}$ and declare it to have prototile type $(\pi_1(\lambda), h)$ where $h = h'\psi(\lambda)$. With these subdivision rules it follows from our remarks in the previous paragraph that the set $\overline{\Lambda}$ serves as the 'relative position indexing set' for the new tiling system for which the maps $\overline{\pi}_1, \overline{\pi}_2$ indicated in the statement of the proposition serve as the analogues of π_1 and π_2. The inflation identification between (s, σ) (regarded as an element of $\overline{S}_\ell$) and (s, σ) (regarded as an element of $\overline{S}_{\ell+1}$) is defined to be the same as that between s (regarded as an element of S_ℓ) and s (regarded as an element of $S_{\ell+1}$).

2.2.2. Choose any infinite sequence $\overline{\lambda}_1, \dots, \overline{\lambda}_i, \dots$, such that

$$\overline{\lambda}_i \in \overline{\Lambda} \text{ and } \overline{\pi}_2(\overline{\lambda}_i) = \overline{\pi}_1(\overline{\lambda}_{i+1}).$$

Writing $\overline{\lambda}_i = \{(s_i, \sigma_i), \lambda_i, (s'_i, \sigma'_i)\}$, this gives rise to a sequence $(\lambda_1, \dots, \lambda_i, \dots)$ in Λ, with $\pi_2(\lambda_i) = \pi_1(\lambda_{i+1})$. It can be arranged so that the latter sequence arises from an admissible sequence T_i in the original hierarchical tiling system. In other words, we may assume that (in the context of the original tiling system)

i) we can choose T_i-tiles $\delta_i, i = 0, 1, 2, \dots$, such that δ_i occurs in the subdivison of δ_{i+1} in relative position λ_{i+1} and

ii) any given T_0-tile lies inside δ_i for sufficiently large i.

2.2.3. We regard the tiles of $T_i, i = 0, 1, 2, \dots$, as tiles of the new tiling system $\overline{T}_i, i = 0, 1, 2, \dots$ with prototile types identified as explained below. When a T_i-tile τ is regarded as a $\overline{T}_i$-tile we denote the same by $\overline{\tau}$. We declare the prototile type of $\overline{\delta}_i$ to be $\overline{\pi}_1(\overline{\lambda}_i)$. Let τ be a T_ℓ-tile. To define the prototile type of $\overline{\tau}$ we choose a large enough i such that τ lies inside δ_i. Consider the (unique) sequence of tiles $\tau = \tau_\ell, \tau_{\ell+1}, \dots, \tau_j, \tau_{j+1}, \dots, \tau_i = \delta_i$ such that τ_j is a T_j-tile occurring in the subdivision of τ_{j+1}. Let $\mu_j \in \Lambda$ be the relative position of τ_j in the subdivision of τ_{j+1}. By our remarks in the beginning of the proof, there is a unique sequence $\overline{\mu}_\ell, \overline{\mu}_{\ell+1}, \dots, \overline{\mu}_i = \overline{\lambda}_i$ of elements of $\overline{\Lambda}$ of the form

$$\overline{\mu}_j = \{(s_j, h_j), \mu_j, (s'_j, h'_j)\} \text{ satisfying } \overline{\pi}_2(\overline{\mu}_j) = \overline{\pi}_1(\overline{\mu}_{j+1}).$$

We declare $\overline{\pi}_1(\overline{\mu}_\ell)$ to be the prototile type of $\overline{\tau}$.

The assumption that $\psi(\Lambda)$ generates $\mathfrak{S}$ ensures that for sufficiently large i every $\overline{S}_0$-prototile type occurs in the i-fold subdivision of each $\overline{T}_i$-prototile. This can be seen by imitating the proof of [EP, proposition 7 (iii) and (iv)].

This completes the proof of the proposition. □

2.3. **Lemma.** *Let $\overline{\mathcal{A}}$ denote the C^*-algebra of the hierarchical tiling system $\overline{T}_i$ given by Proposition 2.2. The group $\mathfrak{S}$ acts naturally as automorphisms of the C^*-algebra $\overline{\mathcal{A}}$. There is a natural inclusion $\eta : \mathcal{A} \to \overline{\mathcal{A}}$. $\mathfrak{S}$ acts as identity on $\eta(\mathcal{A})$.*

