ON A CHARACTERIZATION OF MEASURES OF DISPERSION

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Abstract. Measures of dispersion are characterized by the set of all bounded random variables whose dispersion is minimized when taken around the origin.

1. Introduction

Let $\varphi$ be a real valued function on $\mathbb{R}$, $X$ a bounded random variable (b.r.v.), and $a$ a real number. The functional $E\varphi(X - a)$ may be used as a measure of dispersion of $X$ around $a$. The base of the measure is the set of all b.r.v. $X$ such that

$$\min_a E\varphi(X - a) = E\varphi(X).$$

(1)

For example, the base of the first absolute moment $E|X - a|$ is the set of all b.r.v. with zero median; the base of the second moment $E(X - a)^2$ is the set of all b.r.v. with zero mean value.

In this paper, we consider a characterization of the measures of dispersion by their bases. Kagan and Shepp [2] proved that if $\varphi$ is continuous and the base of the measure $E\varphi(X - a)$ contains all b.r.v. with $EX = 0$, then $\varphi(x) = \alpha x^2 + \varphi(0)$ with some $\alpha \geq 0$, and they also obtained a multivariate version of the result.

In what follows all the functions are real valued; $f$ is a non-negative continuous function on $\mathbb{R}$ with $f(0) = 0$; $B_\varphi$ denotes the base of the measure $E\varphi(X - a)$ (so $B_\varphi$ is the set of all b.r.v.).

Theorem 1. Let $f$ satisfy the following conditions:

(2) $f(x)$ does not vanish identically on $(-\infty, 0)$ or on $(0, \infty)$ and

$$y \int_0^z \{f(x + y) - f(x) - f(y)\} dx \geq 0 \quad \text{for any } y, z \in \mathbb{R}.$$

(3)

If

(4) $\varphi$ is continuous on $\mathbb{R}$ and $B_f \subseteq B_\varphi$,

then

(5) $\varphi(x) = \alpha f(x) + \varphi(0)$

with some $\alpha \geq 0$.
In particular, if \( f \) is convex on \( \mathbb{R} \), then (2) is equivalent to \( f(\pm \infty) = \infty \). Moreover, in this case, the difference \( f(x + y) - f(x) \) is an increasing function of \( x \) for any fixed \( y > 0 \) (see, for example, [1, 3.18]). Therefore, (3) is fulfilled and we have the following:

**Corollary.** If \( f \) is convex on \( \mathbb{R} \) and \( f(\pm \infty) = \infty \), then (4) implies (5).

The bases of convex measures are described in the last section. Note that convexity of \( f \) on \( \mathbb{R} \) is not necessary for (3). For example, the function

\[
f(x) = x^2(x^2 - 3x + 3)
\]

satisfies (2) and (3) but is not convex on \( \mathbb{R} \).

**Theorem 2.** Let \( f \) be absolutely continuous on each finite interval and satisfy (2). Moreover, let \( g \) be defined on \( \mathbb{R} \), bounded on each finite interval, \( g(0) = 0 \) and \( g(x) = f'(x) \) at all the points of differentiability of \( f \) (hence, almost everywhere). If \( \varphi \) is continuous on \( \mathbb{R} \) and \( B_\varphi \) contains all binary r.v. with zero median, then we have (5) with \( \lambda = \varphi(f) \) and \( \lambda(x) = \varphi(x) \).

Condition (2) is essential. The functions

\[
f(x) = (x + |x|)^2, g(x) = 4(x + |x|) \quad \text{and} \quad \varphi(x) = (x + |x|)^3
\]

satisfy all the conditions of Theorems 1 and 2 except (2). Moreover,

\[B_f = B_\varphi = \{X \in B_0 : P(X > 0) = 0\},\]

and \( E\varphi(X) = 0 \) is equivalent to \( X \in B_\varphi \). However, (5) is obviously not valid in this case.

The functions \( f(x) = |x| \) and \( g(x) = \text{sign} x \) satisfy all the conditions of Theorem 2. It follows from \( E\text{sign}X = 0 \) that \( X \) has zero median. So if \( B_\varphi \) contains all b.r.v. with zero median, then we have (5) with \( f(x) = |x| \) (this also follows from the Corollary). The result holds under more general conditions (in particular, the function \( \varphi \) may be a priori discontinuous).

**Theorem 3.** Let \( \varphi \) be a function on \( \mathbb{R} \) bounded from either above or below on some interval and let \( 0 < p < 1 \). If \( B_\varphi \) contains all binary r.v. \( X \) with min \( X \leq 0 \leq \max X \) and \( P'(X = \min X) = p \), then (5) holds with

\[
f(x) = |x| + (2p - 1)x.
\]

Note that in this case \( B_f = \{X \in B_0 : P(X < 0) \leq p \leq P(X \leq 0)\} \) (so that \( B_f \) consists of all bounded r.v. with zero quantile of order \( p \)).