Proof. For this we recall that $\mathcal{A}$ is the inductive limit of certain finite dimensional algebras $\mathcal{A}_k$. The analogous $\overline{\mathcal{A}}_k$ is the following: Let $\overline{X}_k$ be the set of finite sequences

$$\overline{X}_k = \{\overline{w} = (\overline{\lambda}_1, \overline{\lambda}_2, \dots, \overline{\lambda}_k) \mid \overline{\lambda}_i \in \overline{\Lambda}, \overline{\pi}_2(\overline{\lambda}_i) = \overline{\pi}_1(\overline{\lambda}_{i+1})\}.$$

$\overline{\mathcal{A}}_k$ is the algebra of (complex-valued) scalar matrices $(a_{\overline{w},\overline{w}'})_{\overline{w},\overline{w}'\in\overline{X}_k}$ satisfying $a_{\overline{w},\overline{w}'} = 0$ if $\overline{\pi}_2(\overline{\lambda}_k) \neq \overline{\pi}_2(\overline{\lambda}_k')$ where $\overline{\lambda}_k$ and $\overline{\lambda}_k'$ are the k-th components of $\overline{w}$ and $\overline{w}'$. The projection $\overline{\lambda} \mapsto \lambda$ from $\overline{\Lambda}$ to Λ induces maps $\overline{w} \mapsto w$ from $\overline{X}_k$ to X_k. Let $\{a_{w,w'}\}_{w,w'\in X_k} \in \mathcal{A}_k$. Then the natural inclusion $\eta_k : \mathcal{A}_k \to \overline{\mathcal{A}}_k$ is given by $\eta_k(\{a_{w,w'}\}) = \{a_{\overline{w},\overline{w}'}\}_{\overline{w},\overline{w}'\in\overline{X}_k}$ where

$$a_{\overline{w},\overline{w}'} = \begin{cases} 0 & \text{if } \overline{\pi}_2(\overline{\lambda}_k) \neq \overline{\pi}_2(\overline{\lambda}_k'), \\ a_{w,w'} & \text{if } \overline{\pi}_2(\overline{\lambda}_k) = \overline{\pi}_2(\overline{\lambda}_k'), \end{cases}$$

where $\overline{\lambda}_k$ and $\overline{\lambda}_k'$ are the k-th components of $\overline{w}$ and $\overline{w}'$ respectively.

The group $\mathfrak{S}$ acts on $\overline{\Lambda}$: If $g \in \mathfrak{S}$ and $\overline{\lambda} = \{(s,\sigma), \lambda, (s',\sigma')\}$, then $g\overline{\lambda} = \{(s, g\sigma), \lambda, (s', g\sigma')\}$. In turn this gives rise to an action on X_k coordinatewise. For $g \in \mathfrak{S}$ and an element x of $\overline{\mathcal{A}}_k$ represented by a matrix $(a_{\overline{w},\overline{w}'})_{\overline{w},\overline{w}'\in\overline{X}_k}$ we define gx to be the matrix $(a^g{}_{\overline{w},\overline{w}'})_{\overline{w},\overline{w}'\in\overline{X}_k}$ where

$$a^g{}_{\overline{w},\overline{w}'} = (a_{g^{-1}\overline{w},g^{-1}\overline{w}'}).$$

It is evident that $\mathfrak{S}$ acts as identity on $\eta_k(\mathcal{A}_k)$. The inclusions η_k are compatible with the inclusions $\phi_k : \mathcal{A}_k \to \mathcal{A}_{k+1}$ and $\overline{\phi}_k : \overline{\mathcal{A}}_k \to \overline{\mathcal{A}}_{k+1}$, i.e., we have $\eta_{k+1} \cdot \phi_k = \overline{\phi}_k.\eta_k$. As $\mathcal{A}$ and $\overline{\mathcal{A}}$ are C^*-inductive limits of $\mathcal{A}_k$ and $\overline{\mathcal{A}}_k$ respectively, the maps η_k give rise to the desired inclusion. It is also easy to check that the $\mathfrak{S}$ action is compatible with the inclusion of $\overline{\mathcal{A}}_k$ into $\overline{\mathcal{A}}_{k+1}$. This completes the proof of the lemma. $\square$

In [EP] we considered the problem of colouring the T_0-tiles of a given hierarchical tiling system satisfying some conditions like symmetry and frequency of occurrence of colours, *etc.* We recall these notions below. We will see that Proposition 2.2 can be applied to deduce that the colouring scheme described in [EP] in fact yields a hierarchical tiling system. The frequency data implemented in that construction is here interpreted in terms of the unique 'trace' of the associated C^*-algebras.