2. **Proof of Theorems 1 and 2**

Let \( Y_w \) denote an r.v. equal to \( w \) with probability 1,

\[M = \{x \in \mathbb{R} : f(x) > 0\}\]

and \( [M] \) is the closure of \( M \). Set, moreover, for \( u, v \in M \) and \( u < 0 < v \) (there exist the such \( u \) and \( v \) in view of (2))

\[\lambda = \lambda(u, v) = \{vf(u) - uf(v)\}^{-1}.
\]

Let \( Y = Y(u, v) \) be an r.v. with the distribution function \( F(x) = F(x, u, v) \) and

\[
F(x) = \begin{cases} 
\lambda f(v)(x - u) & \text{for } x \in [u, 0], \\
\lambda f(u)x - uf(v) & \text{for } x \in [0, v].
\end{cases}
\]
Lemma 1. Let $f$ satisfy (2). If $B_\varphi$ contains $Y_0, Y_w$ for $w \notin [M]$ and $Y(u, v)$ for $u, v \in M, u < 0 < v$, then (5) holds with some $\alpha \geq 0$.

Proof. It follows from $Y_w \in B_\varphi$ that

$$\varphi(w) = E \varphi(Y_w) = \min_a E \varphi(Y_w - a) = \min_t \varphi(t).$$

Therefore,

$$\varphi(0) = \varphi(w) = \min_t \varphi(t)$$

for all $w \notin [M]$ and we obtain (5) for all $x \notin [M]$. Now let $u, v \in M, u < 0 < v$. Putting for any integrable function $r$

$$E_r(z) = E r(Y + z) = \lambda \{ f(v) \int_{z+u}^z r(x) dx + f(u) \int_{z}^{z+v} r(x) dx \},$$

and taking into account that $Y(u, v) \in B_\varphi$, we get $E_\varphi'(0) = 0$, since $\varphi$ is continuous so $E_\varphi(z)$ is differentiable. Hence

$$s(u) = s(v) \quad \text{for } u, v \in M, u < 0 < v,$$

where

$$s(x) = \frac{\varphi(x) - \varphi(0)}{f(x)}.$$

It follows that $s(x)$ has the same value $\alpha$ for all $x \in M$, so we have (5) for all such $x$. Since $f$ and $\varphi$ are continuous, it implies (5) for all $x \in [M]$ and thus for all real $x$. It follows from (5) and (7) that

$$\min_x \alpha f(x) = 0$$

so $\alpha \geq 0$.

To prove Theorem 1, it is enough now to show that

$$Y_0, Y_w, Y(u, v) \in B_f \quad \text{for any } u, v \in M, u < 0 < v, \quad \text{and any } w \notin [M].$$

Since

$$f(w) = f(0) = 0 = \min_t f(t),$$

we have $Y_0, Y_w \in B_f$. It follows from (3) and (8) that

$$\int_0^z \frac{f(x + u) - f(x)}{f(u)} dx \leq z \leq \int_0^z \frac{f(x + v) - f(x)}{f(v)} dx$$

and

$$E_f(z) \geq E_f(0) \quad \text{for all } z \in \mathbb{R},$$

so $Y(u, v) \in B_f$ for $u, v \in M, u < 0 < v$.

Similarly, to prove Theorem 2, it is enough to show that $E g(X) = 0$ for

$$X = Y_0, Y_w, Y(u, v), \quad \text{where } u, v \in M, u < 0 < v, \quad \text{and } w \notin [M].$$

Indeed, $E g(Y_0) = g(0) = 0$. If $w \notin [M]$, then $f(x) = 0$ in some open interval containing $w$; therefore, also in this interval, $g(x) = f'(x) = 0$, so

$$E g(Y_w) = g(w) = 0.$$
Moreover, it follows from (8) that

\[ E_g \{ Y(u,v) \} = E_g(0) = E'_f(0) = 0 \]

because \( f \) is absolutely continuous and so \( f(x) = \int_0^x g(t) \, dt \) (see, for example, \[4\]
11.7).

3. Proof of Theorem 3

Continuity of \( \varphi \) is essential for the proof of Theorems 1 and 2. Therefore, we now use another approach.

Let \( U = U_p(u,v) \) \((u<v)\) be a binary r.v. defined by

\[ P(U = u) = p, \quad P(U = v) = q = 1 - p. \]

Let \( x > 0 \) and \( u \in [0, x] \). Then the r.v. \( U_p(0,x) \) and \( U_p(-u,x-u) \) satisfy the conditions of Theorem 3. It follows from (1) and (9) that

\[ p\varphi(-u) + q\varphi(x-u) \geq p\varphi(0) + q\varphi(x) \]

and

\[ p\varphi(0) + q\varphi(x) \geq p\varphi(-u) + q\varphi(x-u), \]

whence

\[ p\{ \varphi(-u) - \varphi(0) \} = q\{ \varphi(x) - \varphi(x-u) \}. \]

In particular, we have by setting \( x = u \) that

\[ p\{ \varphi(-u) - \varphi(0) \} = q\{ \varphi(u) - \varphi(0) \}. \]