2.4. **Definition. 'Local (G,γ,M)-symmetry'.** Let G be a finite group acting on a finite set $M : \gamma : G \times M \longrightarrow M$ (the elements of M will be the colours). Assume that the tiles in T_0 are coloured with the elements of M. We say that the colouring of T_0 has *local* (G,γ,M)-*symmetry* (or simply that T_0 has (G,γ,M)-*symmetry*) if given $g \in G$ and given a patch Σ of the tiles of T_0 in any bounded region, there exists $R > 0$ such that in any disc of radius R one can find a copy Σ' of the patch Σ, the only change being that the colouring of Σ' is obtained from that of Σ on applying the permutation $\gamma(g)$.

2.5. **'Frequency data for colour occurrence'.** We may wish to see colours in certain frequencies by prescribing the desired ratios. Let $C_1, C_2, \ldots, C_N$ be the distinct G-orbits in M. Suppose that one wants to see the colours in $C_1, \ldots, C_N$ in the frequency ratios $a_1, \ldots, a_N$ (positive integers). Later in the sequel we use the notation $\{a_\mu\}_{\mu\in M}$ for this frequency data, naturally with the assumption that $a_\mu = a_{\mu'}$ if μ and μ' belong to the same orbit. Let S' be a set which is a disjoint union of a_1 copies of C_1, a_2 copies of C_2, …, a_N copies of C_N. The group G acts on S'. Let $\mathfrak{S} = \mathrm{Aut}(S')$. For any function $\psi : \Lambda \to \mathfrak{S}$ whose image generates $\mathfrak{S}$, Proposition 2.2 gives a hierarchical tiling system with prototiles indexed by $S \times \mathfrak{S}$

and having relative position indexing set

$$\overline{\Lambda} = \left\{ \{(s,\sigma), \lambda, (s',\sigma')\} \mid \lambda \in \Lambda, \pi_1(\lambda) = s, \pi_2(\lambda) = s' \text{ and } \psi(\lambda) = {\sigma'}^{-1}\sigma \right\}.$$

By Lemma 2.3, the group $\mathfrak{S} = \mathrm{Aut}(S')$ and in particular the group G act as automorphisms of the C^*-algebra of the tiling system.

2.6. Let us colour a $\overline{T}_0$-tile $\overline{\delta}$ having relative position index $\{(s,\sigma), \lambda, (s',\sigma')\}$ by $\sigma(x)$ where $x \in S'$ is fixed. In other words, we colour an $\overline{S}_0$-prototile (s,σ) by $\sigma(x)$ and then colour $\overline{T}_0$-tiles according to the prototile type. Since S' is a disjoint union of copies of G-orbits in M, clearly this is a colouring by elements of M. This colouring has local (G,γ,M)-symmetry. To see this, let Σ be any patch of tiles in $\overline{T}_0$. We can assume that this patch is contained in some $\overline{T}_j$-tile, say, $\overline{\kappa}$ of prototile type, say, (s,σ). For sufficiently large i, any $\overline{T}_{j+i}$-tile $\overline{\tau}$ has $\overline{T}_j$-tiles of all prototypes, in particular of prototile type (s,g) for any $g \in \mathrm{Aut}(S')$. Local (G,γ,M)-symmetry is an easy consequence of this.

To give a meaning to the frequency of occurrence of colours in the tiling in terms of the associated C^*-algebra we gather below some definitions and known facts about projections in C^*-algebras and traces.

A projection in a C^*-algebra $\mathcal{A}$ is a self-adjoint element $p = p^*$ such that $p = p^2$. Two projections p_1, p_2 are said to be equivalent (in symbols $p_1 \sim p_2$) if there exists $y \in \mathcal{A}$ such that $p_1 = yy^*, p_2 = y^*y$. An element x is positive if $x = yy^*$ for some $y \in \mathcal{A}$. The order relation in self-adjoint elements of $\mathcal{A}$ is defined by $x \le y$ if $y - x$ is positive. A trace in $\mathcal{A}$ is a continuous (complex) scalar-valued linear map f such that

i) $f(xy) = f(yx), \forall x, y \in \mathcal{A}$,

ii) $f(x) \ge 0$, if x is positive.

For the C^*-algebra $\mathcal{A}=\lim_k(\mathcal{A}_k)$ associated to a hierarchical tiling system, a non-zero trace exists and is unique (up to normalization). The uniqueness is a consequence of the assumption that every T_0-prototile type occurs in the iterated subdivision of any T_i-tile for sufficiently large i.

We describe below this unique trace.