It follows from (10) and (11) that

\[ \varphi(x) - \varphi(x-u) = \varphi(u) - \varphi(0) \quad \text{for} \ x \geq 0, u \in [0, x] \]

and (replacing \( x \) by \( u + v \))

\[ \psi(u + v) = \psi(u) + \psi(v) \quad \text{for any} \ u, v \geq 0, \]

where \( \psi(x) = \varphi(x) - \varphi(0) \). So both the functions \( \psi \) and \( -\psi \) are convex on \([0, \infty)\) \[1\] 3.20. Since one of them is bounded from above on some interval, they are continuous \[1\] 3.18 and therefore linear \[1\] 3.19. Thus \( \psi(x) = \beta x \), where \( \beta \) is a constant, and

\[ \psi(x) = \frac{\beta}{2p}\{ |x| + (2p - 1)x \} \quad \text{for} \ x \geq 0. \]

In view of (11), the last equality is also valid for \( x < 0 \). Setting \( \alpha = \beta/2p \), we obtain (5) with \( f \) defined by (6). Finally, it follows from (5) and (1) for \( X = U_p(0,1) \) that \( \alpha \geq 0 \).

Remark. According to the known Blumberg-Sierpinski theorem \[3\], every measurable convex function is continuous. So the proof shows that the condition on \( \varphi \) in Theorem 3 may be replaced by measurability of \( \varphi \).
4. Convex measures of dispersion

A convex measure of dispersion is a measure $E\varphi(X - a)$ generated by a convex continuous function $\varphi$. The bases of the such measures may be described as follows.

**Theorem 4.** If $\varphi$ is convex and continuous on $\mathbb{R}$, then

$$B_\varphi = \{X \in B_0 : E\varphi_-(X) \leq 0 \leq E\varphi_+(X)\},$$

where $\varphi_-$ and $\varphi_+$ denote the left and right derivatives of $\varphi$, respectively.

In particular, if $\varphi$ is convex and differentiable on $\mathbb{R}$, then

$$B_\varphi = \{X \in B_0 : E\varphi'(X) = 0\}.$$

**Proof.** The proof of Theorem 4 is based on the following lemmas.

**Lemma 2.** Let functions $\psi_n(x)$ ($n = 1, 2, \ldots$) and their variations be uniformly bounded on an interval $[a, b]$ and let

$$\lim_{n \to \infty} \psi_n(x) = \psi(x) \text{ for each } x \in [a, b].$$

If $K(x)$ is a function of bounded variation on $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b \psi_n(x) \, dK(x) = \int_a^b \psi(x) \, dK(x).$$

It is enough to prove it for the cases in which $\psi(x) \equiv 0$ and $K(x)$ is either continuous or discrete on $[a, b]$. In the first case, it follows from the known Helly’s theorem by integration by parts. In the second case,

$$I_n = \int_a^b \psi_n(x) \, dK(x) = \sum_m \psi_n(x_m) h_m,$$

where $m = 1, 2, \ldots$, $x_m$ runs over all the points of discontinuity of $K(x)$ on $[a, b]$ and $h_m$ are the corresponding jumps, so that

$$\sum_m |h_m| < \infty.$$

Let $A > 0$, $|\psi_n(x)| \leq A$ for all $x \in [a, b]$, $n = 1, 2, \ldots$, and let $\varepsilon > 0$ and

$$\sum_{m > N} |h_m| \leq \varepsilon/A,$$

where $N = N(\varepsilon)$. Then

$$|I_n| \leq \sum_{m \leq N} |\psi_n(x_m) h_m| + \varepsilon,$$

whence it follows that

$$\limsup_{n \to \infty} |I_n| \leq \varepsilon, \quad \text{so} \quad \lim_{n \to \infty} I_n = 0,$$

because $\psi_n(x) \rightarrow 0$ and $\varepsilon > 0$ is arbitrary.
Lemma 3. Let the functions $\psi_n(x)$ increase on $\mathbb{R}$ and be uniformly bounded on each finite interval. If
\[
\lim_{n \to \infty} \psi_n(x) = \psi(x) \quad \text{for all real } x,
\]
then
\[
\lim_{n \to \infty} E\psi_n(X) = E\psi(X) \quad \text{for all } X \in B_0.
\]
It follows immediately from the previous lemma.

Lemma 4. Let $\tau(x)$ be a convex continuous function on $\mathbb{R}$. Then:
(i) the ratio
\[
\frac{\tau(x+h) - \tau(x)}{h} \quad (h \neq 0)
\]
is an increasing function of $x$ and $h$, bounded for bounded $x$ and $h$;
(ii) the equality
\[
\tau(x_0) = \min_x \tau(x)
\]
is equivalent to
\[
\tau'_-(x_0) \leq 0 \leq \tau'_+(x_0).
\]
It follows from known properties of convex functions [1, 3.18].

To prove Theorem 4, note that the function $\mu(x) = E\phi(X + x)$ is also convex and continuous on $\mathbb{R}$ for any fixed $X \in B_0$. By Lemmas 3 and 4,
\[
\mu'_\pm(0) = \lim_{h \to \pm 0} \frac{E\phi(X + h) - \phi(X)}{h} = E\phi'_\pm(X).
\]
By Lemma 4, $X \in B_\phi$ if and only if $\mu'_-(0) \leq 0 \leq \mu'_+(0)$. Taking (14) into account, we obtain (13). 

References