2.7. **The trace on the C^*-algebra of tiling spaces.** We recall the set X_k of finite sequences of elements of Λ:

$$X_k = \{w = (\lambda_1, \lambda_2, \dots, \lambda_k) \mid \lambda_i \in \Lambda, \pi_2(\lambda_i) = \pi_1(\lambda_{i+1})\};$$

$\mathcal{A}_k$ is the algebra of (complex-valued) scalar matrices $(a_{w,w'})_{w,w' \in X_k}$ satisfying $a_{w,w'} = 0$, if $\pi_2(\lambda_k) \ne \pi_2(\lambda'_k)$ where λ_k and λ'_k are the k-th components of w and w'. For $w, w' \in X_k$ denote by $\dot{w}, \dot{w}' \in X_{k-1}$ the truncation. We recall the inclusion ϕ_{k-1} of $\mathcal{A}_{k-1}$ into $\mathcal{A}_k$ which is defined as follows: Let $x = (a_{v,v'})_{v,v' \in X_{k-1}}$ be an element of $\mathcal{A}_{k-1}$. Let $w = (\lambda_1, \dots, \lambda_{k-1}, \lambda_k), w' = (\lambda'_1, \dots, \lambda'_{k-1}, \lambda'_k) \in X_k$. The image $\phi_{k-1}(x)$ of x is the matrix $(a_{w,w'})$ where

$$a_{w,w'} = \begin{cases} 0 & \text{if } \lambda_k \ne \lambda'_k, \\ a_{\dot{w},\dot{w}'} & \text{if } \lambda_k = \lambda'_k. \end{cases}$$

2.7.1. Let ξ denote the Perron-Frobenius eigenvalue [Ga] of the multiplicity matrix A and denote the corresponding eigenvector by $\vec{u} = (u_s)_{s \in S}$. (Here, $S = \{1, 2, \dots, n\}$ and u_s are positive real numbers such that $\sum_s u_s = 1$.) To define a trace, first we define linear maps $\mathrm{Tr}_k : \mathcal{A}_k \to \mathbb{C}$ $(k = 1, 2, \dots)$. With notation as above, for a (complex) scalar-valued matrix $x \in \mathcal{A}_{k-1}$ of the form $x = (a_{v,v'})_{v,v' \in X_{k-1}}$ we set

$$\mathrm{Tr}_{k-1}(x) = \frac{1}{\xi^{k-1}} \sum_{v \in X_{k-1}} u_{\pi_2(\lambda_{k-1})} a_{v,v}$$

where λ_{k-1} is the $(k-1)$-th component of v. Viewing $\mathcal{A}_{k-1}$ as a product of matrix algebras, Tr_{k-1} is a linear combination of the natural traces in these matrix algebras.

2.8. **Proposition.** *Let ϕ_{k-1} denote the inclusion of $\mathcal{A}_{k-1}$ into $\mathcal{A}_k$ and let I be the identity element of any $\mathcal{A}_k$. Then:*

i) *The above-defined traces satisfy the compatibility condition $Tr_k(\phi_{k-1}(x)) = Tr_{k-1}(x)$ and give rise to a trace $Tr: \mathcal{A} \to \mathbb{C}$.*

ii) *For any k, $Tr_k(I) = 1$.*

Proof. i) Let $x = (x_{v,v'})_{v,v' \in X_{k-1}} \in \mathcal{A}_{k-1}$. Put $\phi_{k-1}(x) = y = (y_{w,w'})_{w,w' \in X_k}$. Compute Tr_k of the right side. It equals

$$(*) \qquad \begin{aligned} \mathrm{Tr}_k(y) &= \frac{1}{\xi^k} \sum_{w \in X_k} u_{\pi_2(\lambda_k)} y_{w,w} \\ &= \frac{1}{\xi^k} \sum_{w \in X_k} u_{\pi_2(\lambda_k)} x_{\dot{w},\dot{w}} \\ &= \frac{1}{\xi^k} \sum_{v \in X_{k-1}} \left\{ \sum_{w \in X_k, \dot{w} = v} u_{\pi_2(\lambda_k)} x_{v,v} \right\}. \end{aligned}$$

But for a fixed $v \in X_{k-1}$ whose last coordinate is λ_{k-1}, we have

$$\{w \in X_k : \dot{w} = v\} = \{(v,\lambda), \lambda \in \Lambda : \pi_1(\lambda) = \pi_2(\lambda_{k-1})\}.$$

Therefore

$$\begin{aligned} \sum_{w \in X_k, \dot{w} = v} u_{\pi_2(\lambda_k)} x_{v,v} &= \sum_{\lambda \in \Lambda, \pi_1(\lambda) = \pi_2(\lambda_{k-1})} u_{\pi_2(\lambda)} x_{v,v} \\ &= \left(\sum_{j \in \{1,\dots,n\}} a_{\pi_2(\lambda_{k-1}),j} u_j \right) x_{v,v} \\ &= \xi u_{\pi_2(\lambda_{k-1})} x_{v,v}. \end{aligned}$$

Substituting in $(*)$, we obtain

$$\begin{aligned} \mathrm{Tr}_k(\phi_{k-1}(x)) &= \frac{1}{\xi^k} \sum_{v \in X_{k-1}} \xi u_{\pi_2(\lambda_{k-1})} x_{v,v} \\ &= \frac{1}{\xi^{k-1}} \sum_{v \in X_{k-1}} u_{\pi_2(\lambda_{k-1})} x_{v,v} \\ &= \mathrm{Tr}_{k-1}(x). \end{aligned}$$

ii) By i), $\mathrm{Tr}_k(I) = \mathrm{Tr}_{k-1}(I) = \cdots = \mathrm{Tr}_1(I)$. So it is sufficient to prove that $\mathrm{Tr}_1(I) = 1$. We have

$$\begin{aligned}\mathrm{Tr}_1(I) &= \frac{1}{\xi}\sum_{\lambda\in\Lambda} u_{\pi_2(\lambda)}\cdot 1\\ &= \frac{1}{\xi}\sum_{j\in\{1,\dots,n\}}\left(\sum_{i\in\{1,\dots,n\}} u_j a_{ij}\right)\\ &= \frac{1}{\xi}\sum_{i\in\{1,\dots,n\}}\left(\sum_{j\in\{1,\dots,n\}} u_j a_{ij}\right)\\ &= \frac{1}{\xi}\sum_{i\in\{1,\dots,n\}} \xi u_i\\ &= 1.\end{aligned}$$

This completes the proof of the proposition. □

Let $\overline{\mathrm{Tr}} : \overline{\mathcal{A}} \to \mathbb{C}$ denote the similarly constructed trace on the C^*-algebra $\overline{\mathcal{A}}$ of the tiling system $\{\overline{T}_i\}_i$ given by Proposition 2.2. The explicit description of the multiplicity matrix $\overline{A}$ allows us to relate the Perron-Frobenius eigenvectors [Ga] of A and $\overline{A}$.

2.9. **Lemma.** *Let $\vec{\theta} = (\theta_s)_{s\in S}$ be an eigenvector for the multiplicity matrix A corresponding to an eigenvalue β. Define a vector $\vec{\overline{\theta}} = (\overline{\theta}_{(s,\sigma)})_{s\in S,\sigma\in\mathfrak{S}}$ by $\overline{\theta}_{(s,\sigma)} = \frac{1}{\#(\mathfrak{S})}\theta_s$. Then $\vec{\overline{\theta}}$ is an eigenvector for $\overline{A}$ for which the corresponding eigenvalue equals β. In particular, if $\vec{\theta}$ is the Perron-Frobenius eigenvector for A, then $\vec{\overline{\theta}}$ is the Perron-Frobenius eigenvector for $\overline{A}$.*

Proof. From the hypotheses of the lemma, $\sum_{s'\in S} a_{s,s'}\theta_{s'} = \beta\theta_s$. We have

$$\sum_{s'\in S,\sigma'\in\mathfrak{S}} a_{(s,\sigma),(s',\sigma')}\overline{\theta}_{(s',\sigma')} = \frac{1}{\#(\mathfrak{S})}\sum_{s'\in S,\sigma'\in\mathfrak{S}} a_{(s,\sigma),(s',\sigma')}\theta_{s'}.$$

But for a fixed $s' \in S$,

$$\begin{aligned}\sum_{\sigma'\in\mathfrak{S}} a_{(s,\sigma),(s',\sigma')} &= \sum_{\sigma'\in\mathfrak{S}} \#\{\lambda\in\Lambda \mid \pi_1(\lambda) = s, \pi_2(\lambda) = s' \text{ and } \psi(\lambda) = {\sigma'}^{-1}\sigma\}\\ &= \#\{\lambda\in\Lambda \mid \pi_1(\lambda) = s, \pi_2(\lambda) = s'\} = a_{s,s'}.\end{aligned}$$

Since every $\overline{T}_0$-prototile type occurs in the k-fold subdivision of each $\overline{T}_k$-prototile type for sufficiently large k, the matrix $\overline{A}^k$ has all entries positive. From the Perron-Frobenius theory [Ga] one knows that the Perron-Frobenius eigenvector is the only eigenvector with all entries positive.

The lemma follows. □

2.10. **Corollary.** *For $x \in \mathcal{A}$ one has $Tr(x) = \overline{Tr}(\eta(x))$ where $\eta : \mathcal{A} \to \overline{\mathcal{A}}$ is the natural inclusion.*

Proof. This is a consequence of the explicit description of the traces in subsection 2.7. □

Define projections $\Pi^\mu \in \overline{\mathcal{A}}_1, (\mu \in M)$ as follows: Recall from subsection 2.6 that $\overline{S}_0$-tiles are coloured by elements of M. Recall further that T_0-tiles are coloured by regarding such a tile δ as a tile $\overline{\delta}$ of $\overline{T}_0$ and then identifying the prototile type of $\overline{\delta}$ in $\overline{S}_0$. We set

$$\Pi^\mu = (a_{\overline{w},\overline{w}'})_{\overline{w},\overline{w}' \in \overline{X}_1}$$

where

$$a_{\overline{w},\overline{w}'} = \begin{cases} 0 & \text{if } \overline{w} \neq \overline{w}', \\ 1 & \text{if } \overline{S}_0\text{-tile } \overline{\pi}_1(\overline{\lambda}_1) \text{ has colour } \mu, \text{ where } \overline{w} = (\overline{\lambda}_1). \end{cases}$$

With these preparations, we are ready to demonstrate the theorem which gives an interpretation of the frequency of occurrence of the colours μ ($\mu \in M$) among the T_0-tiles. The G-action on $\overline{\mathcal{A}}$ which figures in the theorem below has already been discussed in subsection 2.5 and as mentioned there arises from the group action explained in Lemma 2.3.

2.11. **Theorem.** i) *The projections Π^μ split the identity $1 \in \eta(\mathcal{A}) \subset \overline{\mathcal{A}}$. For $g \in G, g\Pi^\mu = \Pi^{g\mu}$ $(\forall \mu)$. The numbers $\{\overline{Tr}(\Pi^\mu)\}_{\mu \in M}$ are in the ratio $\{a_\mu\}_\mu$ namely, the prescribed frequency ratio of subsection* 2.5.

ii) *Let P, P' be projections of $\mathcal{A}$ such that $P \leq P'$. Then there exist projections $Q^\mu, Q'^\mu \in \overline{\mathcal{A}}, (\mu \in M)$ such that*

(a) $Q^\mu \leq Q'^\mu \leq \Pi^\mu$,

(b) *for $g \in G$, $gQ^\mu = Q^{g\mu}, gQ'^\mu = Q'^{g\mu}$ $(\forall \mu)$,*

(c) *if $Q = \sum_\mu Q^\mu$ and $Q' = \sum_\mu Q'^\mu$, then $Q, Q' \in \mathcal{A}$ and $P \sim_{\mathcal{A}} Q, P' \sim_{\mathcal{A}} Q'$,*

(d) *$\{\overline{Tr}(Q^\mu)\}_\mu$ (resp. $\{\overline{Tr}(Q'^\mu)\}_\mu$) are in the ratio $\{a_\mu\}_\mu$.*

Proof. From the definition of Π^μ, the sum $\sum_{\mu \in M} \Pi^\mu$ is the element of $\overline{\mathcal{A}}_1$ given by

$$a_{\overline{w},\overline{w}'} = \begin{cases} 0 & \text{if } \overline{w} \neq \overline{w}', \\ 1 & \text{if } \overline{w} = \overline{w}', \end{cases}$$

which is the unit element. The equation $g\Pi^\mu = \Pi^{g\mu}$ is a consequence of the definition of the group action in Lemma 2.3 and the definition of Π^μ above. As in subsection 2.7.1, let ξ denote the Perron-Frobenius eigenvalue of the multiplicity matrix A with corresponding eigenvector $\vec{u} = (u_s)_{s \in S}$ such that $\sum_s u_s = 1$. For a (complex) scalar-valued matrix $x \in \mathcal{A}_k$ of the form $x = (a_{v,v'})_{v,v' \in X_k}$ by the definition of Tr,

$$\mathrm{Tr}(x) = \xi^{-k} \sum_{v \in X_k} u_{\pi_2(\lambda_k)} a_{v,v}$$

where λ_k is the k-th component of v.

2.11.1. From Lemma 2.9, if $\vec{\overline{u}} = (\overline{u}_{(s,\sigma)})_{s \in S, \sigma \in \mathfrak{S}}$ is the vector given by $\overline{u}_{(s,\sigma)} = \frac{1}{\#(\mathfrak{S})} u_s$, then $\vec{\overline{u}}$ is the Perron-Frobenius eigenvector for $\overline{A}$ and evidently

$$\sum_{s \in S, \sigma \in \mathfrak{S}} \overline{u}_{s,\sigma} = 1.$$

Thus for a (complex) scalar-valued matrix $\overline{x} \in \overline{\mathcal{A}}_k$ of the form $\overline{x} = (a_{\overline{v},\overline{v}'})_{\overline{v},\overline{v}' \in \overline{X}_k}$ by the definition of $\overline{Tr}$,

$$\overline{Tr}(\overline{x}) = \xi^{-k} \sum_{\overline{v} \in \overline{X}_k} u_{\overline{\pi}_2(\overline{\lambda}_k)} a_{\overline{v},\overline{v}}$$

where $\overline{\lambda}_k$ is the k-th component of $\overline{v}$.

Observe the following about the structure of algebras $\mathcal{A}_k$: For $s \in S$ let $\mathcal{A}_k^{(s)}$ be the subalgebra of matrices $(a_{w,w'})_{w,w' \in X_k}$ for which $a_{w,w'} \neq 0$ only if both $\pi_2(\lambda_k)$ and $\pi_2(\lambda'_k)$ equal s, where λ_k and λ'_k are the k-th components of w and w'. Then $\{\mathcal{A}_k^{(s)}\}_s$ are matrix algebras and $\mathcal{A}_k$ is the product $\prod_{s \in S} \mathcal{A}_k^{(s)}$. Similarly, $\overline{\mathcal{A}}_k$ is also a product of matrix algebras: For $(s,\sigma) \in S \times \mathfrak{S}$, let $\overline{\mathcal{A}}_k^{(s,\sigma)}$ be the subalgebra of matrices $(a_{\overline{w},\overline{w}'})_{\overline{w},\overline{w}' \in \overline{X}_k}$ for which $a_{\overline{w},\overline{w}'} \neq 0$ only if $\overline{\pi}_2(\overline{\lambda}_k)$ and $\overline{\pi}_2(\overline{\lambda}'_k)$ both equal (s,σ), where $\overline{\lambda}_k$ and $\overline{\lambda}'_k$ are the k-th components of $\overline{w}$ and $\overline{w}'$. Then $\{\overline{\mathcal{A}}_k^{(s,\sigma)}\}_{(s,\sigma)}$ are matrix algebras and $\overline{\mathcal{A}}_k$ is the product $\prod_{(s,\sigma) \in S \times \mathfrak{S}} \overline{\mathcal{A}}_k^{(s,\sigma)}$.

C^*-algebras $\mathcal{A}$ which are inductive limits of finite dimensional C^*-algebras $\mathcal{A}_k$ are also called AF-algebras and their structure has been studied in detail. For a comprehensive treatise see [Ef] or [Go]. The following facts are thus widely known.

Any projection P of $\mathcal{A}$ is equivalent to a projection Q of $\mathcal{A}_k$ for sufficiently large k. Moreover, given two projections $P, P' \in \mathcal{A}$ such that $P \leq P'$, we can choose projections $Q, Q' \in \mathcal{A}_k$ for large k such that $Q \leq Q', P \sim Q$ and $P' \sim Q'$. Given two projections $Q, Q' \in \mathcal{A}_k$ such that $Q \leq Q'$ one can choose orthogonal minimal projections $e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$ such that $\sum_{i=1}^m e_i = Q$ and $\sum_{j=1}^{m+n} e_j = Q'$. Two projections in a matrix algebra are equivalent iff they have the same rank; minimal projections have rank 1. Any minimal projection of $\mathcal{A}_k$ lies in one of the factors in the product $\mathcal{A}_k = \prod_{s \in S} \mathcal{A}_k^{(s)}$.

2.11.2. Given any minimal projection $e_s \in \mathcal{A}_k^{(s)}$ its image $\eta_k(e_s)$ under the inclusion $\eta : \mathcal{A}_k \to \overline{\mathcal{A}}_k$ equals a sum of orthogonal minimal projections $(e_{s,\sigma})_{\sigma \in \mathfrak{S}}$ where $e_{s,\sigma} \in \overline{\mathcal{A}}_k^{(s,\sigma)}$. We have $\mathrm{Tr}(e_s) = \xi^{-k} u_s$ and $\overline{Tr}(e_{s,\sigma}) = \frac{1}{\#(\mathfrak{S})} \xi^{-k} u_s$.

2.11.3. For $w \in X_k$ define the minimal projection $e_w \in \mathcal{A}_k$ to be the matrix $(a_{v,v'})_{v,v' \in X_k}$ given by

$$a_{v,v'} = \begin{cases} 1 & \text{if } v = v' = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(e_w)_{w \in X_k}$ is a maximal orthogonal family of minimal projections in $\mathcal{A}_k$. For a subset I of X_k let e_I denote the projection in $\mathcal{A}_k$ given by $e_I = \sum_{w \in I} e_w$. Given projections $Q, Q' \in \mathcal{A}_k$ such that $Q \leq Q'$ we can choose subsets $I \subset I' \subset X_k$ such that $Q \sim e_I$ and $Q' \sim e_{I'}$. Similarly, define minimal projections $e_{\overline{w}} \in \overline{\mathcal{A}}_k$ to be the matrix $(a_{\overline{v},\overline{v}'})_{\overline{v},\overline{v}' \in \overline{X}_k}$ given by

$$a_{\overline{v},\overline{v}'} = \begin{cases} 1 & \text{if } \overline{v} = \overline{v}' = \overline{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(e_{\overline{w}})_{\overline{w} \in \overline{X}_k}$ is a maximal orthogonal family of minimal projections in $\overline{\mathcal{A}}_k$. We have $g e_{\overline{w}} = e_{g\overline{w}}, \forall g \in G$. For a subset $\overline{I}$ of $\overline{X}_k$ let $e_{\overline{I}}$ denote the projection in $\overline{\mathcal{A}}_k$ given by $e_{\overline{I}} = \sum_{\overline{w} \in \overline{I}} e_{\overline{w}}$.

2.11.4. We will now make explicit the decomposition 2.11.2 for a minimal projection $e_w, w \in X_k$. Write $w = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $\sigma \in \mathfrak{S}$. By our remarks in the beginning of the proof of Proposition 2.2, there is a unique sequence $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_k$ of elements of $\overline{\Lambda}$ of the form $\overline{\lambda}_j = \{(s_j, h_j), \lambda_j, (s'_j, h'_j)\}$ satisfying $h'_k = \sigma$ and $\overline{\pi}_2(\overline{\lambda}_j) = \overline{\pi}_1(\overline{\lambda}_{j+1})$. This represents a unique element $\overline{w}_\sigma \in \overline{X}_k$. Then $\eta(e_w) = \sum_{\sigma \in \mathfrak{S}} e_{\overline{w}_\sigma}$. We now have most of the notation needed to specify the splitting of projections in the form stated in the theorem. We first do this for the minimal projections $e_w \in \mathcal{A}_k$. It suffices to define e^μ_w to be the sum of $e_{\overline{w}_\sigma}$ over those $\sigma \in \mathfrak{S}$ such that μ is the colour (according to subsection 2.6) of the $\overline{T}_0$-tile in relative position $\overline{w}_\sigma$ in a $\overline{T}_k$-tile of prototile type $(\pi_2(\lambda_k), \sigma)$. Here, λ_k is the k-th component of w. This gives rise to a splitting $e_w = \sum_{\mu \in M} e^\mu_w$ where e^μ_w are projections in $\overline{\mathcal{A}}_k$. Subsection 2.6 can be used to translate this into an algebraic expression for e^μ_w. It is immediate from the definition of group action in Lemma 2.3 that for $g \in G, ge^\mu_w = e^{g\mu}_w$. By using subsection 2.11.1 one deduces that $\{\overline{Tr}(e^\mu_w)\}_\mu$ are in the ratio $\{a_\mu\}_\mu$.

For the projections $e_I = \sum_{w \in I} e_w$ for a subset $I \subset X_k$ we define $e^\mu_I = \sum_{w \in I} e^\mu_w$. Then the fact that $ge^\mu_I = e^{g\mu}_I, \forall g \in G$ and $\{\overline{Tr}(e^\mu_I)\}_\mu$ are in the ratio $\{a_\mu\}_\mu$ are immediate consequences of the corresponding facts for minimal projections. If P is a projection in $\mathcal{A}$, then for large enough k, $P \sim e_I$ for some $e_I \in \mathcal{A}_k$ as above. Further, if $P, P' \in \mathcal{A}$ are two projections such that $P \leq P'$, then for a suitably large k we can choose subsets $I \subset J \subset X_k$ so that $P \sim e_I$ and $P' \sim e_J$. When the above splittings are done for the projection $e_I \in \mathcal{A}_1$, taking $I = X_1$ we get a decomposition of the unit element. We obtain the same splitting when this is carried out regarding the unit element as $e_I \in \mathcal{A}_k$ for $I = X_k$. This gives rise to the part of the statement in the theorem which asserts "$Q^\mu \leq Q'^\mu \leq \Pi^\mu$". □

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